MATHEMATICS

ON THE CONVERGENCE OF SUCCESSIVE APPROXIMATIONS IN THE THEORY OF ORDINARY DIFFERENTIAL EQUATIONS, II

BY

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1. *Introduction*

Consider the initial value problem

$$x'(t)=f(t, x(t)), \quad x(t_0)=x_0, \tag{1.1}$$

where t and x are real variables and f is a real valued function defined and continuous on the rectangle

$$R: \quad |t-t_0|\leqslant a, \quad |x-x_0|\leqslant b, \quad a, b>0.$$

O. KOOI [1] recently proved the uniqueness of a solution of the problem (1.1) and the uniform convergence of sequences of functions obtained by Picard's method of successive approximations to this unique solution on an appropriate interval around t_0 if $f(t, x)$ satisfies on R the conditions:

$$|t-t_0|\cdot|f(t, x_1)-f(t, x_2)|\leqslant k\,|x_1-x_2|, \quad k>0, \tag{1.2}$$

$$|t-t_0|^{\beta}\,|f(t, x_1)-f(t, x_2)|\leqslant A\,|x_1-x_2|^{\alpha}, \quad A>0, \tag{1.3}$$

where the constants k, α and β satisfy the inequalities

$$0<\alpha<1, \quad \beta<\alpha \text{ and } k(1-\alpha)<1-\beta. \tag{1.4}$$

For $\beta=0$ we obtain a set of conditions which were recently introduced by M. A. KRASNOSELSKIĬ and I. G. KREIN [2]. They proved that this set of conditions implies the uniqueness of a solution of the problem (1.1). In [3], the present author completed this result as he showed that this set of conditions also implies the convergence of sequences of functions obtained by Picard's method of successive approximations to the unique solution of (1.1). As a matter of fact, two proofs of this result were given, which in method are both different from the method used by Kooi. The object of this paper is to show that one of our methods can be used to prove a general result about contractions, which are defined in generalised complete metric spaces (not every two points have necessarily a finite distance), and to show how Kooi's more general result can be obtained from it. As a consequence of this deduction we obtain the result that the sequences of successive approximations converge not only uniformly but even in a stronger sense. Furthermore we shall show by means of an example that the statement about the convergence of successive approximations may be false if $k(1-\alpha) \geqslant 1-\beta$, even though the problem (1.1) has a unique solution.

2. *A Theorem about Contractions*

Let X be an abstract set, the elements of which are denoted by $x, y, \ldots$, and assume that on the Cartesian product $X \times X$ a distance function $d(x, y)$ $(0 \leqslant d(x, y) \leqslant \infty)$ is defined, satisfying the following conditions:

(D1) $d(x, y) = 0$ if and only if $x = y$,

(D2) $d(x, y) = d(y, x)$ (symmetry),

(D3) $d(x, y) \leqslant d(x, z) + d(z, y)$ (triangle inequality),

(D4) every d-Cauchy sequence in X is d-convergent, i.e. $\lim_{n, m \to \infty} d(x_n, x_m) = 0$ for a sequence $x_n \in X$ $(n = 1, 2, \ldots)$ implies the existence of an element $x \in X$ with $\lim_{n \to \infty} d(x, x_n) = 0$, ($x$ is unique by (D1) and (D3)).

This concept differs from the usual concept of a complete metric space by the fact that not every two points in X have necessarily a finite distance. One might call such a space a generalised complete metric space.

Assume now that T is a mapping defined on the whole of X, and which maps X into itself and satisfies the following conditions:

(C1) There exists a constant $0 < q < 1$, such that

$$d(Tx, Ty) \leqslant q d(x, y)$$

for all (x, y) with $d(x, y) < \infty$.

(C2) For every sequence of successive approximations $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$, where x_0 is an arbitrary element of X, there exists an index $N(x_0)$ such that $d(x_N, x_{N+l}) < \infty$ for all $l = 1, 2, \ldots$.

(C3) If x and y are two fix points of T, i.e. $Tx = x$ and $Ty = y$, then $d(x, y) < \infty$.

