On the semisimplicity of the cyclotomic Brauer algebras, II

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Abstract

In this paper, we give a necessary and sufficient condition for the semisimplicity of cyclotomic Brauer algebras $B_{m,n}(\delta)$ of types $G(m, 1, n)$ with $m \geqslant 2$. This generalizes [H. Rui, A criterion on the semisimple Brauer algebras, J. Combin. Theory Ser. A 111 (2005) 78–88, 1.2–1.3] and [H. Rui, M. Si, A criterion on the semisimple Brauer algebras, II, J. Combin. Theory Ser. A 113 (2006) 1199–1203, 2.5] on Brauer algebras.

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1. Introduction

The cyclotomic Brauer algebras $B_{m,n}(\delta)$ have been introduced by Häring–Oldenburg in [10] as classical limits of cyclotomic Birman–Murakami–Wenzl algebras. When $m = 1$, they are Brauer algebras $B_n(\delta)$ [2].

The main purpose of this paper is to give a necessary and sufficient condition for the semisimplicity of $B_{m,n}(\delta)$ under the assumption $m \geqslant 2$. For $m = 1$, such a criterion has been given in [11, 1.2–1.3] and [12, 2.5].

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Unless otherwise stated, we assume that $F$ is a splitting field of $x^m - 1$, which contains $\delta_i, 1 \leq i \leq m$. By assumption, there are $u_i \in F$ such that $x^m - 1 = \prod_{i=1}^{m} (x - u_i)$. Define
\[
e = \begin{cases} +\infty, & \text{if char } F = 0, \\ \text{char } F, & \text{if char } F > 0. \end{cases} \tag{1.1}\]

Following [12], we define $Z_{m,n} = \{ma | a \in \tilde{\mathbb{Z}}_{m,n}\}$, where $\tilde{\mathbb{Z}}_{m,n}$ is given as follows:

1. $\tilde{\mathbb{Z}}_{2,n} = \tilde{\mathbb{Z}}_{1,n} = \{k \in \mathbb{Z} | 3 - n \leq k \leq n - 3\} \cup \{2k - 3 | 3 \leq k \leq n, k \in \mathbb{Z}\}$.
2. $\tilde{\mathbb{Z}}_{m,n} = \tilde{\mathbb{Z}}_{1,n} \cup \{2 - n, n - 2\}$ if $m \geq 3$ and $n \geq 2$.

Suppose that $x_1, x_2, \ldots, x_m$ are indeterminates over $F$. If $F$ contains $\xi$, a primitive $m$th root of unity, then we define
\[
\bar{x}_i = \sum_{j=1}^{m} x_j \xi^{ji}, \quad 0 \leq i \leq m - 1. \tag{1.2}\]

Note that $F$ contains $\xi$ if $e \nmid m$ [8, 8.2]. The following is the main result of this paper.

**Theorem A.** Fix two positive integers $m, n$ with $m > 1$. Let $\mathcal{B}_{m,n}(\delta)$ be a cyclotomic Brauer algebra over $F$.

(a) Suppose $n \geq 2$. If $\delta_i \neq 0$ for some $i, 0 \leq i \leq m - 1$, then $\mathcal{B}_{m,n}(\delta)$ is (split) semisimple if and only if
\[
(1) \ e \nmid m \cdot n!, \\
(2) \ \epsilon_{i,0} m - \bar{\delta}_i \notin \mathbb{Z}_{m,n}, 0 \leq i \leq m - 1, \text{ where } \epsilon_{i,0} \text{ is the Kronecker function.}
\]

(b) Suppose $n \geq 2$. If $\delta_i = 0, 0 \leq i \leq m - 1$, then $\mathcal{B}_{m,n}(0)$ is not (split) semisimple.

(c) $\mathcal{B}_{m,1}(\delta)$ is (split) semisimple if and only if $e \nmid m$.

In what follows, we write $\delta_j = \delta_i$ if $i, j \in \mathbb{Z}$ and $i \equiv j \mod m$.

Let $\mathcal{H}_{i,k}$ be the hyperplane in $F^m$, which is determined by the linear function $\epsilon_{i,0} m - \bar{x}_i = k, 0 \leq i \leq m - 1$ and $k \in \mathbb{Z}_{m,n}$. Condition (2) in Theorem A(a) is equivalent to the fact that $(\delta_0, \delta_1, \ldots, \delta_{m-1}) \notin \cup_{0 \leq i \leq m-1, k \in \mathbb{Z}_{m,n}} \mathcal{H}_{i,k}$. When $m = 1$, $\mathcal{H}_{i,k}$ collapses to a point in 1-dimensional $F$-space. This result has been proved in [11, 1.2-1.3] and [12, 2.5]. We remark that certain sufficient conditions for semisimplicity of complex Brauer algebras have been given in [3,4,14].


In [6], Graham and Lehrer have introduced the notion of cellular algebra which is defined over a poset $\Lambda$. Such an algebra has a nice basis, called a cellular basis. For each $\lambda \in \Lambda$, one can define $\Delta(\lambda)$, called a cell module. Graham and Lehrer have shown that there is a symmetric, associative bilinear form $\phi_\lambda$ defined on $\Delta(\lambda)$. It has been proved in [6, 3.8] that a cellular algebra is (split) semisimple if and only if $\phi_\lambda$ is non-degenerate for any $\lambda \in \Lambda$. It is well known that a cellular algebra is split semisimple if and only if it is semisimple. Therefore, one can determine whether a cellular algebra is semisimple by deciding if all $\phi_\lambda$ are non-degenerate.
In [6], Graham–Lehrer have proved that a Brauer algebra $B_n(\delta)$ over a commutative ring is a cellular algebra over the poset $\Lambda$ which consists of all pairs $(f, \lambda)$, with $0 \leq f \leq \lfloor n/2 \rfloor$ and $\lambda$ being a partition of $n - 2f$. Here $\lfloor n/2 \rfloor$ is the maximal integer which is no more than $n/2$. Therefore, one can study the semisimplicity of $B_n(\delta)$ by deciding whether $\phi_{f,\lambda}$ is non-degenerate or not for any $(f, \lambda) \in \Lambda$. Unfortunately, it is difficult to determine whether $\phi_{f,\lambda}$ is degenerate or not for a fixed $(f, \lambda)$.

In [11], the first author has proved that the semisimplicity of $B_n(\delta)$ is completely determined by $\phi_{f,\lambda}$ for all partitions $\lambda$ of $n - 2f$ with $f = 0, 1$. Using [4, 3.3–3.4], he has decided whether such $\phi_{f,\lambda}$’s are degenerate or not in [11]. This gives a complete solution of the problem of semisimplicity of $B_{m,n}(\delta)$ over an arbitrary field. This method will be used to study the semisimplicity of $B_{m,n}(\delta)$ in the current paper.

The contents of this paper are organized as follows. In Section 2, we state some results on cyclotomic Brauer algebras, and complex reflection group $W_{m,2}$. In Section 3, we describe explicitly the zero divisors of the discriminants for certain cell modules. Theorem A will be proved in Section 4.

2. Cyclotomic Brauer algebras

Let $R$ be a commutative ring which contains the identity $1_R$ and $\delta_i$, $1 \leq i \leq m$. The cyclotomic Brauer algebra $B_{m,n}(\delta)$ with parameters $\delta_i$, $1 \leq i \leq m$, is the associative $R$-algebra which is free as $R$-module with basis which consists of all labeled Brauer diagrams [10]. $B_{m,n}(\delta)$ can also be defined as the $R$-algebra generated by $\{s_i, e_i, t_j | 1 \leq i < n$ and $1 \leq j \leq n\}$ subject to the relations:

(a) $s_i^2 = 1$, for $1 \leq i < n$.
(b) $s_is_j = s_js_i$ if $|i - j| > 1$.
(c) $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$, for $1 \leq i < n - 1$.
(d) $s_it_j = t_js_i$ if $j \neq i, i + 1$.
(e) $e_i^2 = \delta_0 e_i$, for $1 \leq i < n$.
(f) $s_ie_j = e_is_j$, if $|i - j| > 1$.
(g) $e_ie_j = e_je_i$, if $|i - j| > 1$.
(h) $e_it_j = t_je_i$, if $j \neq i, i + 1$.
(i) $t_it_j = t_jt_i$, for $1 \leq i, j \leq n$.
(j) $s_it_i = t_{i+1}s_i$, for $1 \leq i < n$.
(k) $e_je_i = e_i e_j$, for $1 \leq i < n - 1$.
(l) $s_ie_{i+1}e_i = e_i s_{i+1}e_i$, for $1 \leq i < n - 2$.
(m) $e_{i+1}s_is_{i+1} = e_is_{i+1}s_i$, for $1 \leq i < n - 2$.
(n) $e_je_i e_j = e_j e_i e_j$, if $|i - j| = 1$.
(o) $e_it_it_{i+1} = e_i t_{i+1}e_i$, for $1 \leq i < n$.
(p) $e_it_i^a e_i = \delta_a e_i$, for $1 \leq a \leq m - 1$ and $1 \leq i \leq n - 1$.
(q) $t_i^m = 1$, for $1 \leq i \leq n$.

