Linearly ordered extensions of GO spaces *

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Received 14 February 1992

Abstract


We prove a main theorem: Theorem. There always exists a minimal linearly ordered d-extension of a GO space, where a LOTS Y is said to be a linearly ordered d-extension of a GO space \( \langle X, \tau, \prec \rangle \) if Y contains X as a dense subspace and the ordering of Y extends the ordering \( \prec \) of X. As some applications of the Theorem, (1) we give a partial negative answer to a problem: "Does every perfect GO space have a perfect orderable d-extension?" (2) For a discrete space \( \langle X, \tau \rangle \) of cardinality \( \omega_1 \), there is a linear ordering \( \preceq \) of X such that \( \langle X, \tau, \preceq \rangle \) is a GO space and whose every linearly ordered d-extension contains an order preserving copy of the ordinal space \( \omega_1 \) as a dense subspace.

Keywords: GO space; LOTS; (minimal) Linearly ordered d-extension; Orderable space; Perfect GO space.

AMS (MOS) Subj. Class: 54F05, 54A10.

1. Introduction

A linearly ordered topological space (abbreviated LOTS) is a triple \( \langle X, \lambda, \leq \rangle \), where \( \langle X, \leq \rangle \) is a linearly ordered set and \( \lambda \) is the usual interval topology defined by \( \leq \) (i.e., \( \lambda \) is the topology generated by \( \{ (a, \rightarrow): a \in X \} \cup \{ (\leftarrow, a): a \in X \} \) as a subbase), where \( (a, \rightarrow) = \{ x \in X: a < x \} \) and \( (\leftarrow, a) = \{ a \in X: a > x \} \). Similarly \( (a, b) = \{ x \in X: a < x < b \}, [a, b) = \{ x \in X: a \leq x < b \}, [a, b] = \{ x \in X: a \leq x \leq b \} \), etc. If necessary, we write \( \leq_x, (a, b)_X \) instead of \( \leq, (a, b) \). Throughout this paper, \( \lambda \) or \( \lambda_X \)

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* Dedicated to Professor Akihito Okuyama on his 60th birthday.
denote the usual interval topology on a linearly ordered set \( \langle X, \leq \rangle \). Furthermore we use the notations \( \mathbb{N}, \mathbb{Z}, \mathbb{R} \) as the set of all natural numbers, all integers, and all real numbers, respectively.

A **generalized ordered space** (abbreviated **GO space**) is a triple \( \langle X, \tau, \leq \rangle \), where \( \langle X, \leq \rangle \) is a linearly ordered set and \( \tau \) is a topology on \( X \) such that \( \lambda \subset \tau \) and \( \tau \) has a base of open sets each of which is order convex, where a set \( A \) of \( X \) is called **order convex** if \( x \in A \) for every \( x \) lying between two points of \( A \). For a GO space \( \langle X, \tau, \leq \rangle \) and \( Y \subset X \), \( \tau|Y \) denotes the subspace topology \( \{U \cap Y : U \in \tau\} \) for \( Y \) and \( \leq |Y \) denotes the restricted ordering of \( \leq \) for \( Y \). If it will cause no confusion, we shall omit to mention of \( \lambda \) (or \( \tau \)) and \( \leq \), and write simply “Let \( X \) be a LOTS (GO space)”. A topological space \( \langle X, \tau \rangle \), where \( \tau \) is a topology on a set \( X \), is said to be **orderable** if \( \langle X, \tau, \leq \rangle \) is a LOTS for some linear ordering \( \leq \) on \( X \). Similarly, we write simply “Let \( X \) be an orderable space” if it will cause no confusion. A LOTS \( Y = \langle Y, \lambda, \leq \rangle \) is said to be a **linearly ordered extension** of a GO space \( X = \langle X, \tau, \leq \rangle \) if \( X \subset Y \), \( \tau = \lambda | X \) and \( \leq = \leq | X \). Furthermore if \( X \) is closed (respectively dense) in the space \( \langle Y, \lambda \rangle \), then \( Y \) is said to be a **linearly ordered c-extension** (respectively **d-extension**) of \( X \). Similarly an orderable space \( Y = \langle Y, \tau_Y \rangle \) is said to be an **orderable** (c-, d-) **extension** of a GO space \( X = \langle X, \tau_X, \leq \rangle \) if \( X \) is a (closed, dense) subset of \( Y \) and \( \tau_X = \tau_Y | X \). Note that every GO space has a compact linearly ordered d-extension \([2,3.12.3]\).