Under these conditions we have the following theorem:

Theorem. *The equation $Tx = x$ has one and only one solution, and every sequence of successive approximations $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$, where x_0 is an arbitrary element of X, is convergent in distance to this unique solution.*

Proof. The existence of a solution of the equation $Tx = x$ can be proved as follows: Let $x_0 \in X$ and form the sequence $x_n = Tx_{n-1}$ $(n = 1, 2, \ldots)$; then by (C2) there exists an index $N(x_0)$ such that $d(x_N, x_{N+l}) < \infty$, $l = 1, 2, \ldots$, and hence by (D3) we have $d(x_n, x_{n+l}) < \infty$ for $n \geqslant N$ and $l = 1, 2, \ldots$. Then (C1) implies $d(x_{N+1}, x_{N+2}) \leqslant q d(x_N, Tx_N)$ and generally $d(x_n, x_{n+1}) \leqslant q^{n-N} d(x_N, Tx_N)$ for $n \geqslant N$. Since by (D3) we have

$$d(x_n, x_{n+l}) \leqslant \sum_{i=1}^{l} d(x_{n+i}, x_{n+i-1}),$$

we obtain by using the preceding inequality,

$$(2.1)\qquad d(x_n, x_{n+l}) \leqslant \{q^{n-N}(1-q^l)/(1-q)\} d(x_N, Tx_N), \quad n \geqslant N \text{ and } l = 1, 2, \ldots,$$

which proves x_n to be a d-Cauchy sequence. From (D4) it follows then that there exists an element $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$. For this element x we conclude by means of (D3) that $d(x, Tx) \leqslant d(Tx, x_n) + d(x_n, x) \leqslant qd(x, x_{n-1}) + d(x_n, x)$ for $n \geqslant N$, and hence $d(x, Tx) = 0$, which is equivalent to $Tx = x$ by property (D1) of d. So x is a fix point of T. To complete the proof we have to show that T has only one fix point. Assume $Tx = x$ and $Ty = y$ with $x \neq y$; then by (C3) we know that $d(x, y) < \infty$, and hence by (C1) we obtain $0 \leqslant d(x, y) = d(Tx, Ty) \leqslant qd(x, y)$. This shows that $d(x, y) = 0$, which by (D1) contradicts $x \neq y$.

Remark 1. If we let l tend to infinity in (2.1) we obtain

$$(2.2)\qquad d(x, x_n) \leqslant \{q^{n-N}/(1-q)\}\, d(x_N, Tx_N),$$

which gives an estimate of the rate of convergence of the sequence x_n to its limit x.

Remark 2. This theorem is a slightly more general form of the usual theorem about contractions one finds in the literature (see e.g. PETROVSKII, I. G., Vorlesungen über die Theorie der Gewöhnlichen Differentialgleichungen), where it is assumed that d is finite and T satisfies condition (C1) only, since in this case (C2) and (C3) are trivially satisfied.

Remark 3. Condition (C3) of T is necessary for the conclusion that the mapping T has at most one fix point, as the following example shows: Let X_1 and X_2 be two complete metric spaces with distance functions d_1 and d_2 respectively. Assume that on X_1 and X_2 contractions T_1 and T_2 are defined, each of which has only one fix point by our theorem. Now let X be the union of X_1 and X_2, and let the following distance function d on X be defined: $d(x, y) = d_1(x, y)$ if x and $y \in X_1$; $d(x, y) = d_2(x, y)$ if x and $y \in X_2$, and $d(x, y) = \infty$ if $x \in X_1$ and $y \in X_2$. Let T be the mapping which on X_1 coincides with T_1 and on X_2 coincides with T_2. Then all the conditions except (C3) of our theorem are satisfied, but T has obviously two fix points.

3. *Proof of the Theorem of Section* 1

Let R, $f(t, x)$, a, b, α, β, k and A be as in section 1 and let $M = \max(|f(t,x)|;\ (t, x) \in R)$. By I we denote the interval $|t - t_0| \leqslant c$, where $c = \min(a, b/M)$. Then we shall prove, by application of the theorem about contractions, the existence and uniqueness of a solution, and the convergence of Picard's successive approximations of problem (1.1) on this interval I. For this purpose we shall exhibit a space X with a metric d and a mapping T satisfying the conditions given in the preceding section. For the space X we choose the set of all continuous functions $\varphi(t)$, defined on I and satisfying:

$$(3.1)\qquad \varphi(t_0) = x_0, \quad |\varphi(t) - x_0| \leqslant b \text{ for all } t \in I.$$