One can prove that the two definitions of $B_{m,n}(\delta)$ are equivalent by the arguments similar to those for Brauer algebras in [9].

The following result can be proved easily by checking the defining relations of $B_{m,n}(\delta)$.

**Lemma 2.1.** Let $B_{m,n}(\delta)$ be a cyclotomic Brauer algebra over $R$. There is an $R$-linear anti-involution $* : B_{m,n}(\delta) \to B_{m,n}(\delta)$ such that $h^* = h$ for all $h \in \{e_i, s_i, t_j | 1 \leq i < n, 1 \leq j \leq n\}$.

Recall that $F$ is a splitting field of $x^m - 1$. In the remaining part of this section, we assume $e \nmid m \cdot n!$. By [8, 8.2], $F$ contains $\xi$, a primitive $m$th root of unity.

We will decompose an $FW_{m,2}$-module in Proposition 2.5, where $W_{m,2}$ is the complex reflection group of type $G(m, 1, n)$. Note that $W_{m,2}$ is generated by $s_i, t_i$ satisfying the relations

- $s_i^2 = t_i^m = 1$ for $1 \leq i \leq n - 1$. 

The contents of this paper are organized as follows. In Section 2, we state some results on cyclotomic Brauer algebras, and complex reflection group $W_{m,2}$. In Section 3, we describe explicitly the zero divisors of the discriminants for certain cell modules. Theorem A will be proved in Section 4.
The order of $W_{m,n}$ is $m^n \cdot n!$. By Maschke’s theorem, the group algebra $FW_{m,n}$ is (split) semisimple.

Let $\Lambda^+_m(n)$ be the set of $m$-partitions of $n$. When $m = 1$, we use $\Lambda^+(n)$ instead of $\Lambda^+_1(n)$. For any $\lambda \in \Lambda^+_m(n)$, let $S^\lambda$ be the classical Specht module with respect to $\lambda$ (see [5, 2.1]).

For any $\lambda \in \Lambda^+(n)$, let $\mu = (\mu_1, \mu_2, \ldots)$ with $\mu_i = \# \{j \mid \lambda_j \geq i \}$. Then $\mu$, which will be denoted by $\lambda'$, is called the dual partition of $\lambda$. If $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)}) \in \Lambda^+_m(n)$, we write $\lambda' = (\lambda^{(m')}, \lambda^{(m-1')}, \ldots, \lambda^{(1')})$ and call $\lambda'$ the dual partition of $\lambda$.

**Remark 2.2.** All modules considered in this paper are left modules. I.e. $S^\lambda = FW_{m,n}y_{\lambda'}w_{\lambda'}x_{\lambda}$ if we keep the notation in [5]. In [13], we have assumed $u_i = \xi^i$, $1 \leq i \leq m$. In this paper, we keep this assumption in order to use results in [13] directly.

Since $FW_{m,n}$ is the Ariki–Koike algebra [1] with $q = 1$ and $x_1^m - 1 = \prod_{i=1}^m (x_1 - u_i)$, the following result is a special case of the result in [5].

**Lemma 2.3.** The set $\{S^\lambda \mid \lambda \in \Lambda^+_m(n)\}$ is a complete set of pairwise non-isomorphic irreducible $FW_{m,n}$-modules.

**Definition 2.4.** Let $m$ be a positive integer. If $m$ is even, we define $\wp_m(2) = \{\eta_i \mid \frac{m}{2} \leq i \leq m\}$, where

$$
\eta_i = \begin{cases} 
(0, \ldots, 0, 2), & \text{if } i = m, \\
(0, \ldots, 0, \frac{m}{2}, \ldots), & \text{if } i = m, \\
(0, \ldots, 0, 1, \ldots, 0, i), & \text{if } \frac{m}{2} < i \leq m - 1.
\end{cases}
$$

If $m$ is odd, we define $\wp_m(2) = \{\eta_i \mid \frac{m+1}{2} \leq i \leq m\}$, where

$$
\eta_i = \begin{cases} 
(0, \ldots, 0, 2), & \text{if } i = m, \\
(0, \ldots, 0, 1, \ldots, 0, i), & \text{if } \frac{m+1}{2} \leq i \leq m - 1.
\end{cases}
$$

**Proposition 2.5.** Let $\mathbb{Z}_m \cdot B_1$ be the subgroup of $W_{m,2}$ generated by $s_1, t_1t_2$. As $FW_{m,2}$-modules, $\text{Ind}_{\mathbb{Z}_m \cdot B_1}^{W_{m,2}} 1 \cong \bigoplus_{\eta \in \wp_m(2)} S^\eta$.

**Proof.** Since $\{1, t_1, \ldots, t_1^{m-1}\}$ is a complete set of left coset representatives of $\mathbb{Z}_m \cdot B_1$ in $W_{m,2}$, $\{t_1^k \sum_{i=0}^{m-1} (t_1t_2)^i (1 + s_1) \mid 0 \leq k \leq m - 1\}$ is an $F$-basis of $\text{Ind}_{\mathbb{Z}_m \cdot B_1}^{W_{m,2}} 1$. By assumption, $F$ contains
a primitive $m$th root of unity, say $\xi$. Since we are assuming that $u_i = \xi^i$, $1 \leq i \leq m$, $\text{Ind}_{\mathbb{Z}_m \wr \mathbb{S}_2} \mathbb{C}^1$ has a basis $\{w_i \mid 1 \leq i \leq m\}$, where

$$w_i = \prod_{j \neq i, \ 1 \leq j \leq m} (t_1 - u_j) \sum_{l=0}^{m-1} (t_1 t_2)^l (1 + s_1).$$

Since $\prod_{i=1}^m (t_1 - \xi^i) = 0$,

$$w_i = \prod_{j \neq i} (t_1 - u_j) \prod_{1 \leq j \leq m-1} (u_i t_2 - u_j) (1 + s_1).$$

By rescaling the above elements, $\{v_i \mid 1 \leq i \leq m\}$ is a basis of $\text{Ind}_{\mathbb{Z}_m \wr \mathbb{S}_2} \mathbb{C}^1$, where

$$v_i = \prod_{j \neq i} (t_1 - u_j) \prod_{j \neq m-i} (t_2 - u_j) (1 + s_1).$$

We have:

- $Fv_m$ is an $FW_{m,2}$-module with $s_1 v_m = t_1 v_m = v_m$. By [5, 2.1], $Fv_m \cong S^{n_2}$.

- Suppose $2 \mid m$. If $\frac{m-1}{2} \leq i \leq m-1$, then $\xi^i \neq \xi^{m-i}$. The subspace $Fv_i \oplus Fv_{m-i}$ is an $FW_{m,2}$-module such that $t_1 v_j = u_j v_j$ for $j = i, m - i$, and $s_1 v_i = v_{m-i}$. Therefore, $Fv_i \oplus Fv_{m-i} \cong S^{n_i}$, $\frac{m-1}{2} \leq i \leq m - 1$.

- Suppose $2 \not\mid m$. If $\frac{m}{2} < i \leq m - 1$, then $Fv_i \oplus Fv_{m-i}$ is an $FW_{m,2}$-module such that $t_1 v_j = u_j v_j$ for $j = i, m - i$, and $s_1 v_i = v_{m-i}$. Therefore, $Fv_i \oplus Fv_{m-i} \cong S^{n_i}$, $\frac{m}{2} < i \leq m - 1$.

- Suppose $i = \frac{m}{2}$. Then $Fv_i$ is an $FW_{m,2}$-module such that $s_1 v_i = v_i$ and $t_1 v_i = u_i v_i$. Therefore, $Fv_i \cong S^{n_i}$.

Consequently, $\text{Ind}_{\mathbb{Z}_m \wr \mathbb{S}_2} \mathbb{C}^1 \cong \bigoplus_{\eta \in \mathbb{S}_m(2)} S^{\eta}$ no matter whether $m$ is even or odd.

**Remark 2.6.** Proposition 2.5 is a special case of [13, (4.4)]. The decomposition given there involves certain $m$-partitions $\eta$. In fact, we have to put more restrictions on $\eta$. The reason is that $\sum_{l=0}^{m-1} t_1^l w e_a$ may be equal to zero for general $a$ (Here, we keep the notation in [13]). Therefore, the first equality in [13, (4.3)] is not true in general. If we denote by $c_\eta$ the multiplicity of $S^{\eta}$ in $\text{Ind}_{\mathbb{Z}_m \wr \mathbb{S}_{2k}} \mathbb{C}^1$, [13, (4.1), 6.2] are still true although we do not know the explicit description of $c_\eta$. Proposition 2.5 gives us the explicit information for $\eta$ and $c_\eta$ when $k = 1$.

In the remaining part of this section, we recall the result in [13], which says that $\mathcal{B}_{m,n}(\delta)$ is a cellular algebra in the sense of [6]. We also prove Theorem 2.9, which will play the key role in the proof of Theorem A.