Let \( X = \langle X, \tau, \leq \rangle \) be a GO space and \( \lambda \) the usual order topology on \( X \). Define a subset \( X^* = \langle X, \tau, \leq \rangle^* \) of \( X \times \mathbb{Z} \) by

\[
X^* = X \times \{0\} \cup \{(x, n) : x \in X, (x, \to) \in \tau - \lambda \text{ and } n < 0\}
\cup \{(x, m) : x \subset X, (\prec, x) \subset \tau - \lambda \text{ and } m > 0\}
\]

\([3, \text{Definition 2.5}]\). Then \( X^* \) is considered as a linearly ordered c-extension of \( X \) by identifying \( X = X \times \{0\} \).

For many topological properties \( P \), it is known that a GO space with a property \( P \) has an orderable extension which also has property \( P \). For example, the following are known.

(a) If a GO space \( X \) is metrizable, then so is \( X^* \) \([3, \text{Proposition 5.5}]\).

(b) If a GO space is (hereditarily) paracompact, then so is \( X^* \) \([3, \text{Theorem 4.2}]\). But the situation \( P = \text{"perfect"} \) is unclear, where a topological space is **perfect** if each of its closed subsets is a \( G_\delta \)-set. The following problem was posed in \([1, \text{Question 1}]\).

**Problem 1.1.** Does every perfect GO space have a perfect orderable extension?

In connection with this, the following is known from \([3, \text{Theorem 5.9} \text{ and Example 7.2}]\).

(c) The Sorgenfrey line \( S \) is a perfect GO space, but does not have a perfect orderable c-extension.
However, $S$ does not provide an example necessary to answer Problem 1.1 negatively, since the LOTS $\mathbb{R} \times (0, 1)$ with the lexicographic ordering is a perfect linearly ordered $d$-extension of $S$.

The following problem which is a special case of Problem 1.1 was posed in [4, Question (VI)].

**Problem 1.2.** Does every perfect GO space have a perfect orderable $d$-extension?

In this paper, we shall prove in Section 2 that there always exists a minimal linearly ordered $d$-extension of a GO space. In Section 3, we shall give an example which is a partial negative answer for Problem 1.2. In Section 4, we pose a problem: "What properties are hereditary to linearly ordered $d$-extensions?", and we shall show: For a discrete space $\langle X, \tau \rangle$ of cardinality $\omega_1$, there is a linear ordering $\prec$ of $X$ such that $\langle X, \tau, \prec \rangle$ is a GO space and whose every linearly ordered $d$-extension contains an order preserving copy of the ordinal space $\omega_1$ as a dense subspace. This shows that metrizability, hereditary paracompactness and perfectness are not hereditary properties to linearly ordered $d$-extensions.

## 2. Existence of a minimal linearly ordered $d$-extension

First we define a LOTS $\tilde{X}$ for a GO space $X$. Let $X = \langle X, \tau, \leq \rangle$ be a GO space and let $X$ be the usual order topology on $X$. Define a subset $\tilde{X} = \langle X, \tau, \leq \rangle$ of $X \times \{0, 1\}$ by

$$\tilde{X} = X \times \{0\} \cup \{ (x, -1): x \in X \text{ and } [x, -) \in \tau - \lambda \} \cup \{ (x, 1): x \in X \text{ and } (,-x) \in \tau - \lambda \}.$$ 