Then, on the Cartesian product $X \times X$, we define the following distance function:

$$(3.2) \qquad d(\varphi_1, \varphi_2) = \sup\,(|\varphi_1(t) - \varphi_2(t)|/|t-t_0|^{pk};\quad |t-t_0| \leqslant c;\ \varphi_1, \varphi_2 \in X),$$

where $p > 1$, but such that $pk(1-\alpha) < 1-\beta$, which is possible since by (1.6) we have $k(1-\alpha) < 1-\beta$. It is clear that this d-function satisfies the properties (D1), (D2) and (D3) of section 2. To prove (D4), we first remark that if we write

$$(3.3) \qquad d_1(\varphi_1, \varphi_2) = \max\,(|\varphi_1(t) - \varphi_2(t)|;\ |t-t_0| \leqslant c;\ \varphi_1, \varphi_2 \in X)$$

(the metric of uniform convergence), we have

$$(3.4) \qquad c^{-pk} d_1(\varphi_1, \varphi_2) \leqslant d(\varphi_1, \varphi_2) \text{ for all } \varphi_1, \varphi_2 \in X.$$

This shows that d-convergence implies uniform convergence and d-convergence is in general stronger than uniform convergence. Now let $\varphi_n \in X$, $n = 1, 2, \ldots$, be a d-Cauchy sequence, i.e. $\lim d(\varphi_n, \varphi_m) = 0$. Then, there exists a subsequence ψ_n such that $d(\psi_n, \psi_{n+1}) \leqslant 2^{-n}$. The inequality (3.3) implies that this sequence ψ_n is uniformly convergent to a function $\varphi(t)$ on I, which obviously satisfies (3.1). By putting $\varphi - \psi_n = \sum_{k=n}^{\infty} (\psi_{k+1} - \psi_k)$, dividing through $|t-t_0|^{pk}$ and taking the suprema of the absolute values of each term of this sum, we obtain

$$(3.5) \qquad d(\varphi, \psi_n) \leqslant \sum_{k=n}^{\infty} d(\psi_k, \psi_{k+1}) \leqslant 2^{-n+1},$$

and hence $\lim\limits_{n\to\infty} d(\varphi, \psi_n) = 0$. By (D3) we have $d(\varphi, \varphi_n) \leqslant d(\varphi, \psi_k) + d(\psi_k, \varphi_n)$ and the required property $\lim\limits_{n\to\infty} d(\varphi, \varphi_n) = 0$ follows.

For T we choose the mapping

$$(3.6) \qquad T\varphi(t) = x_0 + \int_{t_0}^{t} f(u, \varphi(u))\,du, \qquad \varphi \in X.$$

Then, by the choice of the interval I, this T maps X into itself. Indeed, $T\varphi(t_0) = x_0$ is trivial and $|T\varphi(t) - x_0| = |\int_{t_0}^{t} f(u, \varphi(u))du| \leqslant M|t-t_0| \leqslant Mc \leqslant b$. Furthermore it is easy to see that a function φ is a fix point of T if and only if φ is a solution of problem (1.1). If we form the sequence $\varphi_n = T\varphi_{n-1}$, $n = 1, 2, \ldots$, where φ_0 is an arbitrary element of X, we obtain a sequence of successive approximations of Picard

$$(3.7) \qquad \varphi_n(t) = x_0 + \int_{t_0}^{t} f(u, \varphi_{n-1}(u))\,du, \qquad \varphi_0 \in X.$$

We shall show now that T satisfies properties (C1), (C2) and (C3) of section 2.