Recall that a dotted Brauer diagram $D$ with $k$ horizontal arcs is determined by a pair of labeled $(n, k)$-parenthesis diagrams $\alpha, \beta$ and $w \in W_{m,n-2k}$, and vice versa [13]. In this situation, we write $D = \alpha \otimes w \otimes \beta$ if

- $\alpha$ (respectively $\beta$) is the top (respectively bottom) row of $D$. 

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• \( w \) corresponds to the dotted Brauer diagram (or braid diagram) which is obtained from \( D \) by removing the horizontal arcs at top and bottom rows of \( D \).

We denote by \( P(n, k) \) the set of all labeled \((n, k)\)-parenthesis diagrams.

A Young diagram \( Y(\lambda) \) for a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) is a collection of boxes arranged in left-justified rows with \( \lambda_i \) boxes in the \( i \)th row of \( Y(\lambda) \). Suppose \( \lambda \in \Lambda_m^+/(n) \) with \( \lambda = (\lambda(1), \lambda(2), \ldots, \lambda(m)) \). The Young diagram \( Y(\lambda) = (Y(\lambda(1)), Y(\lambda(2)), \ldots, Y(\lambda(m))) \) is a bijection \( t = (t_1, \ldots, t_{m-1}, t_m) : (Y(\lambda(1)), \ldots, Y(\lambda(m-1)), Y(\lambda(m))) \rightarrow \{1, 2, \ldots, n\} \). If the entries in each \( t_i \), \( 1 \leq i \leq m \) increase from left to right in each row and from top to bottom in each column, then \( t \) is called a standard \( \lambda \)-tableau. Let \( T^s(\lambda) \) be the set of all standard \( \lambda \)-tableaux.

Let \( \{y^\lambda_{s,t} | \lambda \in \Lambda_m^+(n), s, t \in T^s(\lambda)\} \) be the Murphy basis for \( FW_{m,n} \) \([5, 2.8]\).

Define

\[
C^{(k,\lambda)}_{(\alpha,s), (\beta,t)} = \alpha \otimes y^\lambda_{s,t} \otimes \beta, \quad \alpha, \beta \in P(n, k), \ s, t \in T^s(\lambda). \tag{2.7}
\]

Recall that \( R \) is a commutative ring containing the identity 1 and \( \delta_1, \ldots, \delta_m \).

**Theorem 2.8.** (See \([13, 5.11]\).) Suppose \( R \) contains \( u_1, \ldots, u_m \) such that \( x^m - 1 = (x - u_1) \times (x - u_2) \cdots (x - u_m) \). Let \( \Lambda = \{(f, \lambda) | 0 \leq f \leq [n/2], \lambda \in \Lambda_m^+(n - 2f)\} \). Then

\[
\{C^{(k,\lambda)}_{(\alpha,s), (\beta,t)} | \alpha, \beta \in P(n, k), s, t \in T^s(\lambda), (k, \lambda) \in \Lambda\}
\]

is a cellular basis of \( B_{m,n}(\delta) \). The \( R \)-linear anti-involution defined on \( B_{m,n}(\delta) \) is that defined in Lemma 2.1.

Following \([6, 2.1]\), we have the cell modules for \( B_{m,n}(\delta) \) with respect to the cellular basis provided in Theorem 2.8. Let \( \Delta(k, \lambda) \) be the cell module for \( B_{m,n}(\delta) \) with respect to \((k, \lambda) \in \Lambda \). Let \( \Delta(\lambda) \) be the cell module for \( FW_{m,n} \) with respect to the cellular basis \( \{y^\lambda_{s,t} | \lambda \in \Lambda_m^+(n), s, t \in T^s(\lambda)\} \).

It has been proved in \([5, 2.7]\) that \( \Delta(\lambda) \cong S^{\lambda'} \), where \( \lambda' \) is the dual partition of \( \lambda \). By \([6, 2.1]\), \( \Delta(k, \lambda) \) is spanned by \( \alpha \otimes v_j \otimes \alpha_0 \mod B_{m,n}(\delta)^{\rightarrow(k,\lambda)} \), where \( v_j \) ranges over the basis elements of \( S^{\lambda'} \).

Suppose \( \lambda \in \Lambda_m^+(n) \) and \( \mu \in \Lambda_m^+(n - 1) \). If there is a pair \((i, j)\) such that \( \lambda_i^{(j)} = \mu_i^{(j)} + 1 \) and \( \lambda_j^{(k)} = \mu_j^{(k)} \) for any \((k, l) \neq (j, i)\), then we write \( \mu \rightarrow \lambda \) and say that \( \mu \) is obtained from \( \lambda \) by removing a box. In this situation, we also say that \( \lambda \) can be obtained from \( \mu \) by adding a box.

**Theorem 2.9.** Let \( B_{m,n}(\delta) \) be a cyclotomic Brauer algebra over \( F \). If \( \mu \in \Lambda_m^+(n - 2) \) and \( \lambda \in \Lambda_m^+(n) \), then either \( [\Delta(1, \mu') : \Delta(\lambda')] = 0 \) or \([\Delta(1, \mu') : \Delta(\lambda')] = 1 \). Furthermore, \([\Delta(1, \mu') : \Delta(\lambda')] = 1 \) if and only if one of the following conditions holds true.

1. \( \lambda^{(j)} = \mu^{(j)} \), \( j \neq m \) and two boxes in the skew Young diagram \( Y(\lambda^{(m)}/\mu^{(m)}) \) are not in the same column.
2. Suppose that \( m \) is odd. There is an \( i \) with \( \frac{m+1}{2} \leq i \leq m - 1 \) such that \( \mu^{(i)} \rightarrow \lambda^{(i)} \) and \( \mu^{(m-i)} \rightarrow \lambda^{(m-i)} \), and \( \lambda^{(j)} = \mu^{(j)} \) for \( j \neq i, m - i \).
3. Suppose that \( m \) is even. There is an \( i \) with \( \frac{m}{2} < i \leq m - 1 \) such that \( \mu^{(i)} \rightarrow \lambda^{(i)} \) and \( \mu^{(m-i)} \rightarrow \lambda^{(m-i)} \), and \( \lambda^{(j)} = \mu^{(j)} \) for \( j \neq i, m - i \).
4. Suppose that \( m \) is even. \( \lambda^{(j)} = \mu^{(j)}, j \neq m/2 \) and two boxes in the skew Young diagram \( Y(\lambda^{(m)/\mu^{(m)})} \) are not in the same column.
Definition 2.10. Suppose $\mu, \lambda \in \Lambda_m^+(n-2)$ and $\lambda \in \Lambda_m^+(n)$. $\lambda$ is called $\mu$-admissible if one of the conditions in Theorem 2.9(1)–(4) holds true. Let $\mathscr{A}(\mu)$ be the set of all $\mu$-admissible $m$-partitions.

3. Zero divisors of certain discriminants

In this section, we assume $\delta_i \in F$ for $1 \leq i \leq m$, where $F$ is a splitting field of $x^m - 1$ and $e \nmid m \cdot n!$. The main purpose of this section is to prove Theorem 3.9, which will give all zero divisors of the discriminants of the Gram matrices $G_{1,\mu'}$ with respect to the cell modules $\Delta(1,\mu')$, $\mu \in \Lambda_m^+(n-2)$.

Recall that $P(n,k)$ is the set of labeled parenthesis Brauer diagrams with $k$ horizontal arcs. In what follows, we assume $\alpha_0 = \text{top}(e_{n-1}) \in P(n,1)$, the top row of $e_{n-1}$. Define $M_1$ and $M_2$ by setting

- $M_1 = \{\alpha \otimes w \otimes \alpha_0 \mid \alpha \in P(n,1), \ w \in W_{m,n-2}\}$.
- $M_2 = \{\alpha \otimes w \otimes \beta \mid \alpha, \beta \in P(n,k), \ w \in W_{m,n-2k}, \ 2 \leq k \leq \lfloor \frac{n}{2} \rfloor \}$.

We consider the quotient $F$-subspace $V = V_1/V_2$, where $V_1$ (respectively $V_2$) is spanned by $M_1 \cup M_2$ (respectively $M_2$). For convenience, we use $\alpha \otimes w \otimes \alpha_0$ instead of $\alpha \otimes w \otimes \alpha_0 + V_2$.

Recall that any dotted Brauer diagram can be written as $\alpha \otimes w \otimes \beta$ where $\alpha, \beta \in P(n,k)$ and $w \in W_{m,n-2k}$. Let $\tilde{\alpha} \in P(n,k)$ be such that

(a) $\alpha$ and $\tilde{\alpha}$ have the same horizontal arcs.
(b) There are $m-i$ dots on a horizontal arc in $\tilde{\alpha}$ if and only if there are $i$ dots on the corresponding horizontal arc in $\alpha$.