Throughout this paper, we identify $X$ with $X \times \{0\}$ and consider $\tilde{X}$ as a LOTS by the lexicographic order $\preceq$ on $\tilde{X}$. The LOTS $\tilde{X}$ is sometimes denoted by $\langle X, \lambda, \preceq \rangle$. Then it is straightforward to show that $\tilde{X}$ is a linearly ordered $d$-extension of $X$. Note that $X^*$ is not a linearly ordered extension of $\tilde{X}$ under the natural correspondence, and that $X^* = \tilde{X} = X$ if $X$ is a LOTS.

In the following theorem, we prove that, for a GO space $X$, $\tilde{X}$ can be considered as a minimal (in the sense of inclusion) linearly ordered $d$-extension of $X$.

**Theorem 2.1.** Let $X = \langle X, \tau, \leq \rangle$ be a GO space, and $Y = \langle Y, \lambda, \leq \rangle$ a linearly ordered $d$-extension of $X$. Then there is an order preserving homeomorphism $f$ from $\tilde{X}$ into $Y$ such that $f(x) = x$ for each $x \in X$.

**Proof.** Using the notation in the definition of $\tilde{X}$, let $\lambda$ be the usual order topology on $X$ and $\tilde{X} = \langle X, \lambda, \preceq \rangle$. 
Claim 1. If \( x \in X \) and \( (x, \rightarrow) \in \tau - \lambda \) (i.e., \( (x, -1) \in \tilde{X} \)), then \( x \) has an immediate predecessor \( x_- \) in \( Y \) with \( x_- \notin X \).

Proof. Assume \( x \in X \) and \( (x, \rightarrow) \in \tau - \lambda \). Since \( (x, \rightarrow) \in \tau \) and \( Y \) is a linearly ordered extension of \( X \), there is \( y, z \in Y \) with \( y <_Y x <_Y z \) such that \( (y, z)_Y \in Y \) \( (x, -1) \in X \), where \( (a, b) \) (respectively \( (a, b)_Y \)) denotes the open interval in \( X \) (respectively in \( Y \)). Then it is easy to show that \( y \) is an immediate predecessor \( x_- \) of \( x \) in \( Y \) (i.e., \( (y, x)_Y = \emptyset \)) using the fact that \( X \) is dense in \( Y \). It follows from \( (x, \rightarrow) \notin \lambda \) that \( x_- \notin X \).

Similarly we have:

Claim 1'. If \( x \in X \) and \( (\leftarrow, x) \in \tau - \lambda \) (i.e., \( (x, 1) \in \tilde{X} \)), then \( x \) has an immediate successor \( x_+ \) in \( Y \) with \( x_+ \notin X \).

Define a function \( f : \tilde{X} \to Y \) by

\[
 f(z) = \begin{cases} 
  z, & \text{if } z = (z, 0) \in X = X \times \{0\}, \\
  x_-, & \text{if } z = (x, 1) \in \tilde{X}, \\
  x_+, & \text{if } z = (x, 1) \in \tilde{X}.
\end{cases}
\]

Obviously, we have \( f(x) = x \) for each \( x \in X \). We show:

Claim 2. \( f \) is order preserving.

