Annals of Pure and Applied Logic 163 (2012) 68–84

Contents lists available at SciVerse ScienceDirect

Annals of Pure and Applied Logic

journal homepage: www.elsevier.com/locate/apal

# Inclusion and exclusion dependencies in team semantics — On some logics of imperfect information

Pietro Galliani *

*Faculteit der Natuurwetenschappen, Wiskunde en Informatica, Institute for Logic, Language and Computation, Universiteit van Amsterdam, P.O. Box 94242, 1090 GE AMSTERDAM, The Netherlands*

ARTICLE INFO

*Article history:*
Received 7 June 2011
Received in revised form 5 August 2011
Accepted 17 August 2011
Available online 25 September 2011
Communicated by P.J. Scott

*MSC:*
03B60
03C80
03C85

*Keywords:*
Dependence
Independence
Imperfect information
Team semantics
Model theory

ABSTRACT

We introduce some new logics of imperfect information by adding atomic formulas corresponding to *inclusion* and *exclusion* dependencies to the language of first order logic. The properties of these logics and their relationships with other logics of imperfect information are then studied. As a corollary of these results, we characterize the expressive power of independence logic, thus answering an open problem posed in Grädel and Väänänen, 2010 [9].



## 1. Introduction

The notions of dependence and independence are among the most fundamental ones considered in logic, in mathematics, and in many of their applications. For example, one of the main aspects in which modern predicate logic can be thought of as superior to medieval term logic is that the former allows for quantifier alternation, and hence can express certain complex patterns of dependence and independence between variables that the latter cannot easily represent.

*Logics of imperfect information* are a family of logical formalisms whose development arose from the observation that not all possible patterns of dependence and independence between variables may be represented in first order logic. Among these logics, *dependence logic* [20] is perhaps the one most suited for the analysis of the notion of dependence itself, since it isolates it by means of *dependence atoms* which correspond, in a very exact sense, to functional dependencies of the exact kind studied in database theory. The properties of this logic, and of a number of variants and generalizations thereof, have been the object of much research in recent years, and we cannot hope to give here an exhaustive summary of the known results. We will content ourselves, therefore, to recall in Section 2.1 those that will be of particular interest for the rest of this work.

*Independence logic* [9] is a recent variant of dependence logic. In this new logic, the fundamental concept that is being added to the first order language is not *functional dependence*, as for the case of dependence logic proper, but

* Tel.: +31 020 525 8260; fax: +31 20 525 5206.
*E-mail address:* pgallian@gmail.com.

0168-0072/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.apal.2011.08.005

*informational independence*; as we will see, this is achieved by considering *independence atoms* $y \perp_x z$, whose informal meaning corresponds to the statement according to which, for any fixed value of $x$, the sets of the possible values for $y$ and $z$ are independent. Just as dependence logic allows us to reason about the properties of functional dependence, independence logic does the same for this notion. Much is not known at the moment about independence logic; in particular, one open problem mentioned in [9] concerns the *expressive power* of this formalism over open formulas.

In this work, we will find an answer to this problem; and furthermore, as a means to do so, we will study some logics obtained by extending the language of first order logic along the same lines of dependence or independence logic.

## 2. Dependence and independence logic

### 2.1. Dependence logic

Dependence logic [20] is, together with independence-friendly (IF) logic [10,19], one of the most widely studied logics of imperfect information. In brief, it can be described as the extension of first order logic obtained by adding *dependence atoms* $=(t_1 \ldots t_n)$ to its language, with the informal meaning of "The value of the term $t_n$ is functionally determined by the values of the terms $t_1 \ldots t_{n-1}$".

We will later recall the full definition of the *team semantics* of dependence logic, an adaptation of Hodges' compositional semantics for IF logic [12], and one of the three equivalent semantics for dependence logic described in [20]. It is worth noting already here, though, that the key difference between Hodges semantics and the usual Tarskian semantics is that in the former the satisfaction relation $\models$ associates to every first order model[1] $M$ and formula $\phi$ a set of *teams*, that is, a set of sets of assignments, instead of just a set of assignments as in the latter.

As discussed in [13], the fundamental intuition behind Hodges' semantics is that a team is a representation of an *information state* of some agent: given a model $M$, a team $X$, and a suitable formula $\phi$, the expression $M \models_X \phi$ asserts that, from the information that the "true" assignment $s$ belongs to the team $X$, it is possible to infer that $\phi$ holds, or, in game-theoretic terms, that the verifier has a strategy $\tau$ which is winning for all plays of the game $G(\phi)$ which start from any assignment $s \in X$.

The satisfaction conditions for dependence atoms are then given by the following semantic rule **TS-dep**.

**Definition 2.1** (*Dependence Atoms*)**.** Let $M$ be a first order model, let $X$ be a team over it, let $n \in \mathbb{N}$, and let $t_1 \ldots t_n$ be terms over the signature of $M$ and with variables in $\text{Dom}(X)$. Then the following holds.

**TS-dep**: $M \models_X =(t_1 \ldots t_n)$ if and only if, for all $s, s' \in X$ such that $t_i\langle s\rangle = t_i\langle s'\rangle$ for $i = 1 \ldots n-1$, $t_n\langle s\rangle = t_n\langle s'\rangle$.