Define an $R$-linear isomorphism $\iota: \mathcal{B}_{m,n}(\delta) \to \mathcal{B}_{m,n}(\delta)$ by declaring that

$$
\iota(\alpha \otimes w \otimes \beta) = \tilde{\beta} \otimes w^{-1} \otimes \tilde{\alpha}.
$$

(3.1)
We remark that $\iota$ is not an algebraic (anti-)homomorphism since $\iota(e_it^k_i e_i) = \delta_k e_i \neq \delta_{m-k} e_i$ in general. However, by straightforward computation, we have
\[
\iota(w(\alpha \otimes w_1 \otimes \beta)) = \iota(\alpha \otimes w_1 \otimes \beta)w^{-1},
\] (3.2)
for any $\alpha, \beta \in P(n, k), w \in W_{m,n}, w_1 \in W_{m,n-2k}$.

Following [7], we have the following definition.

**Definition 3.3.** Suppose $\alpha_i \otimes w \otimes \alpha_0 \in V$ for $i = 1, 2$. Let $(\alpha_1 \otimes w_1 \otimes \alpha_0, \alpha_2 \otimes w_2 \otimes \alpha_0)$ be the coefficient of $e_{n-1}$ in the expression of $\iota(\alpha_1 \otimes w_1 \otimes \alpha_0) \cdot (\alpha_2 \otimes w_2 \otimes \alpha_0)$, where $\iota$ is defined in (3.1). Let $G_{m,n}(\delta)$ be the $f \times f$-matrix with $f = \dim V$ such that the entry in $(\alpha_1 \otimes w_1 \otimes \alpha_0)$th row, $(\alpha_2 \otimes w_2 \otimes \alpha_0)$th column is $(\alpha_1 \otimes w_1 \otimes \alpha_0, \alpha_2 \otimes w_2 \otimes \alpha_0)$.

If either $h_1 \in M_2$ or $h_2 \in M_2$, then $\iota(h_1)h_2 \in V_2$. Since $e_{n-1} \notin V_2$, $(h_1, h_2) = 0$. Hence, $(\cdot, \cdot) : V \times V \to F$ is a well-defined $F$-bilinear form on $V$.

The following lemma can be verified easily.

**Lemma 3.4.** $G_{m,n}(\delta) = (g_{ij})$ is an $f \times f$ matrix such that $g_{ii} = \delta_0, 1 \leq i \leq f$ and $g_{ij} \in \{0, 1, \delta_1, \ldots, \delta_{m-1}\}$ if $i \neq j$.

**Lemma 3.5.** $G_{m,n}(\delta) : V \to V$ is a left $FW_{m,n}$-homomorphism and a right $FW_{m,n-2}$ homomorphism.

**Proof.** We consider $G_{m,n}(\delta)$ as the $F$-linear endomorphism on $V$ such that
\[
G_{m,n}(\delta)(\alpha_1 \otimes w_1 \otimes \alpha_0) = \sum_{\alpha \in P(n, 1), w \in W_{m,n-2}} (\alpha \otimes w \otimes \alpha_0, \alpha_1 \otimes w_1 \otimes \alpha_0)\alpha \otimes w \otimes \alpha_0.
\]
By (3.2),
\[
\langle w(\alpha_1 \otimes w_1 \otimes \alpha_0), w(\alpha_2 \otimes w_2 \otimes \alpha_0) \rangle = \langle \alpha_1 \otimes w_1 \otimes \alpha_0, \alpha_2 \otimes w_2 \otimes \alpha_0 \rangle.
\]
In other words, $G_{m,n}(\delta) : V \to V$ is a left $FW_{m,n}$-homomorphism.

On the other hand, since $(\alpha_1 \otimes w_1 \otimes \alpha_0)y = \alpha_1 \otimes w_1 y \otimes \alpha_0$ for any $y \in W_{m,n-2}$, $e_{n-1}$ appears in $y^{-1}(\alpha_0 \otimes w_1 \otimes \alpha_0)y$ with non-zero coefficient if and only if $w_1 = 1$. Therefore,
\[
\langle (\alpha_1 \otimes w_1 \otimes \alpha_0)y, (\alpha_2 \otimes w_2 \otimes \alpha_0)y \rangle = \langle \alpha_1 \otimes w_1 \otimes \alpha_0, \alpha_2 \otimes w_2 \otimes \alpha_0 \rangle.
\]
Consequently, $G_{m,n}(\delta) : V \to V$ is a right $FW_{m,n-2}$-homomorphism. \(\square\)

Since we are assuming that $F$ is a splitting field of $x^m - 1$ and $e \nmid m \cdot n!$, $FW_{m,k}$ is (split) semisimple for any $k, 1 \leq k \leq n$. Assume that $\lambda \in \Lambda_m^+(k)$, the classical Specht module $S^\lambda$ is a direct summand of $FW_{m,k}$. Consequently, $\Delta(1, \lambda')$ can be realized as a submodule of $V$, which is spanned by $\alpha \otimes v_j \otimes \alpha_0$ (mod $V_2$), where $v_j$ ranges over the basis elements of $S^\lambda$. Note that $G_{m,n}(\delta)$ is a right $FW_{m,n-2}$-module. For any $\lambda \in \Lambda_m^+(n-2)$, the restriction of $G_{m,n}(\delta)$ on $\Delta(1, \lambda')$ induces a linear endomorphism on $\Delta(1, \lambda')$. 
Definition 3.6. For \( \mu \in \Lambda_+^m(n - 2) \), define \( g_{\mu} = \prod_{\lambda \in \mathcal{A}(\mu)} g_{\lambda, \mu} \), where

\[
g_{\lambda, \mu} = \left( \tilde{\delta}_0 - m + m \sum_{p \in Y(\lambda/\mu)} c(p) \right)^{m-1} \prod_{i=1}^{m} \left( \tilde{\delta}_i + m \sum_{p \in Y(\lambda/\mu)} c(p) \right).
\]

(3.7)

It follows from [6, 2.3] that there is a unique symmetric bilinear form defined on each cell module \( \Delta(k, \lambda) \). Via such a bilinear form, one can define a Gram matrix \( G_{k, \lambda} \). Let \( \det G_{k, \lambda} \) be the determinant of \( G_{k, \lambda} \). The following result follows from [6, 3.8] and Theorem 2.8, immediately.

Lemma 3.8. \( \mathcal{B}_{m,n}(\delta) \) is (split) semisimple over \( F \) if and only if \( \det G_{k, \lambda} \neq 0 \) for all \( (k, \lambda) \in \Lambda \).

In general, it is difficult to compute \( \det G_{k, \lambda} \). Assume \( \delta_i \neq 0 \) for some \( 1 \leq i \leq m \). The following result describes all the zero divisors of \( \det G_{1,\lambda} \), \( \lambda \in \Lambda_+^m(n - 2) \). Fortunately, it completely determines \( \mathcal{B}_{m,n}(\delta) \) being (split) semisimple.

Theorem 3.9. Suppose \( \delta_i \neq 0 \) for some \( i, 1 \leq i \leq m \). \( \det G_{1,\mu} = 0 \) if and only if \( g_{\mu} = 0 \).

Proof. \((\Rightarrow)\) If \( \det G_{1,\mu} = 0 \), then we can find an irreducible \( \mathcal{B}_{m,n}(\delta) \)-module \( M \subset \text{Rad} G_{1,\mu}' \), where \( \text{Rad} G_{1,\mu}' = \{ v \in \Delta(1, \mu') \mid G_{1,\mu}'(v) = 0 \} \). It follows from [6, 2.6, 3.4] that any irreducible module of a cellular algebra must be the simple head of a cell module, say \( \Delta(k, \lambda') \). Hence, there is a non-zero homomorphism from \( \Delta(k, \lambda') \) to \( \Delta(1, \mu') \) with \( (k, \lambda') < (1, \mu') \). Therefore, either \( k = 1 \) or \( k = 0 \).

Assume that \( k = 1 \). We use [13, 7.4] to get a non-zero homomorphism from \( \Delta(0, \lambda') \) to \( \Delta(0, \mu') \). Notice that, as \( FW_{m,n} \)-modules, \( \Delta(0, \lambda') \cong S^k \). Since \( FW_{m,n} \) is (split) semisimple, we have \( \lambda = \mu \), a contradiction since \( (1, \lambda') < (1, \mu') \).

If \( k = 0 \), then there is a non-zero \( \mathcal{B}_{m,n}(\delta) \)-homomorphism from \( \Delta(0, \lambda') \) to \( \Delta(1, \mu') \), forcing \( \lambda \in \mathcal{A}(\mu) \). By [13, 8.6, 8.8], \( g_{\lambda, \mu} = 0 \). We have \( g_{\mu} = 0 \) as required.

\((\Leftarrow)\) Suppose \( g_{\mu} = 0 \). Then there is a \( \lambda \in \mathcal{A}(\mu) \) such that \( g_{\lambda, \mu} = 0 \). Since \( \lambda \in \mathcal{A}(\mu) \), by Theorem 2.9, \( [\Delta(1, \mu') : S^k] = 1 \). Hence, there is a unique \( FW_{m,n} \)-submodule of \( \Delta(1, \mu') \) which is isomorphic to \( S^k \). Recall that \( G_{m,n}(\delta)|_{\Delta(1, \mu')} \) is a linear endomorphism on \( \Delta(1, \mu') \). For simplicity, we use \( G_{m,n}(\delta) \) instead of \( G_{m,n}(\delta)|_{\Delta(1, \mu')} \) if there is no confusion.