Proof. Let \( z = (x, n) \) and \( z' = (x', n') \) be points in \( \tilde{X} \) with \( z < z' \). Then it is clear that \( x < x' \). We shall show \( f(z) <_Y f(z') \). There are six cases: (i) \( x < x' \), \( z = (x, 1), \) \( z' = (x', -1) \), (ii) \( x < x', z = (x, 1), \) \( z' = (x', 0) \), (iii) \( x < x', \) \( z = (x, 0), \) \( z' = (x', -1) \), (iv) \( z = (x, -1), \) \( z' = (x, 0) \), (v) \( z = (x, -1), \) \( z' = (x, 1) \), (vi) \( z = (x, 0), \) \( z' = (x, 1) \). Since the proofs of other cases are much more simpler, we only show the case (i). In this case, we have \( f(z) = x_+ \) and \( f(z') = x_- \). Since \( x < x' \) and \( x_+ \notin X \), we have \( x_+ <_Y x' \). Furthermore, since \( x_- \) is an immediate predecessor of \( x' \) in \( Y \), we have \( x_- <_Y x' \). Assume \( x_+ = x' \). Since \( x_+ \notin X \) by Claim 1, \( x' \) is the immediate successor of \( x \) in \( X \) (i.e., \( (x, x') = \emptyset \)). So we have \( [x', \rightarrow) = (x, \rightarrow) \in \lambda \). Then by the definition of \( \tilde{X} \), we have \( z' = (x', -1) \notin \tilde{X} \). This is a contradiction. Therefore we have \( f(z) <_Y f(z') \). This completes the proof of Claim 2.

Claim 3. \( f : \tilde{X} \to f(\tilde{X}) \) is a homeomorphism.

Proof. It suffices to show that \( f \) and \( f^{-1} \) are continuous. Note that the topology on \( f(\tilde{X}) \) is the subspace topology of \( \lambda_Y \).

First, it is easy to show the continuity of \( f^{-1} \) from the fact that \( f((x, \rightarrow)^-) = (f(z), \rightarrow)_Y \cap f(\tilde{X}) \) and \( f((\leftarrow, z)^+) = (\leftarrow, f(z))_Y \cap f(\tilde{X}) \) for each \( z \in \tilde{X} \), where \( (\leftarrow, \rightarrow)^- = \{u \in \tilde{X} : z \leq u\}, (\leftarrow, \rightarrow)^+ = \{u \in \tilde{X} : u < z\} \).

Secondly, we show the continuity of \( f \). It suffices to show \( f^{-1}((y, \rightarrow)_Y) \) and \( f^{-1}((\leftarrow, y)_Y) \) are open in \( (\tilde{X}, \tilde{A}) \) for each \( y \in Y \). We only show the first. Let \( y \in Y \). If \( y \in f(\tilde{X}) \), there is \( u \in \tilde{X} \) with \( f(u) = y \). Then we have \( f^{-1}((y, \rightarrow)_Y) = (u, \tilde{A}) \).
Next, we consider the case "\( y \not\in f(X) \)". It suffices to show the next subclaim.

**Subclaim.** \( f^{-1}((y, \to)_Y) = \bigcup \{(u, \to) : y <_Y u, u \in X\} \) if \( y \not\in f(X) \).

**Proof.** The inclusion "\( \subseteq \)" is evident. To show the inclusion "\( \subset \)", let \( z \in f^{-1}((y, \to)_Y) \). Note that \( y <_Y f(z) \).

**Fact.** \( (y, f(z))_Y \neq \emptyset \).

**Proof of fact.** There are three cases.

**Case 1:** \( z = (\langle x, 1 \rangle) \) for some \( x \in X \).

In this case, since \( f(z) = x_+ \) is the immediate successor of \( x \) in \( Y \) and \( y \in f(X) \), we have \( x \in (y, f(z))_Y \).

**Case 2:** \( z = (\langle x, 0 \rangle) (=x) \) for some \( x \in X \).

Assume that \( (y, f(z))_Y (= (y, x)_Y) \) is empty. First we show that \( x \) has no immediate predecessor in \( X \) (i.e., \( \{x, \to\} \notin \lambda \)). To show this, let \( x' \) be a point in \( X \) with \( x' < x \). Then we have \( x' = f(x') <_Y y <_Y f(z) = f(x) = x \). It follows from \( y \not\in f(X) \) and \( x' \in X \) that \( y \in (x', x)_Y \). Since \( X \) is dense in \( Y \), there is an \( x'' \in (x', x)_Y \cap X \). Then we have \( x'' < x'' < x \). This shows that \( x \) has no immediate predecessor in \( X \). By the assumption "\( (y, x)_Y = \emptyset \)", we have \( \{x, \to\} = (y, \to)_Y \cap X \in \tau \). It follows from the definition of \( X \) that \( \langle x, -1 \rangle \) is a point of \( X \). Since both of \( y \) and \( x_+ \) are immediate predecessors of \( x \) in \( Y \), we have \( y = x_+ \). But this is a contradiction, because \( y \not\in f(X) \) and \( x_+ \in f(X) \). Thus we have \( (y, f(z))_Y \neq \emptyset \).