Proof of (C1). Let φ_1, φ_2 be two arbitrary elements of X. Then, by property (1.2) of f, we have

$|T\varphi_1 - T\varphi_2| \leqslant |\int_{t_0}^{t} |f(u, \varphi_1(u)) - f(u, \varphi_2(u))| du| \leqslant k \int_{t_0}^{t} (|\varphi_1(u) - \varphi_2(u)|/|u - t_0|) |du|$ $= k \int_{t_0}^{t} |u - t_0|^{pk-1} (|\varphi_1(u) - \varphi_2(u)|/|u - t_0|^{pk}) \cdot |du|$. Hence, if $d(\varphi_1, \varphi_2) < \infty$, then $|T\varphi_1 - T\varphi_2| \leqslant k\, d(\varphi_1, \varphi_2) \int_{t_0}^{t} |u - t_0|^{pk-1} |du| = k\, d(\varphi_1, \varphi_2)\, (pk)^{-1}\ |t - t_0|^{pk} = p^{-1}$ $d(\varphi_1, \varphi_2)\ |t - t_0|^{pk}$. This implies, by the definition of d, that $d(T\varphi_1, T\varphi_2) \leqslant$ $\leqslant q d(\varphi_1, \varphi_2)$, where $q = p^{-1} < 1$, by the choice of p. Hence T is a contraction.

Proof of (C2). Let $\varphi_n = T\varphi_{n-1}$, $n = 1, 2, \ldots$, $\varphi_0 \in X$ and arbitrary. Then by the boundedness of f we have

$$|\varphi_2(t) - \varphi_1(t)| \leqslant \int_{t_0}^{t} |f(u, \varphi_1(u)) - f(u, \varphi_0(u))| \cdot |du| \leqslant 2M |t - t_0|,$$

so that by property (1.3) of f we obtain

$|\varphi_3(t) - \varphi_2(t)| \leqslant \int_{t_0}^{t} |f(u, \varphi_2(u)) - f(u, \varphi_1(u))| \cdot |du| \leqslant A \int_{t_0}^{t} |u - t_0|^{-\beta} |\varphi_1(u) - \varphi_2(u)|^{\alpha}$ $|du| \leqslant A(2M)^{\alpha} \int_{t_0}^{t} |u - t_0|^{\alpha-\beta} |du| = A(2M)^{\alpha} (1 + (\alpha - \beta))^{-1} |t - t_0|^{1+\alpha-\beta} \leqslant A(2M)^{\alpha}$ $|t - t_0|^{1+\alpha-\beta}$, since $\alpha - \beta > 0$ by (1.4). Generally

$$(3.7)\quad \begin{cases} |\varphi_{n+1}(t) - \varphi_n(t)| \leqslant A^{1+\alpha+\ldots+\alpha^{n-2}} (2M)^{\alpha^{n-1}} |t - t_0|^{(1-\beta)(1+\alpha+\ldots+\alpha^{n-2})+\alpha^{n-1}} < \\ \qquad < B |t - t_0|^{(1-\beta)(1+\alpha+\ldots+\alpha^{n-2})+\alpha^{n-1}}, \end{cases}$$

where $B = A^{\frac{1}{1-\alpha}} \max(2M, 1)$. In view of $pk(1 - \alpha) < 1 - \beta$ there exists an index $N(p)$ such that $(1 - \beta)(1 + \alpha + \ldots + \alpha^{n-2}) + \alpha^{n-1} > pk$ for all $n \geqslant N$, and hence for these values of n, we have

$|\varphi_{n+1}(t) - \varphi_n(t)|/|t - t_0|^{pk} \leqslant B |t - t_0|^{\gamma_n}$, where $\gamma_n = (1 - \beta)\ (1 + \alpha + \ldots + \alpha^{n-2}) +$ $+ \alpha^{n-1} - pk > 0$. This shows that $d(\varphi_n, \varphi_{n+1}) < \infty$ for all $n \geqslant N(p)$, which completes the proof.

Proof of (C3). If $T\varphi_1 = \varphi_1$ and $T\varphi_2 = \varphi_2$, then as in the preceding proof we have $|\varphi_2(t) - \varphi_1(t)| \leqslant 2M\ |t - t_0|$, and by using (1.3) successively we obtain $|\varphi_1(t) - \varphi_2(t)| \leqslant B\ |t - t_0|^{\frac{1-\beta}{1-\alpha}}$, where B has the same meaning as in the preceding proof. Since $pk < (1 - \beta)/(1 - \alpha)$, this implies $d(\varphi_1, \varphi_2) < \infty$, which finishes the proof of (C3).

After this verification an application of the theorem about contractions in the preceding section gives the desired result. Moreover we have proved that the successive approximations (3.7) converge not only uniformly to the solution of (1.1), but even in the stronger sense of the metric d (see (3.3) and (3.4)).