Since \( G_{m,n}(\delta) \) is an \( FW_{m,n} \)-homomorphism, and \( [\Delta(1, \mu') : S^k] = 1 \), \( G_{m,n}(\delta)(M) \subset M \). By Schur’s Lemma, \( G_{m,n}(\delta)|_M = f(\delta)I \), where \( I \) is dim \( M \times \text{dim} M \) identity matrix and \( f(\delta) := f(\delta_0, \delta_1, \ldots, \delta_{m-1}) \) is a polynomial in \( \delta_i, 0 \leq i \leq m - 1 \).

Take a basis of \( M \) and extend it to get a basis of \( V \) via the elements \( \alpha \otimes w \otimes \alpha_0 \). Then \( G_{m,n}(\delta) \) is conjugate to \( \left( \begin{array}{cc} f(\delta)I & 0 \\ * & B \end{array} \right) \), where any entry in the diagonal of \( B \) is \( \delta_0 \), and the term of the entry of \( B \) elsewhere does not contains \( \delta_0 \). Since the degree of \( \delta_0 \) in \( \det G_{m,n}(\delta) \) is dim \( V \) (see Lemma 3.4), the degree of \( \delta_0 \) in \( f(\delta) \) must be 1. In particular, \( f(\delta) \) is not a constant number.

Take the parameters \( \delta_0, \delta_1, \ldots, \delta_{m-1} \) such that \( f(\delta) = 0 \). Then \( G_{m,n}(\delta)|_M = 0 \).

We claim \( e_{n-1}v = 0 \) for any \( v \in M \). Write \( v = \sum_{\alpha \otimes w} a_{\alpha \otimes w} \alpha \otimes w \otimes \alpha_0 \), where there are \( s \) dots at the left endpoint of the unique arc in \( \alpha \otimes w \). We divide \( P(n, 1) \) into three disjoint subsets \( P_1, P_2, P_3 \) as follows. Recall that a point in \( \alpha \otimes w \) is called a fixed point if it is an endpoint of a horizontal arc of \( \alpha \otimes w \). Otherwise, it is called a free point.
• $P_1$ consists of all $\alpha^x \in P(n, 1)$ such that $(n - 1, n)$ is a unique arc of $\alpha^x$. Then $e_{n-1}(\alpha^x \otimes w \otimes \alpha_0) = \delta_0 \alpha_0 \otimes w \otimes \alpha_0$.

• $P_2$ consists of all $\alpha^x \in P(n, 1)$ such that both $n - 1$ and $n$ are free points in $\alpha$. Then $e_{n-1} \alpha^x \otimes w \otimes \alpha_0 = 0$.

• $P_3$ consists of all $\alpha^x \in P(n, 1)$ such that either $n - 1$ or $n$ is a fixed point. Let $i$ be the left endpoint of the unique arc in $\alpha^x$. By assumption, there are $s$ dots at the endpoint $i$. We define $w_{\alpha^x} \in \mathfrak{S}_{n-2}$ by setting

$$w_{\alpha^x} = \begin{pmatrix} i & i+1 & i+2 & \cdots & n-3 & n-2 \\ n-2 & i & i+1 & \cdots & n-4 & n-3 \end{pmatrix}.$$ 

Define $y_{\alpha^x} := t_i^i w_{\alpha^x}$. Then $e_{n-1} \cdot (\alpha^x \otimes 1 \otimes \alpha_0) = \alpha_0 \otimes y_{\alpha^x} \otimes \alpha_0$. Therefore, the coefficient of $\alpha_0 \otimes w_1 \otimes \alpha_0$ in $e_{n-1} v$ is

$$\sum_{\alpha^x \in P_3} a_{\alpha^x, y_{\alpha^x}^{-1}} w_1 + \sum_{s=0}^{m-1} \delta_s a_{\alpha^x_0, w_1}.$$ 

On the other hand, by direct computation, the coefficient of $\alpha_0 \otimes w_1 \otimes \alpha_0$ in $G_{m,n}(\delta)v$ is

$$\sum_{\alpha^x \in P(n,1), w \in W_{m,n-2}} a_{\alpha^x, w}(\alpha_0 \otimes w_1 \otimes \alpha_0, \alpha^x \otimes w \otimes \alpha_0).$$

We have

$$\langle \alpha_0 \otimes w_1 \otimes \alpha_0, \alpha^x \otimes w \otimes \alpha_0 \rangle = \begin{cases} \delta_s, & \text{if } \alpha^x \in P_1, \ w = w_1, \\ 0, & \text{if } \alpha^x \in P_1, \ w \neq w_1, \\ 0, & \text{if } \alpha^x \in P_2, \\ 1, & \text{if } \alpha^x \in P_3 \text{ and } w = y_{\alpha^x}^{-1} w_1, \\ 0, & \text{if } \alpha^x \in P_3 \text{ and } w \neq y_{\alpha^x}^{-1} w_1. \end{cases}$$

Since $G_{m,n}(\delta)v = 0$, the coefficient of $\alpha_0 \otimes w_1 \otimes \alpha_0$ in $G_{m,n}(\delta)v$ is zero. Therefore,

$$\sum_{\alpha^x \in P(n,1), \ w \in W_{m,n-2}} a_{\alpha^x, w}(\alpha_0 \otimes w_1 \otimes \alpha_0, \alpha^x \otimes w \otimes \alpha_0) = \sum_{\alpha^x \in P_3} a_{\alpha^x, y_{\alpha^x}^{-1}} w_1 + \sum_{s=0}^{m-1} \delta_s a_{\alpha^x_0, w_1} = 0,$$

forcing the coefficient of $\alpha_0 \otimes w_1 \otimes \alpha_0$ in $e_{n-1} v$ to be zero for all $w_1 \in W_{m,n-2}$. This completes the proof of the claim.

Therefore, as $\mathcal{B}_{m,n}(\delta)$-module, $M \cong \Delta(0, \lambda')$. We obtain a non-zero $\mathcal{B}_{m,n}(\delta)$-homomorphism from $\Delta(0, \lambda')$ to $\Delta(1, \mu')$. In particular, $\det G_{1,\mu'} = 0$. By [13, 8.6, 8.8] the parameters $\delta_i$’s must satisfy the equation $g_{\lambda, \mu} = 0$, the condition we have assumed. \hfill $\Box$

4. Proof of Theorem A

In this section, we prove Theorem A, the main result of this paper. Unless otherwise stated, we assume that $F$ is a splitting field of $x^m - 1$, which contains $\delta_i$, $1 \leq i \leq m$. Assume $m > 1$.

**Proposition 4.1.** Suppose $n \geq 2$. If $0 \neq \delta_i \in F$ for some $i$, $1 \leq i \leq m$, then $\mathcal{B}_{m,n}(\delta)$ is (split) semisimple if and only if $\delta_i \mid m \cdot n!$ and $\det G_{1,\lambda} \neq 0$ for any $\lambda \in \Lambda^+_m(k - 2)$, $2 \leq k \leq n$.

---

3 Under our assumption, the group algebra $FW_{m,n}$ is (split) semisimple. Since the proof of [13, 8.6, 8.8] depends only on the fact that $CW_{m,n}$ is (split) semisimple, we can apply these results here.
Proof. \((\Leftarrow)\) Suppose that \(\mathcal{B}_{m,n}(\delta)\) is not (split) semisimple. There is a \((k, \lambda) \in \Lambda\) such that \(\det \mathcal{G}_{k,\lambda} = 0\). Since \(FW_{m,n}\) is (split) semisimple, \(k \neq 0\).

Take an irreducible submodule \(M \subset \mathcal{R} \Delta(k, \mu)\). By [6, 2.6, 3.4], \(M\) must be isomorphic to the simple head of a cell module, say \(\Delta(l, \lambda)\), such that \((l, \lambda) < (k, \mu)\). Furthermore, it results in a non-trivial homomorphism from \(\Delta(l, \lambda)\) to \(\Delta(k, \mu)\).

If \(l = k\), we use [13, 7.4] to get \(\Delta(0, \lambda) \cong \Delta(0, \mu)\). As \(FW_{m,n-2k}\)-modules, \(\Delta(0, \lambda) \cong S^\mu\). Since \(FW_{m,n-2k}\) is (split) semisimple, \(\lambda = \mu\), which contradicts \((l, \lambda) < (k, \mu)\).

Suppose \(l < k\). By [13, 7.4, 7.7], there is a non-trivial homomorphism from \(\Delta(0, \tilde{\lambda})\) to \(\Delta(1, \tilde{\mu})\) for some \(\tilde{\mu} \in \Lambda_m^+(p-2)\) with \(p \leq n\). By assumption, \(\det \mathcal{G}_{1,\tilde{\mu}} \neq 0\). Hence, \(\Delta(1, \tilde{\mu}) = D^{(1, \tilde{\mu})} \cong \Delta(0, \tilde{\lambda})\).