This rule corresponds closely to the definition of *functional dependency* commonly used in database theory [4]: more precisely, if $X(t_1 \ldots t_n)$ is the relation $\{(t_1\langle s\rangle, \ldots, t_n\langle s\rangle) : s \in X\}$ then

$$M \models_X =(t_1 \ldots t_n) \Leftrightarrow X(t_1 \ldots t_n) \models \{t_1 \ldots t_{n-1}\} \to t_n,$$

where the right-hand expression states that, in the relation $X(t_1 \ldots t_n)$, the value of the last term $t_n$ is a function of the values of $t_1 \ldots t_{n-1}$.

The following known results will be of some use for the rest of this work.

**Theorem 2.2** (*Locality [20]*)**.** *Let $M$ be a first order model and let $\phi$ be a dependence logic formula over the signature of $M$ with free variables in $\vec{v}$. Then, for all teams $X$ with domain $\vec{w} \supseteq \vec{v}$, if $X'$ is the restriction of $X$ to $\vec{v}$, then*

$$M \models_X \phi \Leftrightarrow M \models_{X'} \phi.$$

As an aside, it is worth pointing out that the above property does not hold for most variants of IF logic: for example, if $\text{Dom}(M) = \{0, 1\}$ and $X = \{(x := 0, y := 0), (x := 1, y := 1)\}$, it is easy to see that $M \models_X (\exists z/y)z = y$, even though for the restriction $X'$ of $X$ to $\text{Free}((\exists z/y)z = y) = \{y\}$ we have that $M \not\models_{X'} (\exists z/y)z = y$. This is a typical example of *signalling* [10,14], one of the most peculiar and, perhaps, problematic aspects of IF logic.

**Theorem 2.3** (*Downwards Closure Property [20]*)**.** *Let $M$ be a model, let $\phi$ be a dependence logic formula over the signature of $M$, and let $X$ be a team over $M$ with domain $\vec{v} \supseteq \text{Free}(\phi)$ such that $M \models_X \phi$. Then, for all $X' \subseteq X$,*

$$M \models_{X'} \phi.$$

**Theorem 2.4** (*Dependence Logic Sentences and $\Sigma_1^1$ [20]*)**.** *For every dependence logic sentence $\phi$, there exists a $\Sigma_1^1$ sentence $\Phi$ such that*

$$M \models_{\{\emptyset\}} \phi \Leftrightarrow M \models \Phi.$$

*Conversely, for every $\Sigma_1^1$ sentence $\Phi$, there exists a dependence logic sentence $\phi$ such that the above holds.*

[1] In all of this paper, I will assume that first order models have at least two elements in their domain.

**Theorem 2.5** (*Dependence Logic Formulas and $\Sigma_1^1$ [16]*)**.** *For every dependence logic formula $\phi$ and every tuple of variables $\vec{x} \supseteq Free(\phi)$, there exists a $\Sigma_1^1$ sentence $\Phi(R)$, where $R$ is a $|\vec{x}|$-ary relation which occurs only negatively in $\Phi$, such that, for all teams $X$ with domain $\vec{x}$ and for $R = \{s(\vec{x}) : s \in X\}$, it holds that*

$$M \models_X \phi \Leftrightarrow M \models \Phi(R).$$

*Conversely, for all such $\Sigma_1^1$ sentences, there exists a dependence logic formula $\phi$ such that the above holds with respect to all* nonempty *teams $X$.*

### 2.2. Independence logic

Independence logic [9] is a recently developed logic which substitutes the dependence atoms of dependence logic with *independence atoms* $\vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3$, where $\vec{t}_1 \dots \vec{t}_3$ are tuples of terms (not necessarily of the same length).

The intuitive meaning of such an atom is that the values of the tuples $\vec{t}_2$ and $\vec{t}_3$ are informationally independent for any fixed value of $\vec{t}_1$; or, in other words, that all information about the value of $\vec{t}_3$ that can be possibly inferred from the values of $\vec{t}_1$ and $\vec{t}_2$ can already be inferred from the value of $\vec{t}_1$ alone.

More formally, the definition of team semantics for the independence atom is as follows.

**Definition 2.6** (*Independence Atoms*)**.** Let $M$ be a first order model, let $X$ be a team over it, and let $\vec{t}_1, \vec{t}_2$ and $\vec{t}_3$ be three finite tuples of terms (not necessarily of the same length) over the signature of $M$ and with variables in Dom($X$). Then the following holds.

**TS-indep**: $M \models_X \vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3$ if and only if, for all $s, s' \in X$ with $\vec{t}_1\langle s\rangle = \vec{t}_1\langle s\rangle$, there exists an $s'' \in X$ such that $\vec{t}_1\langle s''\rangle\vec{t}_2\langle s''\rangle = \vec{t}_1\langle s\rangle\vec{t}_2\langle s\rangle$ and $\vec{t}_1\langle s''\rangle\vec{t}_3\langle s''\rangle = \vec{t}_1\langle s'\rangle\vec{t}_3\langle s'\rangle$.