**Case 3:** \( z = (\langle x, -1 \rangle) \) for some \( x \in X \).

Since \( y \not\in f(X) \) and \( x_+ = f(z) \in f(X) \), we have \( x_+ = f(z) \in (y, x)_Y \). It follows from the density of \( X \) in \( Y \) that \( (y, x)_Y \cap X \neq \emptyset \). But since \( f(z) = x_+ \) is the immediate predecessor of \( x \) in \( Y \) and \( x_+ \notin X \), we have \( \emptyset \neq (y, f(z))_Y \cap X \subset (y, f(z))_Y \). This completes the proof of the Fact.

Using this Fact and the density of \( X \) in \( Y \), pick a point \( u \) in \( (y, f(z))_Y \cap X \). Then we have \( z \in (u, \to)_Y \). This completes the proof of the Subclaim.

Thus the proof of the theorem is completed. □

**Remark 2.2.** (1) In Theorem 2.1, we usually identify \( \tilde{X} \) with \( f(X) \). Therefore \( \tilde{X} \) can be considered as a minimal linearly ordered \( d \)-extension of a GO space \( X \).

(2) In general, a GO space \( \langle X, \tau, \leq \rangle \) has many linearly ordered \( d \)-extensions. For example, consider the GO space (in fact, a LOTS) \( X = (0, 1) \) with the usual topology and ordering in \( \mathbb{R} \). Then \( (0, 1), [0, 1), (0, 1], [0, 1] \) with the usual topology and ordering are all linearly ordered \( d \)-extensions of \( X \). In this case, \( \tilde{X} = X = (0, 1) \).

3. A partial negative answer for Problem 1.2

In this section, we shall give an example which is a partial negative answer for Problem 1.2. Note that if a GO space \( X \) has countable cellularity, then so does every orderable \( d \)-extension of \( X \), therefore \( X \) is perfect by [3, Theorem 2.10].
Example 3.1. There exists a perfect GO space which does not have any perfect linearly ordered $d$-extension or any orderable $c$-extension.

Construction of an example. Let $X$ be the closed unit interval $[0, 1]$ with the usual ordering $\leq$, and $C$ the Cantor set. The topology $\tau$ of $X$ has local bases $\{[x, x + \varepsilon): \varepsilon > 0\}$ for each point $x \in X - C$, and $\{\{x\}\}$ for each point $x \in C$. It is clear that $X$ is a GO space and $C$ is a discrete closed and open subspace of $X$. We shall prove the following claims.

Claim 1. The space $X$ is perfect.

Proof. Let $U$ be an open set of $X$. Since $C$ is closed discrete in $X$, we may assume $U \subset X - C$. For each $x \in U$, there is an $\varepsilon(x) > 0$ such that $[x, x + \varepsilon(x)) \subset U$. Let $G = \bigcup\{(x, x + \varepsilon(x)): x \in U\}$ and $K = U - G$.

Subclaim. $[x, x + \varepsilon(x)) \cap [y, y + \varepsilon(y)) = \emptyset$ whenever $x, y \in K$ with $x \neq y$.

Proof. We may assume that $x < y$. Suppose that there is a point $z \in [x, x + \varepsilon(x)) \cap [y, y + \varepsilon(y))$. Then $x < y \leq z < x + \varepsilon(x)$, so we have $y \in (x, x + \varepsilon(x)) \subset G$. But this is a contradiction, because $y \in K = U - G$. This completes the proof of the Subclaim.