\((\Rightarrow)\) If \(\mathcal{B}_{m,n}(\delta)\) is (split) semisimple, then [6, 3.8] implies that \(\det \mathcal{G}_{k,\lambda} = 0\) for all \(0 \leq k \leq \lceil \frac{n}{2} \rceil\). Therefore, \(FW_{m,n}\) is (split) semisimple, forcing \(e \nmid m \cdot n\).

Suppose \(\det \mathcal{G}_{1,\mu'} = 0\) for some \(\mu' \in \Lambda_m^+(k-2)\). Then \(k < n\). By Theorem 3.9, there is a \(\mu\)-admissible \(m\)-partition \(\lambda\) such that \(\det \mathcal{G}_{k,\mu} = 0\). Equivalently, there is a non-zero \(\mathcal{B}_{m,n}(\delta)\)-homomorphism from \(\Delta(0, \lambda')\) to \(\Delta(1, \mu')\).

Since we are assuming that \(m \geq 2\), we can find an \(i, 1 \leq i \leq m\), such that \(\chi(i) = \mu(i)\). We can add \(l\) boxes to \(\lambda(i)\) so as to get another partition \(\tilde{\lambda}(i) = \tilde{\mu}(i)\). In this situation, \(\tilde{\lambda}(\tilde{i}) = \tilde{\mu}(\tilde{i})\), where \(\tilde{\lambda}\) (respectively \(\tilde{\mu}\)) can be obtained from \(\lambda\) (respectively \(\mu\)) by using \(\tilde{\lambda}(i)\) instead of \(\lambda(i)\) (respectively \(\mu(i)\)). By definition, \(\tilde{\lambda} \in \mathcal{A}(\tilde{\mu})\). If we take \(l\) such that \(|\lambda| + l = n\), then \(\Delta(0, \tilde{\lambda})\) and \(\Delta(1, \tilde{\mu})\) are \(\mathcal{B}_{m,n}(\delta)\)-modules. By Theorem 3.9, \(\det \mathcal{G}_{1,\tilde{\mu}} = 0\). However, since \(\mathcal{B}_{m,n}(\delta)\) is (split) semisimple, \(\det \mathcal{G}_{1,\mu'} \neq 0\), a contradiction. \(\square\)

Corollary 4.2. Let \(\mathcal{B}_{m,n}(\delta)\) be a cyclotomic Brauer algebra over \(F\), where \(F\) contains a non-zero \(\delta_i\) for some \(i, 1 \leq i \leq m\). \(\mathcal{B}_{m,n}(\delta)\) is (split) semisimple if only if \(\det \mathcal{G}_{k,\lambda} \neq 0\) for all \(\lambda \in \Lambda_m^+(n-2k)\), and \(k = 0, 1\).

Proof. Suppose \(\mathcal{B}_{m,n}(\delta)\) is (split) semisimple. It follows from [6, 3.8] that \(\det \mathcal{G}_{k,\lambda} \neq 0\) for all \(0 \leq k \leq \lceil \frac{n}{2} \rceil\). In particular, \(\det \mathcal{G}_{k,\lambda} \neq 0\) with \(k = 0, 1\) and \(\lambda \in \Lambda_m^+(n-2k)\).

Conversely, if \(\det \mathcal{G}_{0,\lambda} \neq 0\) for all \(\lambda \in \Lambda_m^+(n)\), then \(FW_{m,n}\) is (split) semisimple. Suppose that \(\mathcal{B}_{m,n}(\delta)\) is not (split) semisimple. By Proposition 4.1, there is a \(\mu \in \Lambda_m^+(k-2)\) with \(k < n\) such that \(\det \mathcal{G}_{1,\mu} = 0\). From the proof of Proposition 4.1, we can find a \(\tilde{\mu} \in \Lambda_m^+(n-2)\) such that \(\det \mathcal{G}_{1,\tilde{\mu}} = 0\). This contradicts our assumption. \(\square\)

Corollary 4.2 has been stated as a question in [13, p220]. We remark that Corollary 4.2 is not true if \(m = 1\). In fact, the first author has proved that the Brauer algebra \(\mathcal{B}_n(\delta)\) is (split) semisimple over \(F\) if and only if \(e \nmid n\) and \(\det \mathcal{G}_{1,\lambda} \neq 0\) for all \(\lambda \in \Lambda^+(k-2), 2 \leq k \leq n\). By [4, 3.3–3.4], Corollary 4.2 is not true if \(m = 1\).

Definition 4.3. Suppose that \(m, n \in \mathbb{N}\) with \(n \geq 2\). For \(m \geq 2\), define \(\rho_{m,n} = \{ma \mid a \in \tilde{\rho}_{m,n}\}\), where

\[
\tilde{\rho}_{m,n} = \left\{ k \in \mathbb{Z} \mid k = \sum_{p \in Y(\lambda/\mu)} c(p) \mid \mu \in \Lambda_m^+(n-2), \lambda \in \mathcal{A}(\mu) \right\}.
\]
If \( m = 1 \), we define

\[
\tilde{\rho}_{m,n} = \left\{ r \in \mathbb{Z} \mid r = \sum_{p \in Y(\lambda/\mu)} c(p) \mid \mu \in \Lambda^+(k-2), \lambda \in \Lambda^+(k), 2 \leq k \leq n \right\},
\]

where two boxes in \( Y(\lambda/\mu) \) are not in the same column.

At the end of this paper, we will prove \( \tilde{\rho}_{m,n} = \tilde{\mathbb{Z}}_{m,n} \). Hence, \( \rho_{m,n} = \mathbb{Z}_{m,n} \).

**Theorem 4.4.** Let \( \mathcal{B}_m(n) \) be a cyclotomic Brauer algebra over \( F \), where \( F \) contains a non-zero \( \delta_i \) for some \( i, 1 \leq i \leq m \). Suppose \( n \geq 2 \). \( \mathcal{B}_m(n) \) is (split) semisimple if and only if

\( e \mid m \cdot n! \)

(2) \( \varepsilon_i,0 \rho_{m,n}, 0 \leq i \leq m - 1 \), where \( \varepsilon_{i,0} \) is the Kronecker function.

**Proof.** The result follows from Theorem 3.9 and Corollary 4.2. \( \square \)

In the remaining part of this section, we deal with the case \( \delta_i = 0 \) for all \( 1 \leq i \leq m \). First, we discuss \( \mathcal{B}_{m,3}(0) \).

We want to compute \( \det G_{1,\lambda} \) with \( \lambda = (1, 0, \ldots, 0) \). Note that we have assumed \( u_i = \xi^i \).

\( 1 \leq i \leq m \). In this situation, \( y_{\lambda}: w_{\lambda}: x_{\lambda} = g(t_1) = \prod_{i=1}^{m-1} (t_1 - \xi^i) \).

Write \( v_1^{(0)} = \text{top}(e_1) \), \( v_2^{(0)} = \text{top}(s_1 e_2) \) and \( v_3^{(0)} = \text{top}(e_2) \). Let \( v_i^{(k)} \) be obtained from \( v_i^{(0)} \) by putting \( k \) dots at the left endpoint of the unique horizontal arc in \( v_i^{(0)} \). Then \( \Delta(1, \lambda) \) can be considered as a free \( F \)-module with basis \( \{ v_i^{(k)} \otimes g(t_1) \otimes v_i^{(0)} \mid 1 \leq i \leq 3, 0 \leq k \leq m - 1 \} \). Let \( a = \prod_{i=1}^{m-1} (1 - \xi^i) \). The Gram matrix with respect to this basis is

\[
G_{1,\lambda} = \begin{pmatrix}
0 & A & A \\
A & 0 & A \\
A & A & 0
\end{pmatrix},
\]

where \( A = (a_{ij}) \) is the \( m \times m \) matrix with \( a_{ij} = a, 1 \leq i, j \leq m \). Since we are assuming that \( m > 1 \), \( \det G_{1,\lambda} = 0 \). In other words, \( \text{Rad} \Delta(1, \lambda) \neq 0 \). Take an irreducible submodule \( D \) of \( \text{Rad} \Delta(1, \lambda) \). Note that any irreducible module must be the simple head of a cell module, say \( \Delta(k, \mu) \). Therefore, there is a non-trivial homomorphism from \( \Delta(k, \mu) \) to \( \Delta(1, \lambda) \). By [6, 2.6], \( (k, \mu) < (1, \lambda) \). This proves the following lemma.

**Lemma 4.5.** Suppose \( \lambda = (1, 0, \ldots, 0) \). There is a cell module \( \Delta(k, \mu) \) of \( \mathcal{B}_{m,3}(0) \) with \( (k, \mu) < (1, \lambda) \) such that there is a non-trivial homomorphism from \( \Delta(k, \mu) \) to \( \Delta(1, \lambda) \).