We refer to [9] for a discussion of this interesting class of atomic formulas and of the resulting logic. Here we only mention a few results, found in that paper, which will be useful for the rest of this work[2].

**Theorem 2.7.** *Dependence atoms are expressible in terms of independence atoms: more precisely, for all suitable models $M$, teams $X$, and terms $t_1 \dots t_n$,*

$$M \models_X =(t_1 \dots t_n) \Leftrightarrow M \models_X t_n \perp_{t_1 \dots t_{n-1}} t_n.$$

**Theorem 2.8.** *Independence logic is equivalent to $\Sigma_1^1$ (and therefore, by Theorem 2.4, to dependence logic) over sentences: in other words, for every sentence $\phi$ of independence logic there exists a sentence $\Phi$ of existential second order logic such that*

$$M \models_{\{\emptyset\}} \phi \Leftrightarrow M \models \Phi,$$

*and for every such $\Phi$ there exists a $\phi$ such that the above holds.*

There is no analogue of Theorem 2.3 for independence logic, however, as the classes of teams corresponding to independence atoms are not necessarily downwards closed: for example, according Definition 2.6, the formula $x \perp_\emptyset y$ holds in the team $\{(x := a, y := b) : a, b \in \{0, 1\}\}$ but not in its subteam $\{(x := 0, y := 0), (x := 1, y := 1)\}$.

The problem of finding a characterization similar to that of Theorem 2.5 for the classes of teams definable by formulas of independence logic was left open by Grädel and Väänänen, who concluded their paper by stating that [9]

> *The main open question raised by the above discussion is the following, formulated for finite structures:*
>
> **Open Problem:** *Characterize the NP properties of teams that correspond to formulas of independence logic.*

In this paper, an answer to this question will be given, as a corollary of an analogous result for a new logic of imperfect information.

## 3. Team semantics

### 3.1. First order (team) logic, in two flavors

In this subsection, we will present and briefly discuss the team semantics for first order logic, laying the groundwork for reasoning about its extensions while avoiding, as far as we are able to do so, all forms of semantical ambiguity.

As we will see, some special care is required here, since certain rules which are equivalent with respect to dependence logic proper will not be so with respect to these new logics.

But let us begin by recalling some basic definitions from [20].

---

2 Another interesting result about independence logic, pointed out by Fredrik Engström in [6], is that the semantic rule for independence atoms corresponds to that of *embedded multivalued dependencies*, in the same sense in which the one for dependence atoms corresponds to *functional* ones.

**Definition 3.1** (*Team*). Let $M$ be a first order model, and let $\vec{v}$ be a tuple of variables.[3] Then a *team* $X$ for $M$ with *domain* $\vec{v}$ is simply a set of assignments with domain $\vec{v}$ over $M$.

**Definition 3.2** (*From Teams to Relations*). Let $M$ be a first order model, let $X$ be a team for $M$ with domain $\vec{v}$, and let $\vec{t} = t_1 \dots t_k$ be a tuple of terms with variables in $\vec{v}$. Then we write $X(\vec{t})$ for the relation

$$X(\vec{t}) = \{(t_1\langle s\rangle \dots t_k\langle s\rangle) : s \in X\}.$$

Furthermore, if $\vec{w}$ is contained in $\vec{v}$ we will write $\mathrm{Rel}_{\vec{w}}(X)$ for $X(\vec{w})$; and, finally, if $\mathrm{Dom}(X) = \vec{v}$ we will write $\mathrm{Rel}(X)$ for $\mathrm{Rel}_{\vec{v}}(X)$.

**Definition 3.3** (*Team Restrictions*). Let $X$ be any team in any model, and let $V$ be a set of variables contained in $\mathrm{Dom}(X)$. Then

$$X_{\upharpoonright V} = \{s_{\upharpoonright V} : s \in X\},$$

where $s_{\upharpoonright V}$ is the restriction of $s$ to $V$, that is, the only assignment $s'$ with domain $V$ such that $s'(v) = s(v)$ for all $v \in V$.

The team semantics for the first order fragment of dependence logic is then defined as follows.

**Definition 3.4** (*Team Semantics for First Order Logic [12,20]*). Let $M$ be a first order model, let $\phi$ be a first order formula in negation normal form, and let $X$ be a team over $M$ with domain $\vec{v} \supseteq \mathrm{Free}(\phi)$. Then the following hold.

**TS-atom**: If $\phi$ is a first order literal, $M \models_X \phi$ if and only if, for all assignments $s \in X$, $M \models_s \phi$ in the usual first order sense.
**TS-$\vee_L$**: If $\phi$ is $\psi \vee \theta$, $M \models_X \phi$ if and only if there exist two teams $Y$ and $Z$ such that $X = Y \cup Z$, $M \models_Y \psi$ and $M \models_Z \theta$.
**TS-$\wedge$**: If $\phi$ is $\psi \wedge \theta$, $M \models_X \phi$ if and only if $M \models_X \psi$ and $M \models_X \theta$.
**TS-$\exists_S$**: If $\phi$ is $\exists x\psi$, $M \models_X \phi$ if and only if there exists a function $F : X \to \mathrm{Dom}(M)$ such that $M \models_{X[F/x]} \psi$, where

$$X[F/x] = \{s[F(s)/x] : s \in X\}.$$

**TS-$\forall$**: If $\phi$ is $\forall x\psi$, $M \models_X \phi$ if and only if $M \models_{X[M/x]} \psi$, where

$$X[M/x] = \{s[m/x] : s \in X, m \in \mathrm{Dom}(M)\}.$$

Over singleton teams, this semantics coincides with the usual one for first order logic.