Remark. The condition that $f(t, x)$ is continuous may be replaced by a weaker condition. It is e.g. sufficient to assume that $f(t, x)$ is a Lebesgue measurable function in t for each fixed x, continuous in x if $t = t_0$ (this is not necessary if $\beta = 0$), and that there exists a positive Lebesgue integrable function $M(t)$ on $|t - t_0| \leqslant a$, such that $|f(t, x)| \leqslant M(t)$ for all $(t, x) \in R$. Condition (1.2) or (1.3) of f implies that f is a continuous function in x for each $t \neq t_0$, and hence f is continuous in x for each t (if $\beta = 0$ then (1.3) already implies this). From these properties of f, it follows that if φ is a measurable function, $f(t, \varphi(t))$ is a measurable function of t. In this setting the initial value problem has to be read as

$$(1.1)' \qquad x'(t) = f(t, x(t)) \text{ almost everywhere, } x(t_0) = x_0.$$

If I is the interval $|t - t_0| \leqslant c$, where $c = \max\left(|t|; \left|\int_{t_0}^{t+t_0} M(u)du\right| \leqslant b\right)$, then one can prove, by applying the theorem about contractions, that on this interval I the problem has a unique solution among the Lebesgue measurable functions satisfying (3.1) on I, and that sequences of successive approximations converge uniformly on I to this unique solution. The proof of this result is left to the reader since it is formally the same.

4. *An Example*

In this section we shall show by means of an example that if $k(1-\alpha) \geqslant 1-\beta$, where k, α and β have the same meaning as in section 1 and satisfy $k > 0$, $0 < \alpha < 1$ and $\beta < \alpha$, the sequence of successive approximations may be divergent. For this purpose we define a function $f(t, x)$ as follows:

$$f(t, x) = \begin{cases} 0 & \text{if } \; -1 \leqslant t \leqslant 0, \; -\infty < x < \infty, \\ 0 & \text{if } \; 0 < t \leqslant 1, \; t^{\frac{1-\beta}{1-\alpha}} \leqslant x < \infty, \\ k\, t^{\frac{\alpha-\beta}{1-\alpha}} - k\dfrac{x}{t} & \text{if } \; 0 < t \leqslant 1, \; 0 \leqslant x < t^{\frac{1-\beta}{1-\alpha}}, \\ k\, t^{\frac{\alpha-\beta}{1-\alpha}} & \text{if } \; 0 < t \leqslant 1, \; -\infty < x < 0. \end{cases}$$

It is easy to verify that this function is continuous and that $\max(|f(t,x)|) = k$. We consider the initial value problem $x'(t) = f(t, x(t))$, $x(0) = 0$. In this case $t_0 = 0$, $a = 1$, $b = \infty$, $c = 1$ and $M = k$. We shall prove first that f satisfies the following conditions:

$$|t| \cdot |f(t, x_1) - f(t, x_2)| \leqslant k|x_1 - x_2|,$$
$$|t|^{\beta}\, |f(t, x_1) - f(t, x_2)| \leqslant k\, |x_1 - x_2|^{\alpha},$$

i.e. f satisfies the conditions (1.2) and (1.3) with $A = k$.

For the proof we consider the following cases:

(i) $-1 \leqslant t < 0$, x_1 and x_2 arbitrary; then there is nothing to prove.

(ii) $0 < t \leqslant 1$, x_1 and $x_2 \geqslant t^{\frac{1-\beta}{1-\alpha}}$; then there is also nothing to prove.

(iii) $0 < t \leqslant 1$, $x_1 \geqslant t^{\frac{1-\beta}{1-\alpha}}$ and $0 \leqslant x_2 \leqslant t^{\frac{1-\beta}{1-\alpha}}$. Then we have

$$t\,|f(t, x_1) - f(t, x_2)| = k(t^{\frac{1-\beta}{1-\alpha}} - x_2) \leqslant k\, |x_1 - x_2|,$$

$$t^{\beta}|f(t,x_1) - f(t,x_2)| = k\left(\frac{t^{\frac{1-\beta}{1-\alpha}} - x_2}{t^{1-\beta}}\right) = k\frac{t^{\frac{1-\beta}{1-\alpha}} - x_2}{(t^{\frac{1-\beta}{1-\alpha}})^{(1-\alpha)}} \leqslant k\frac{t^{\frac{1-\beta}{1-\alpha}} - x_2}{(t^{\frac{1-\beta}{1-\alpha}} - x_2)^{1-\alpha}} =$$

$$= k\,(t^{\frac{1-\beta}{1-\alpha}} - x_2)^{\alpha} \leqslant k\,|x_1 - x_2|^{\alpha}.$$