By picking up a rational number in $(x, x + \varepsilon(x))$ for each $x \in K$, it is clear that $|K| \leq \omega$ by the Subclaim. Since $\{(x, x + \varepsilon(x)): x \in U\}$ is an Euclidean open cover of $G$, there is a countable subset $U' \subset U$ such that $G = \bigcup\{(x, x + \varepsilon(x)): x \in U'\}$. Since $K$ is countable and $U = K \cup \bigcup\{(x, x + \varepsilon(x)): x \in U'\}$, $U$ is an $F_\sigma$-set of $X$. This completes the proof of Claim 1.

It is easy to show that $[x, \to) \in \tau - \lambda$ if and only if $x \in (0, 1]$, and $(\leftarrow, x] \in \tau - \lambda$ if and only if $x \in C - \{1\}$, where $\lambda$ is the usual order topology on $X$ (so, the Euclidean topology). Therefore $\tilde{X}$ is represented by $X \times \{0\} \cup (0, 1] \times \{-1\} \cup (C - \{1\}) \times \{1\}$.

Next we show:

Claim 2. The space $\tilde{X}$ is not perfect.

Proof. We naturally consider that $X$ is a subset of $\tilde{X}$. Then the subset $C$ of $X$ is open in $\tilde{X}$. Suppose that $C$ is an $F_\sigma$-set of $\tilde{X}$ such that $C = \bigcup\{F_n: n \in \mathbb{N}\}$ and each $F_n$ is closed in $\tilde{X}$. Then there is an $F_n$ such that $F_n$ is an infinite subset. Since $C$ is compact in the usual topology of $[0, 1]$, $F_n$ has a cluster point $x$ (in the usual topology of $[0, 1]$) and $x \in C$. Then the points $\langle x, 1 \rangle$ or $\langle x, -1 \rangle$ of $\tilde{X}$ are contained in $\text{cl}_{\tilde{X}} F_n$. But since $F_n = \text{cl}_{\tilde{X}} F_n$ and $F_n \subset C$, this is a contradiction. Thus Claim 2 is completely proved.

Claim 3. Any linearly ordered $d$-extension $Y$ of $X$ is not perfect.

Proof. Since perfectness is a hereditary property and $\tilde{X}$ is the minimal linearly ordered $d$-extension of $X$, it follows from Claim 2 that $Y$ is not perfect.
Claim 4. Any orderable c-extension $Y$ of $X$ is not perfect.

Proof. Suppose that $Y$ is a perfect orderable c-extension of $X$. Since $X$ is closed in $Y$, $X$ is a $G_δ$-set of $Y$. Therefore $X$ is a $p$-embedded subspace of $Y$. (For $p$-embedded, see [3, Definition 5.8].) Since $X$ is submetrizable, $X$ has a $G_δ$-diagonal. By [3, Theorem 5.9], $X$ must be metrizable. But the subspace $(\frac{1}{2}, \frac{3}{2})$ of $X$ is homeomorphic to the Sorgenfrey line, therefore $X$ cannot be metrizable. This is a contradiction. Thus the proof of Claim 4 is completed.

4. Hereditary properties to linearly ordered $d$-extensions

In this section, we consider the following problem.

Problem 4.1. What properties are hereditary to linearly ordered $d$-extensions?

For this problem, it is easy to see that separability and countable cellularity are hereditary properties to linearly ordered $d$-extensions. On the other hand, Example 4.2 below shows that metrizability, hereditary paracompactness and perfectness are not hereditary properties to linearly ordered $d$-extensions.