**Proposition 3.5** ([20]). *Let $M$ be a first order model, let $\phi$ be a first order formula in negation normal form over the signature of $M$, and let $s$ be an assignment with $\mathrm{Dom}(s) \supseteq \mathrm{Free}(\phi)$. Then $M \models_{\{s\}} \phi$ if and only if $M \models_s \phi$ with respect to the usual Tarski semantics for first order logic.*

**Proposition 3.6** ([20]). *Let $M$ be a first order model, let $\phi$ be a first order formula in negation normal form over the signature of $M$, and let $X$ be a team with $\mathrm{Dom}(X) \supseteq \mathrm{Free}(\phi)$. Then $M \models_X \phi$ if and only if, for all assignments $s \in X$, $M \models_{\{s\}} \phi$.*

These two propositions show that, for first order logic, all the above machinery is quite unnecessary. We have no need to carry around such complex objects as teams, since we can consider the assignments in a team individually!

However, things change if we add dependence atoms $=\!(t_1 \dots t_n)$ to our language, with the semantics of rule **TS-dep** (Definition 2.1 here). In the resulting formalism, which is precisely *dependence logic* as defined in [20], not all satisfaction conditions over teams can be reduced to satisfaction conditions over assignments: for example, a "constancy atom" $=\!(x)$ holds in a team $X$ if and only if $s(x) = s'(x)$ for all $s, s' \in X$, and verifying this condition clearly requires checking *pairs* of assignments at least!

When studying variants of dependence logic, similarly, it is necessary to keep in mind that semantic rules which are equivalent with respect to dependence logic proper may not be equivalent with respect to these new formalisms. In particular, two alternative definitions of disjunction and existential quantification exist which are of special interest for this work's purposes.

**Definition 3.7** (*Alternative Rules for Disjunctions and Existentials*). Let $M, X, \phi, \psi$, and $\theta$ be as usual. Then the following hold.

**TS-$\vee_S$**: If $\phi$ is $\psi \vee \theta$, $M \models_X \phi$ if and only if there exist two teams $Y$ and $Z$ such that $X = Y \cup Z$, $Y \cap Z = \emptyset$, $M \models_Y \psi$ and $M \models_Z \theta$.
**TS-$\exists_L$**: If $\phi$ is $\exists x\psi$, $M \models_X \phi$ if and only if there exists a function $H : X \to \mathcal{P}(\mathrm{Dom}(M)) \backslash \emptyset$ such that $M \models_{X[H/x]} \psi$, where $X[H/x] = \{s[m/x] : s \in X, m \in H(s)\}$.[4]

3 Or, equivalently, a *set* of variables; but having a fixed ordering of the variables as part of the definition of team will simplify the definition of the correspondence between teams and relations. With an abuse of notation, we will identify this tuple of variables with the underlying set whenever it is expedient to do so.

4 The rule **TS**-$\exists_L$ is also discussed in [6], in which it is shown that it arises naturally from treating the existential quantifier as a *generalized quantifier* [18,17] for dependence logic.

The subscripts $\cdot_S$ and $\cdot_L$ of these rules and of the corresponding ones of Definition 3.4 allow us to discriminate between the *lax* operators $\vee_L$ and $\exists_L$ and the *strict* ones $\vee_S$ and $\exists_S$. From the point of view of game-theoretic semantics, it is not difficult to verify that the choice between lax and strict semantics corresponds to the choice between allowing and disallowing players from using *nondeterministic strategies*; but, even at a glance, this grouping of the rules is justified by the fact that **TS-**$\vee_S$ and **TS-**$\exists_S$ appear to be stronger conditions than **TS-**$\vee_L$ and **TS-**$\exists_L$. We can then define two alternative semantics for first order logic (and for its extensions, of course) as follows.

**Definition 3.8** (*Lax and Strict Semantics*)**.** The relation $M \models_X^L \phi$, where $M$ ranges over all first order models, $X$ ranges over all teams, and $\phi$ ranges over all formulas with free variables in $\mathrm{Dom}(X)$, is defined as the relation $M \models_X \phi$ of Definition 3.4 (with additional rules for further atomic formulas as required), but substituting rule **TS-**$\exists_S$ with rule **TS-**$\exists_L$. Similarly, the relation $M \models_X^S \phi$ is defined as the relation $M \models_X \phi$ of Definition 3.4, but substituting rule **TS-**$\vee_L$ with rule **TS-**$\vee_S$.

For the cases of first order and dependence logic, lax and strict semantics are equivalent.