Let \( J_{m,n}(0) \) be the left ideal of \( \mathcal{B}_{m,n}(0) \) spanned by the dotted Brauer diagrams \( D \) such that \( \{ n - 1, n \} \) is a horizontal arc at the bottom row of \( D \). It is clear that \( J_{m,n}(0) = \mathcal{B}_{m,n}(0)e_{n-1} \).

Following [13], let \( I_{m,n} \) (respectively \( I_{m,n}^k \)) be the vector space generated by \( (n, l) \)-dotted Brauer diagrams with \( l \geq k \) (respectively \( l > k \)). Let \( I_{m,n}^k(0) = \mathcal{B}_{m,n}/I_{m,n}^k \). Then \( I_{m,n}^k(0) \) is a \( \mathcal{B}_{m,n}(0) \)-module. Let \( I_{m,n}^k(0) \) be the subspace of \( I_{m,n}^k(0) \) generated by \( \{ \alpha \otimes w \otimes \beta_0 \mid \alpha \in P(n, k) \),
$w \in W_{m,n-2k}$, where $\beta_0 = \text{top}(e_{n-2k+1} \cdots e_{n-3} e_{n-1})$. Let $\mathcal{B}_{m,n}(0)$-mod be the category of the left $\mathcal{B}_{m,n}(0)$-modules. Let $G : \mathcal{B}_{m,n-2}(0)\text{-mod} \rightarrow \mathcal{B}_{m,n}(0)\text{-mod}$ be the tensor functor defined by declaring that $G(M) = J_{m,n}(0) \otimes \mathcal{B}_{m,n-2}(0) M$, for any $\mathcal{B}_{m,n-2}(0)\text{-mod} M$.

**Proposition 4.6.** Suppose $\lambda \in \Lambda^+_m(n-2k)$.

(a) The functor $G$ sends non-zero $\mathcal{B}_{m,n-2}(0)$-homomorphisms to non-zero ones.

(b) $G(\Delta(k-1, \lambda)) = \Delta(k, \lambda)$.

**Proof.** Suppose $\phi : M_1 \rightarrow M_2$ is a $\mathcal{B}_{m,n-2}(0)$-module homomorphism. Write $\phi_* = G(\phi)$. For any $D_1 \in \mathcal{B}_{m,n}(0)$, $D \in J_{m,n}(0)$ and $m \in M_1$,

$$
\phi_* (D_1 (D \otimes m)) = \phi_* (D_1 D \otimes m) = (D_1 D) \otimes \phi(m) = D_1 (D \otimes \phi(m)) = D_1 \phi_* (D \otimes m)
$$

Therefore, $\phi_*$ is a $\mathcal{B}_{m,n}(0)$-homomorphism. For any $\mathcal{B}_{m,n-2}(0)$-module $M$, define an $F$-linear map $\alpha : J_{m,n}(0) \otimes \mathcal{B}_{m,n-2}(0) M \rightarrow M$ by setting $\alpha(D \otimes m) = (e_{n-1} D)_0 m$, where $(e_{n-1} D)_0$ is obtained from $e_{n-1} D$ by removing the horizontal arcs $\{n-1, n\}$ at the top and bottom rows of $e_{n-1} D$.

Suppose $D^* = s_{n-2} e_{n-1} \in J_{m,n}(0)$. Then $\alpha(D^* \otimes m) = m$. If $\phi \neq 0$, then there is an $m_1 \in M_1$ such that $\phi(m_1) = m_2 \neq 0$. Consequently, $\alpha(D^* \otimes m_2) = m_2 \neq 0$. We have $\phi_* \neq 0$ since $\phi_*(D^* \otimes m_1) = D^* \otimes m_2 \neq 0$. This completes the proof of (a).

(b) can be proved similarly as [13, 7.2]. We include a proof as follows. First, we claim as ($\mathcal{B}_{m,n}(0)$, $W_{m,n-2k}$)-modules

$$
I^k_{m,n} (0) \cong J_{m,n}(0) \otimes \mathcal{B}_{m,n-2}(0) I^{k-1}_{m,n-2}(0).
$$

(4.7)

For the simplification in exposition and notation, we omit $\mathcal{B}_{m,n-2}(0)$ in what follows.

Suppose $D_1 \otimes D_2 \in J_{m,n}(0) \otimes I^{k-1}_{m,n-2}(0)$. Let $e_{i,j} = \alpha \otimes 1 \otimes \alpha$, where $\alpha \in P(n, 1)$ contains a unique horizontal arc $\{i, j\}$. Define $e_{i,j}^x = t_i^x e_{i,j} t_j^x$. We claim that there is a dotted Brauer diagram $D'_1$ in $I^{k-1}_{m,n}(0)$ such that $D_1 \otimes D_2 = D'_1 \otimes e_{i_1,j_1}^{s_1,t_1} \cdots e_{i_{k-1},j_{k-1}}^{s_{k-1},t_{k-1}} D_2$, where $e_{i_l,j_l}^{s_l,t_l} \in \mathcal{B}_{m,n-2}(0)$, $1 \leq l \leq k-1$.

In fact, if the bottom row of $D_1$ contains a horizontal arc $\{i, j\}$, which is different from $\{n-1, n\}$ and if there are $t$ dots at the left endpoint $i$ of $\{i, j\}$, then we can find another horizontal arc $\{i', j'\}$ at the top row of $D_1$ such that there are $s$ dots at the left endpoint $i'$ of $\{i', j'\}$. Using vertical arcs $\{i, i'\}$ and $\{j, j'\}$ instead of the horizontal arcs $\{i, j\}$ and $\{i', j'\}$ in $D_1$, we get another dotted Brauer diagram $\tilde{D}_1$. We have $D_1 = \tilde{D}_1 e_{i,j}^x$. Note that the number of horizontal arcs in $\text{top}(\tilde{D}_1)$ is $k-1$ if the number of horizontal arcs in $\text{top}(D_1)$ is $k$. Using this method repeatedly, we have $D_1 \otimes D_2 = D'_1 \otimes e_{i_1,j_1}^{s_1,t_1} \cdots e_{i_{k-1},j_{k-1}}^{s_{k-1},t_{k-1}} D_2$.

Since $D_2 \in I^{k-1}_{m,n-2}(0)$, the number of the horizontal arcs in the top row of the composite of $e_{i_1,j_1}^{s_1,t_1} \cdots e_{i_{k-1},j_{k-1}}^{s_{k-1},t_{k-1}}$ and $D_2$ is at least $k-1$. If it is bigger than $k$, then $e_{i_1,j_1}^{s_1,t_1} \cdots e_{i_{k-1},j_{k-1}}^{s_{k-1},t_{k-1}} D_2 = 0$
First, we assume that $\prod_{i=1}^{n-1} D_2 = 0$ since $\delta_i = 0$, $0 \leq i \leq m - 1$. We have $D_1 \otimes D_2 = 0$. In the remaining case, $\prod_{i=1}^{n-1} D_2 = w \cdot e_{n-3} e_{n-5} \cdots e_{n-2k+1}$ for some $w \in W_{m,n-2}$. Note that $e_{n-1} w = w e_{n-1}$,

$$
D_1 \otimes D_2 = D_1' \otimes e_{i_1,j_1} \cdots e_{i_{k-1},j_{k-1}} D_2 = D_1' w \otimes e_{n-3} e_{n-5} \cdots e_{n-2k+1}.
$$

Since $(n-1, n)$ is the unique horizontal arc at the bottom row of $D_1'$, $D_1' w = w_1 e_{n-1}$ for some $w_1 \in W_{m,n}$. Hence, $D_1 \otimes D_2 = w_1 e_{n-1} \otimes e_{n-3} e_{n-5} \cdots e_{n-2k+1}$. We can identify $D_1 \otimes D_2$ with $w_1 e_{n-1} e_{n-3} \cdots e_{n-2k+1} \in I_{m,n}^k(0)$ and vice versa. This proves $\dim_F U_0 = \dim_F I_{m,n}^k(0)$, where

$$U_0 = J_{m,n}(0) \otimes_{\mathcal{B}_{m,n-2}} I_{m,n-2}^k(0).$$

On the other hand, for any $\alpha \in I_{m,n-2}^k(0)$, let $\alpha_2^{\lambda}$ be obtained from $\alpha_2$ by adding two vertical arcs $\{n-1, n-1\}$ and $\{n, n\}$. The $F$-linear map $\phi: U_0 \to I_{m,n}^k(0)$ sending $\alpha_1 \otimes \alpha_2$ to $\alpha_1 \cdot \alpha_2$ is surjective. Since $\dim_F U_0 = \dim_F I_{m,n}^k(0)$, it must be injective. By the definition of the product of two dotted Brauer diagrams in [13], we can verify that $\phi$ is a $(\mathcal{B}_{m,n}(0), W_{m,n-2k})$-homomorphism. This completes the proof of the claim.