(iv) $0<t\leqslant 1$, $x_1\geqslant t^{\frac{1-\beta}{1-\alpha}}$ and $x_2\leqslant 0$. Then we have

$t\,|f(t, x_1)-f(t, x_2)|=k\,t^{\frac{1-\beta}{1-\alpha}}\leqslant k\,|x_1-x_2|$ since $x_1-x_2\geqslant t^{\frac{1-\beta}{1-\alpha}}$,

$t^\beta\,|f(t, x_1)-f(t, x_2)|=k(t^{\frac{1-\beta}{1-\alpha}})^\alpha\leqslant k\,|x_1-x_2|^\alpha$ for the same reason.

(v) $0<t\leqslant 1$, $0\leqslant x_1$, $x_2\leqslant t^{\frac{1-\beta}{1-\alpha}}$. Then we have

$t\,|f(t, x_1)-f(t, x_2)|=k\,|x_1-x_2|$,

$t^\beta\,|f(t, x_1)-f(t, x_2)|=k\,t^{\beta-1}\,|x_1-x_2|$, and since $|x_1-x_2|\leqslant t^{\frac{1-\beta}{1-\alpha}}$ we see that $t^{\beta-1}\leqslant|x_1-x_2|^{\alpha-1}$, hence $t^\beta\,|f(t, x_1)-f(t, x_2)|\leqslant k\,|x_1-x_2|^\alpha$.

(vi) $0<t\leqslant 1$, $0\leqslant x_1\leqslant t^{\frac{1-\beta}{1-\alpha}}$ and $-\infty<x_2<0$. Then we have

$t\,|f(t, x_1)-f(t, x_2)|=k\,x_1\leqslant k\,|x_1-x_2|$ since $x_2<0$,

$t^\beta|f(t,x_1)-f(t,x_2)|=k\,t^{\beta-1}x_1\leqslant k\,x_1^{\alpha-1}x_1=kx_1^{\alpha}\leqslant k\,|x_1-x_2|^\alpha$ for the same reason.

Let $\gamma=k(1-\alpha)/(1-\beta)$ and let $\varphi_0(t)=0$ for $-1\leqslant t\leqslant 1$; then the successive approximations are for $\gamma<1$:

$\varphi_n(t)=0$ if $-1\leqslant t\leqslant 0$ and $\varphi_n(t)=(\gamma-\gamma^2+\ldots+(-1)^{n-1}\gamma^n)\,t^{\frac{1-\beta}{1-\alpha}}$ if $0<t\leqslant 1$, and for $\gamma\geqslant 1$,

$\varphi_{2n-1}(t)=0$ if $-1\leqslant t\leqslant 0$ and $\varphi_{2n-1}(t)=\gamma\,t^{\frac{1-\beta}{1-\alpha}}$ if $0<t\leqslant 1$,

$\varphi_{2n}(t)=0$ if $-1\leqslant t\leqslant 1$, $n=1, 2, \ldots$.

This shows that if $\gamma<1$, the successive approximations converge uniformly to the solution

(4.1) $$\varphi(t)=0 \text{ if } -1\leqslant t\leqslant 0,\quad \varphi(t)=\{\gamma/(1+\gamma)\}\,t^{\frac{1-\beta}{1-\alpha}} \text{ if } 0<t\leqslant 1.$$

If $\gamma\geqslant 1$, there is no convergence at all and the functions φ_{2n} and φ_{2n-1} are no solutions of (1.1). The solution of this initial value problem for all possible values of γ is (4.1). This solution is unique since to the left of $t=0$, the function f is equal to zero and to the right of $t=0$, this function has the property that it is non-increasing in x for each t. It follows that in the case $\gamma\geqslant 1$, even if the solution is unique, the successive approximations are not necessarily convergent.

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