**Proposition 3.9.** *Let $\phi$ be any formula of dependence logic. Then*

$$M \models_X^S \phi \Leftrightarrow M \models_X^L \phi$$

*for all suitable models $M$ and teams $\phi$.*

**Proof.** This is easily verified by structural induction over $\phi$, using the downwards closure property (Theorem 2.3) to take care of disjunctions and existentials (and, moreover, applying the Axiom of Choice for the case of existentials). □

As we will argue in Section 4.2, for the logics that we will study for which a difference exists between lax and strict semantics the former will be the most natural choice; therefore, from this point until the end of this work, the symbol $\models$ written without superscripts will stand for the relation $\models^L$.

### *3.2. Constancy logic*

In this section, we will present and examine a simple fragment of dependence logic. This fragment, which we will call *constancy logic*, consists of all the formulas of dependence logic in which only dependence atoms of the form $=\!(t)$ occur; or, equivalently, it can be defined as the extension of (team) first order logic obtained by adding *constancy atoms* to it, with the semantics given by the following definition.

**Definition 3.10** (*Constancy Atoms*)**.** Let $M$ be a first order model, let $X$ be a team over it, and let $t$ be a term over the signature of $M$ and with variables in $\mathrm{Dom}(X)$. Then the following holds.

**TS-const**: $M \models_X =\!(t)$ if and only if, for all $s, s' \in X$, $t\langle s\rangle = t\langle s'\rangle$.

Clearly, constancy logic is contained in dependence logic. Furthermore, over open formulas it is more expressive than first order logic proper, since, as already mentioned, the constancy atom $=\!(x)$ is a counterexample to Proposition 3.6.

The question then arises whether constancy logic is properly contained in dependence logic, or if it coincides with it. This will be answered through the following results.

**Proposition 3.11.** *Let $\phi$ be a constancy logic formula, let $z$ be a variable not occurring in $\phi$, and let $\phi'$ be obtained from $\phi$ by substituting one instance of $=\!(t)$ with the expression $z = t$.*

*Then*

$$M \models_X \phi \Leftrightarrow M \models_X \exists z(=\!(z) \wedge \phi').$$

**Proof.** The proof is an easy induction on $\phi$. □

As a corollary of this result, we get the following normal form theorem for constancy logic[5].

**Corollary 3.12.** *Let $\phi$ be a constancy logic formula. Then $\phi$ is logically equivalent to a constancy logic formula of the form*

$$\exists z_1 \ldots z_n \left( \bigwedge_{i=1}^{n} =\!(z_i) \wedge \psi(z_1 \ldots z_n) \right)$$

*for some tuple of variables $\vec{z} = z_1 \ldots z_n$ and some first order formula $\psi$.*

[5] This normal form theorem is very similar to the one of dependence logic proper found in [20]. See also [5] for a similar, but not identical result, developed independently, which Arnaud Durand and Juha Kontinen use in that paper in order to characterize the expressive powers of subclasses of dependence logic in terms of the maximum allowed width of their dependence atoms.

**Proof.** Repeatedly apply Proposition 3.11 to "push out" all constancy atoms from $\phi$, thus obtaining a formula, equivalent to it, of the form

$$\exists z_1(=(z_1) \wedge \exists z_2(=(z_2) \wedge \cdots \wedge \exists z_n(=(z_n) \wedge \psi(z_1 \ldots z_n)))$$

for some first order formula $\psi(z_1 \ldots z_n)$. It is then easy to see, from the semantics of our logic, that this is equivalent to

$$\exists z_1 \ldots z_n(=(z_1) \wedge \cdots \wedge =(z_n) \wedge \psi(z_1 \ldots z_n)),$$

as required. □

The following result shows that, over sentences, constancy logic is precisely as expressive as first order logic.

**Corollary 3.13.** *Let $\phi = \exists \vec{z} \left(\bigwedge_i =(z_i) \wedge \psi(\vec{z})\right)$ be a constancy logic sentence in normal form.*
*Then $\phi$ is logically equivalent to $\exists \vec{z}\psi(\vec{z})$.*

**Proof.** By the rules of our semantics, $M \models_{\{\emptyset\}} \psi$ if and only if there exists a family $A_1 \ldots A_n$ of nonempty sets of elements in $\text{Dom}(M)$ such that, for

$$X = \{(z_1 := m_1 \ldots z_n := m_n) : (m_1 \ldots m_n) \in A_1 \times \cdots \times A_n\},$$

it holds that $M \models_X \psi$. But $\psi$ is first order, and therefore, by Proposition 3.6, this is the case if and only if for all $m_1 \in A_1, \ldots, m_n \in A_n$ it holds that $M \models_{\{(z_1:=m_1,\ldots z_n:=m_n)\}} \psi$.

But then $M \models_{\{\emptyset\}} \phi$ if and only if there exist $m_1 \ldots m_n$ such that this holds[6]; and therefore, by Proposition 3.5, $M \models_{\{\emptyset\}} \phi$ if and only if $M \models_\emptyset \exists z_1 \ldots z_n \psi(z_1 \ldots z_n)$ according to Tarski's semantics, or, equivalently, if and only if $M \models_{\{\emptyset\}} \exists z_1 \ldots z_n \psi(z_1 \ldots z_n)$ according to team semantics. □

Since, by Theorem 2.4, dependence logic is strictly stronger than first order logic over sentences, this implies that constancy logic is strictly weaker than dependence logic over sentences (and, since sentences are a particular kind of formulas, over formulas too).