Example 4.2. Let $\langle \omega_1, \lambda, \leq \rangle$ be the LOTS with the usual ordering $\leq$ and its interval topology on $\omega_1$. Let $X = \{\alpha < \omega_1; \alpha = 0$ or $\alpha$ is a successor ordinal]. Putting $\tau_X = \lambda \upharpoonright X$ and $\leq_X = \leq \upharpoonright X$, $X = \langle X, \tau_X, \leq_X \rangle$ can be considered as a GO space. Note that the topological space $\langle X, \tau_X \rangle$ is discrete, it is orderable by identifying $X = \omega_1 \times \mathbb{Z}$ with the lexicographic order. But $\langle X, \tau_X, \leq_X \rangle$ is not a LOTS. In fact, let $\lambda_X$ be the usual interval topology on the ordered set $\langle X, \leq_X \rangle$. Then it is easy to show that the topological space $\langle X, \lambda_X \rangle$ is homeomorphic to the ordinal space $\omega_1$. So $\bar{X} - X \neq \emptyset$. Next we decide $\bar{X}$. First observe that $(\langle X, \tau_X \rangle X, \leq_X)$ for each $\alpha E X$, where $(\alpha, X)$ denotes the interval in $X$. Therefore $\langle \alpha, 1 \rangle \notin \bar{X}$ for each $\alpha \in X$.

Claim. For each $\alpha \in X$, $\langle \alpha, \to \rangle \in \tau_X - \lambda_X$ if and only if $\alpha$ is an immediate successor of a limit ordinal $\alpha_\infty$ in $\omega_1$.

Proof. The “only if” part: Assume that $\alpha \in X$ and $\alpha = \alpha_\infty + 1$ for some nonlimit ordinal $\alpha_\infty$ (i.e., $\alpha_\infty \in \lambda_X$). Then we have $\langle \alpha, \to \rangle \in \lambda_X$. The “if” part: Let $\alpha \in X$ and $\alpha = \alpha_\infty + 1$ in $\omega_1$ for some limit ordinal $\alpha_\infty$ (i.e., $\alpha_\infty \in \lambda_X$). Then $\langle \alpha, \to \rangle \in \lambda_X$. Assume $\langle \alpha, \to \rangle \in \lambda_X$. Then there is an immediate predecessor $\alpha''$ of $\alpha$ in $X$. Since $\alpha'$ is the immediate predecessor of $\alpha$ in $\omega_1$ and $\alpha' \notin X$, we have $\alpha'' < \alpha'$. Since $\alpha'$ is limit, we have $\alpha'' < \alpha_\infty + 1 < \alpha_\infty < \alpha$. This contradicts that $\alpha''$ is the immediate predecessor of $\alpha$ in $X$ by $\alpha'' + 1 \notin X$. This completes the proof of the Claim.
By the above observation and the Claim, $\tilde{X}$ can be represented by $X \times \{0\} \cup \{\langle \alpha, -1 \rangle : \alpha \in X \text{ and } \alpha \text{ is the immediate successor of the limit ordinal } \alpha_- \text{ in } \omega_1 \}$. Define $f: \tilde{X} \to \omega_1$ by

$$f(\langle \alpha, n \rangle) = \begin{cases} \alpha, & \text{if } n = 0, \\ \alpha_-, & \text{if } n = -1. \end{cases}$$

Then it is straightforward to show that $f$ is an order preserving isomorphism. So we can identify $\tilde{X} - \omega_1$.

Thus, by Theorem 2.1, every linearly ordered $d$-extension of the discrete GO space $X$ contains an order preserving copy of the ordinal space $\omega_1$ as a dense subspace. Therefore all linearly ordered $d$-extensions of $X$ are neither metrizable, hereditarily paracompact nor perfect (cf. The results (a), (b) in Section 1).

The proof of Example 4.2 shows the following.

**Theorem 4.3.** For every discrete space $\langle X, \tau \rangle$, there is a linear ordering $\leq$ of $X$ such that $\langle X, \tau, \leq \rangle$ is a GO space and whose every linearly ordered $d$-extension contains an order preserving copy of the ordinal space $|X|$ as a dense subspace.

**References**