By [13, (6.3)], $I_{m,n}^k(0) \otimes_{W_{m,n-2k}} S^\lambda \cong S^{(k, \lambda)}$. Therefore,

$$
G(S^{(k-1, \lambda)}) = \mathcal{B}_{m,n}(0) e_{n-1} \otimes_{\mathcal{B}_{m,n-2}} (I_{m,n-2}^k(0) \otimes_{W_{m,n-2k}} S^\lambda) = \mathcal{B}_{m,n}(0) e_{n-1} \otimes_{\mathcal{B}_{m,n-2}} I_{m,n-2}^k(0) \otimes_{W_{m,n-2k}} S^\lambda \

\cong I_{m,n}^k(0) \otimes_{W_{m,n-2k}} S^\lambda \cong S^{(k, \lambda)}.\quad \Box
$$

The following theorem is Theorem A(b).

**Theorem 4.8.** If $n \geq 2$, then $\mathcal{B}_{m,n}(0)$ is not (split) semisimple over $F$.

**Proof.** First, we assume that $n$ is even. A direct computation shows that the Gram matrix $G_{\mathcal{Z},0}$ with respect to $\Delta(\mathcal{Z}, 0)$ is zero. In particular, $\det G_{\mathcal{Z},0} = 0$. By [6, 3.8], $\mathcal{B}_{m,n}(0)$ is not (split) semisimple. Suppose $n$ is odd. We have $n \geq 3$. By Lemma 4.5, there is a non-zero $\phi \in \text{Hom}_{\mathcal{B}_{m,n}}(\Delta(k, \mu), \Delta(1, \lambda))$, where $\lambda = ((1), 0, 0, \ldots, 0) \in \Delta_m^+(1)$. Write $n = 3 + 2l$ for some $l \in \mathbb{N}$. Applying Proposition 4.6 $l$ times, we get a non-zero homomorphism from $\Delta(k+l, \mu)$ to $\Delta(1+l, \lambda)$. By [6, 3.8], $\mathcal{B}_{m,n}(0)$ is not (split) semisimple. \Box

In order to complete the proof of Theorem A(a), we need verify $\tilde{\rho}_{m,n} = \tilde{\mathcal{Z}}_{m,n}$.

**Proposition 4.9.** Suppose $m, n \in \mathbb{N}$ with $n \geq 2$.

1. $\tilde{\rho}_{2,n} = \tilde{\rho}_{1,n} = \{k \in \mathbb{Z} \mid 3 - n \leq k \leq n - 3\} \cup \{2k - 3 \mid 3 \leq k \leq n, k \in \mathbb{Z}\}$.
2. $\tilde{\rho}_{m,n} = \tilde{\rho}_{1,n} \cup \{2 - n, n - 2\}$ if $m \geq 3$.

**Proof.** First, we assume $m = 2$. If $\mu \in \Delta_m^+(n-2)$ and $\lambda \in \mathcal{A}(\mu)$, then either $\lambda^{(1)} = \mu^{(1)}$, $\lambda^{(2)} \in \mathcal{A}(\mu^{(2)})$ or $\lambda^{(2)} = \mu^{(2)}$, $\lambda^{(1)} \in \mathcal{A}(\mu^{(1)})$. We can assume $\lambda^{(1)} \in \mathcal{A}(\mu^{(1)})$ without loss of generality. Suppose $|\mu^{(1)}| = k$. Then $k$ can be any integer between 0 and $n - 2$. If $r = \sum_{p \in \mathcal{Y}(\lambda, \mu)} c(p)$,
then \( r \in \tilde{\rho}_{1,n} \), forcing \( \tilde{\rho}_{2,n} \subseteq \tilde{\rho}_{1,n} \). Identifying \( \lambda \in \Lambda^+(k) \) with bipartition \((\lambda, (n-k))\), we have \( \tilde{\rho}_{2,n} \supseteq \tilde{\rho}_{1,n} \). This proves the first equality in (1). Following [11], we define

\[
\mathcal{Z}(n) = \left\{ r \in \mathbb{Z} \mid r = 1 - \sum_{p \in Y(\lambda/\mu)} c(p), \lambda \in \Lambda^+(k), \mu \in \Lambda^+(k-2), 2 \leq k \leq n \right\},
\]

where two boxes in \( Y(\lambda/\mu) \) are not in the same column. By [12, 2.4],

\[
\mathcal{Z}(n) = \{ i \in \mathbb{Z} \mid 4 - 2n \leq i \leq n - 2 \} \setminus \{ i \in \mathbb{Z} \mid 4 - 2n < i \leq 3 - n, 2 \notdivides i \}.
\]

Therefore, the second equality in (1) follows.

Suppose \( m \geq 3 \). If \( \mu \) and \( \lambda \) satisfy one of the conditions in Theorem 2.9(1) and Theorem 2.9(4), then \( \sum_{p \in Y(\lambda/\mu)} c(p) \in \tilde{\rho}_{1,n} \). If \( \mu \) and \( \lambda \) satisfy the conditions (2) or (3) in Theorem 2.9, then there is an \( i \), such that \( \mu^{(i)} \to \lambda^{(i)} \) and \( \mu^{(m-i)} \to \lambda^{(m-i)} \). In this situation, \( \sum_{p \in Y(\lambda/\mu)} c(p) \in \Xi_a + \Xi_b \) with \( |\mu^{(i)}| = a \) and \( |\mu^{(m-i)}| = b \), where

- \( \Xi_a = \{ \sum_{p \in Y(\lambda/\mu)} c(p) \mid \mu \in \Lambda^+(a), \mu \to \lambda \} \), and
- \( \Xi_a + \Xi_b = \{ i \mid i = x + y, x \in \Xi_a, y \in \Xi_b \} \).

Note that we can choose a suitable \( \mu \) such that \( a + b = i \) for all \( i, 0 \leq i \leq n - 2 \). We claim

\[
\Xi_a = \begin{cases} 
\{0\}, & \text{if } a = 0, \\
\{ i \in \mathbb{Z} \mid -a \leq i \leq a \} \setminus \{0\}, & \text{if } a = 1, 2, \\
\{ i \in \mathbb{Z} \mid -a \leq i \leq a \}, & \text{otherwise.}
\end{cases}
\]

In fact, one can verify the above result directly when \( a \in \{0, 1, 2, 3\} \).

Suppose \( \mu \in \Lambda^+(k + 1) \) and \( \mu \to \lambda \). If \( \lambda \) has at least two removable nodes, then we can find a box \( q \) which is a removable node for both \( \lambda \) and \( \mu \). Let \( \tilde{\lambda} \) (respectively \( \tilde{\mu} \)) be obtained from \( \lambda \) (respectively \( \mu \)) by removing \( q \). Then

\[
\sum_{p \in Y(\lambda/\mu)} c(p) = \sum_{p \in Y(\tilde{\lambda}/\tilde{\mu})} c(p) \in \Xi_k = \{-k \leq i \leq k\},
\]

the last equality follows from the induction assumption.

If \( \lambda \) has a unique removable node, then \( \lambda = (\lambda_1, \ldots, \lambda_r) \) with \( \lambda_i = \lambda_j, 1 \leq i, j \leq r \). We have \( \sum_{p \in Y(\lambda/\mu)} c(p) = \lambda_1 - r \). Note that \(-1 - k \leq \lambda_1 - r \leq k + 1 \). In any case, we have \( \Xi_{k+1} \subset \{ i \in \mathbb{Z} \mid -1 - k \leq i \leq 1 + k \} \).

Conversely, by the induction assumption, we can write \( i = \sum_{p \in Y(\lambda/\mu)} c(p) \), for some \( \lambda \in \Lambda^+(k + 1) \) and \( \mu \to \lambda \) if \(-k \leq i \leq k \). Since any Young diagram of a partition has at least two addable nodes, we can choose an addable node \( q \) for both \( \lambda \) and \( \mu \) such that \( q \) and \( \lambda/\mu \) are not in the same row. In other words, \( i \in \Xi_{k+1} \). We have

- \( \sum_{p \in Y(\lambda/\mu)} c(p) = -(k+1) \) if \( \lambda = (1, \ldots, 1) \in \Lambda^+(k+2) \) and \( \mu = (1, \ldots, 1) \in \Lambda^+(k+1) \).
- \( \sum_{p \in Y(\lambda/\mu)} c(p) = k + 1 \) if \( \lambda = (k+2) \) and \( \mu = (k+1) \).
Consequently, $\mathcal{S}_{k+1} \supset \{i \in \mathbb{Z} \mid -k - 1 \leq i \leq k + 1\}$. This completes the proof of the claim. Therefore,

$$\bigcup_{0 \leq a+b \leq n-2} T_a + T_b = \{i \in \mathbb{Z} \mid 2 - n \leq i \leq n - 2\}.$$

Note that $i \in \tilde{\rho}_{1,n}$ if $3 - n \leq i \leq n - 3$. (2) follows immediately. \Box

**Proof of Theorem A(a) and (c).** Theorem A(a) follows from Theorem 4.4 and Proposition 4.9. Theorem A(c) follows from Maschke’s theorem. \Box

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**References**