The relation between first order logic and constancy logic, in conclusion, appears somewhat similar to that between dependence logic and independence logic; that is, in both cases we have a pair of logics which are reciprocally translatable on the level of sentences, but such that one of them is strictly weaker than the other on the level of formulas. This discrepancy between translatability on the level of sentences and translatability on the level of formulas is, in the opinion of the author, one of the most intriguing aspects of logics of imperfect information, and it deserves further investigation.

## 4. Inclusion and exclusion in logic

### *4.1. Inclusion and exclusion dependencies*

Functional dependencies are the forms of dependency which attracted the most interest from database theorists, but they certainly are not the only ones ever considered in that field.

Therefore, studying the effect of substituting the dependence atoms with ones corresponding to other forms of dependency, and examining the relationship between the resulting logics, may be, in the author's opinion, at least, a very promising, and hitherto not sufficiently explored, direction of research in the field of logics of imperfect information.[7]

The present paper will, for the most part, focus on *inclusion* [8,2] and *exclusion* [3] dependencies and on the properties of the corresponding logics of imperfect information. Let us start by recalling and briefly discussing these dependencies.

**Definition 4.1** (*Inclusion Dependencies*)**.** Let $R$ be a relation, and let $\vec{x}, \vec{y}$ be tuples of attributes of $R$ of the same length. Then $R \models \vec{x} \subseteq \vec{y}$ if and only if $R(\vec{x}) \subseteq R(\vec{y})$, where

$$R(\vec{z}) = \{r(\vec{z}) : r \text{ is a tuple in } R\}.$$

In other words, an inclusion dependency $\vec{x} \subseteq \vec{y}$ states that all values taken by the attributes $\vec{x}$ are also taken by the attributes $\vec{y}$.

*Exclusion* dependencies [3], instead, assert that two tuples of attributes have no values in common.

**Definition 4.2** (*Exclusion Dependencies*)**.** Let $R$ be a relation, and let $\vec{x}, \vec{y}$ be tuples of attributes of $R$ of the same length. Then $R \models \vec{x} \mid \vec{y}$ if and only if $R(\vec{x}) \cap R(\vec{y}) = \emptyset$, where $R(\vec{x})$ and $R(\vec{y})$ are as stated in the previous definition.

6 Indeed, if this is the case we can just choose $A_1 = \{m_1\}, \ldots, A_n = \{m_n\}$, and conversely if $A_1 \ldots A_n$ exist with the required properties we can simply select arbitrary elements of them for $m_1 \ldots m_n$.

7 Apart from the present paper, [6], which introduces *multivalued dependence atoms*, is also a step in this direction. The resulting "multivalued dependence logic" is easily seen to be equivalent to independence logic.

We will not discuss here the significance of inclusion and exclusion dependencies in the context of database theory, nor the results that were found in that area of study about their properties.

What interests us here is that it is not difficult to transfer the definitions of inclusion and exclusion dependencies to team semantics, thus obtaining *inclusion atoms* and *exclusion atoms*.

**Definition 4.3** (*Inclusion and Exclusion Atoms*)**.** Let $M$ be a first order model, let $\vec{t}_1$ and $\vec{t}_2$ be two finite tuples of terms of the same length over the signature of $M$, and let $X$ be a team whose domain contains all variables occurring in $\vec{t}_1$ and $\vec{t}_2$. Then the following hold.

**TS-inc**: $M \models_X \vec{t}_1 \subseteq \vec{t}_2$ if and only if for every $s \in X$ there exists an $s' \in X$ such that $\vec{t}_1\langle s\rangle = \vec{t}_2\langle s'\rangle$.
**TS-exc**: $M \models_X \vec{t}_1 \mid \vec{t}_2$ if and only if, for all $s, s' \in X$, $\vec{t}_1\langle s\rangle \neq \vec{t}_2\langle s'\rangle$.

We will also consider another kind of atom, which can be seen as a symmetric version of inclusion.

**Definition 4.4** (*Equiextension Atoms*)**.** Let $M$ be a first order model, let $\vec{t}_1$ and $\vec{t}_2$ be two finite tuples of terms of the same length over the signature of $M$, and let $X$ be a team whose domain contains all variables occurring in $\vec{t}_1$ and $\vec{t}_2$. Then the following holds.

**TS-equ**: $M \models_X \vec{t}_1 \bowtie \vec{t}_2$ if and only if $X(\vec{t}_1) = X(\vec{t}_2)$.

It is easy to see that $\vec{t}_1 \bowtie \vec{t}_2$ is equivalent to $\vec{t}_1 \subseteq \vec{t}_2 \wedge \vec{t}_2 \subseteq \vec{t}_1$ and that it is strictly weaker than the first order formula

$$\vec{t}_1 = \vec{t}_2 := \bigwedge_i ((\vec{t}_1)_i = (\vec{t}_2)_i).$$

As we will see later, it is possible to recover inclusion atoms from equiextension atoms and the connectives of our logics.

### 4.2. Inclusion logic

In this section, we will begin to examine the properties of *inclusion logic*, that is, the logic obtained adding to (team) first order logic the *inclusion atoms* $\vec{t}_1 \subseteq \vec{t}_2$ with the semantics of Definition 4.3.

A first, easy observation is that this logic does not respect the downwards closure property. For example, consider the two assignments $s_0 = (x := 0, y := 1)$ and $s_1 = (x := 1, y := 0)$; then, for $X = \{s_0, s_1\}$ and $Y = \{s_0\}$, it is easy to see by rule **TS-inc** that $M \models_X x \subseteq y$ but $M \not\models_Y x \subseteq y$.

Hence, the proof of Proposition 3.9 cannot be adapted to the case of inclusion logic. The question then arises whether inclusion logic with strict semantics and inclusion logic with lax semantics are different; and, as the next two propositions will show, this is indeed the case.

**Proposition 4.5.** *There exist a model $M$, a team $X$, and two formulas $\psi$ and $\theta$ of inclusion logic such that $M \models^L_X \psi \vee \theta$ but $M \not\models^S_X \psi \vee \theta$.*

**Proof.** Let $\mathrm{Dom}(M) = \{0, 1, 2, 3, 4\}$, let $X$ be the team

$X =$

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| $s_0$ | 0 | 1 | 2 |
| $s_1$ | 1 | 0 | 3 |
| $s_2$ | 4 | 3 | 0 |

,

and let $\psi = x \subseteq y, \theta = y \subseteq z$.

- $M \models^L_X \psi \vee \theta$:
  Let $Y = \{s_0, s_1\}$ and $Z = \{s_1, s_2\}$. Then $Y \cup Z = X$, $Y(x) = \{0, 1\} = Y(y)$ and $Z(y) = \{0, 3\} = Z(z)$.
  Hence, $M \models^L_Y x \subseteq y$ and $M \models^L_Z y \subseteq z$, and therefore $M \models^L_X x \subseteq y \vee y \subseteq z$ as required.
- $M \not\models^S_X \psi \vee \theta$:
  Suppose that $X = Y \cup Z$, $Y \cap Z = \emptyset$, $M \models^S_X x \subseteq y$ and $M \models^S_Z y \subseteq z$.
  Now, $s_2$ cannot belong in $Y$, since $s_2(x) = 4$ and $s_i(y) \neq 4$ for all assignments $s_i$; therefore, we necessarily have that $s_2 \in Z$. But since $M \models^S_Z y \subseteq z$, this implies that there exists an assignment $s_i \in Y$ such that $s_i(z) = s_2(y) = 3$. The only such assignment in $X$ is $s_1$, and therefore $s_1 \in Y$.
  Analogously, $s_0$ cannot belong in $Z$: indeed, $s_0(y) = 1 \neq s_i(z)$ for all $i \in 0 \ldots 2$. Therefore, $s_0 \in Y$; and since $M \models^S_Y x \subseteq y$, there exists an $s_i \in Y$ with $s_i(y) = s_0(x) = 0$. But the only such assignment in $X$ is $s_1$, and therefore $s_1 \in Y$.
  In conclusion, $Y = \{s_0, s_1\}$, $Z = \{s_1, s_2\}$ and $Y \cap Z = \{s_1\} \neq \emptyset$, which contradicts our hypothesis. □

**Proposition 4.6.** *There exist a model $M$, a team $X$, and a formula $\phi$ of inclusion logic such that $M \models^L_X \exists x\phi$ but $M \not\models^S_X \exists x\phi$.*

**Proof.** Let $\mathrm{Dom}(M) = \{0, 1\}$, let $X$ be the team

$$X = \begin{array}{c|cc} & y & z \\ \hline s_0 & 0 & 1 \end{array},$$

and let $\phi$ be $y \subseteq x \wedge z \subseteq x$.

- $M \models^L_X \exists x\phi$:
  Let $H : X \to \mathcal{P}(\mathrm{Dom}(M))$ be such that $H(s_0) = \{0, 1\}$.
  Then

  $$X[H/x] = \begin{array}{c|ccc} & y & z & x \\ \hline s'_0 & 0 & 1 & 0 \\ s'_1 & 0 & 1 & 1 \end{array},$$

  and hence $X[H/x](y), X[H/x](z) \subseteq X[H/x](x)$, as required.
- $M \not\models^S_X \exists x\psi$:
  Let $F$ be any function from $X$ to $\mathrm{Dom}(M)$. Then

  $$X[F/x] = \begin{array}{c|ccc} & y & z & x \\ \hline s''_0 & 0 & 1 & F(s_0) \end{array}.$$

  But $F(s_0) \neq 0$ or $F(s_0) \neq 1$; and in the first case $M \not\models^S_{X[F/x]} y \subseteq x$, while in the second case $M \not\models^S_{X[F/x]} z \subseteq x$. □

Therefore, when studying the properties of inclusion logic it is necessary to specify whether we are using strict or lax semantics for disjunction and existential quantification. However, only one of these choices preserves *locality* in the sense of Theorem 2.2, as the two following results show.

**Proposition 4.7.** *The strict semantics does not respect locality in inclusion logic (or in any extension thereof). In other words, there exists a model M, a team X, and two formulas $\psi$ and $\theta$ such that $M \models^S_X \psi \vee \theta$, but for $X' = X_{\restriction Free(\phi\vee\psi)}$ it holds that $M \not\models^S_{X'} \psi \vee \theta$ instead; and, analogously, there exists a model M, a team X, and a formula $\xi$ such that $M \models^S_X \exists x\xi$, but for $X' = X_{\restriction Free(\exists x\xi)}$ we have that $M \not\models^S_{X'} \exists\xi$ instead.*

**Proof.** 1. Let $\mathrm{Dom}(M) = \{0 \ldots 4\}$, let $\psi$ and $\theta$ be $x \subseteq y$ and $y \subseteq z$, respectively, and let

$$X = \begin{array}{c|cccc} & x & y & z & u \\ \hline s_0 & 0 & 1 & 2 & 0 \\ s_1 & 1 & 0 & 3 & 0 \\ s_2 & 1 & 0 & 3 & 1 \\ s_3 & 4 & 3 & 0 & 0 \end{array}.$$

Then $M \models^S_X \psi \vee \theta$; indeed, for $Y = \{s_0, s_1\}$ and $Z = \{s_2, s_3\}$, we have that $X = Y \cup Z$, $Y \cap Z = \emptyset$, $M \models_Y \psi$, and $M \models_Z \theta$, as required. However, the restriction $X'$ of $X$ to $\mathrm{Free}(\psi \vee \theta) = \{x, y, z\}$ is the team considered in the proof of Proposition 4.5, and, as shown in that proof, $M \not\models^S_X \psi \vee \theta$.

2. Let $\mathrm{Dom}(M) = \{0, 1\}$, let $\xi$ be $y \subseteq x \wedge z \subseteq x$, and let

$$X = \begin{array}{c|ccc} & y & z & u \\ \hline s_0 & 0 & 1 & 0 \\ s_1 & 0 & 1 & 1 \end{array}.$$

Then $M \models^S_X \exists x\xi$; indeed, for $F : X \to \mathrm{Dom}(M)$ defined as

$$F(s_0) = 0;$$
$$F(s_1) = 1,$$

we have that

$$X[F/x] = \begin{array}{c|cccc} & y & z & u & x \\ \hline s'_0 & 0 & 1 & 0 & 0 \\ s'_1 & 0 & 1 & 1 & 1 \end{array},$$

and it is easy to check that this team satisfies $\xi$. However, the restriction $X'$ of $X$ to $\mathrm{Free}(\exists x\xi) = \{y, z\}$ is the team considered in the proof of Proposition 4.6, and, again, as shown in that proof, $M \not\models^S_X \exists x\psi$. □

**Theorem 4.8** (*Inclusion Logic with Lax Semantics is Local*)**.** *Let M be a first order model, let $\phi$ be any inclusion logic formula, and let V be a set of variables with $Free(\phi) \subseteq V$. Then, for all suitable teams X,*

$$M \models^L_X \phi \Leftrightarrow M \models^L_{X_{\restriction V}} \phi.$$

**Proof.** The proof is by structural induction on $\phi$.

In Section 4.5, Theorem 4.22, we will prove the same result for an extension of inclusion logic; so we refer to that theorem for the details of the proof. □

Since, as we saw, inclusion logic is not downwards closed, by Theorem 2.3, it is not contained in dependence logic. It is then natural to ask whether dependence logic is contained in inclusion logic, or if dependence and inclusion logic are two incomparable extensions of first order logic.

This is answered by the following result, and by its corollary.

**Theorem 4.9** (*Union Closure for Inclusion Logic*)**.** *Let $\phi$ be any inclusion logic formula, let $M$ be a first order model, and let $(X_i)_{i\in I}$ be a family of teams with the same domain such that $M \models_{X_i} \phi$ for all $i \in I$. Then, for $X = \bigcup_{i\in I} X_i$, we have that $M \models_X \phi$.*

**Proof.** The proof is an easy structural induction over $\phi$. □

**Corollary 4.10.** *There exist constancy logic formulas which are not equivalent to any inclusion logic formula.*

**Proof.** This follows at once from the fact that the constancy atom $=(x)$ is not closed under unions.

Indeed, let $M$ be any model with two elements 0 and 1 in its domain, and consider the two teams $X_0 = \{(x := 0)\}$ and $X_1 = \{(x := 1)\}$; then $M \models_{X_0} =(x)$ and $M \models_{X_1} =(x)$, but $M \not\models_{X_0 \cup X_1} =(x)$. □

Therefore, not only does inclusion logic not contain dependence logic, it does not even contain constancy logic!

Now, by Theorem 2.7, we know that dependence logic is properly contained in independence logic. As the following result shows, inclusion logic is also (properly, because dependence atoms are expressible in independence logic) contained in independence logic.

**Theorem 4.11.** *Inclusion atoms are expressible in terms of independence logic formulas. More precisely, an inclusion atom $\vec{t}_1 \subseteq \vec{t}_2$ is equivalent to the independence logic formula*

$$\phi := \forall v_1 v_2 \vec{z}((\vec{z} \neq \vec{t}_1 \wedge \vec{z} \neq \vec{t}_2) \vee (v_1 \neq v_2 \wedge \vec{z} \neq \vec{t}_2) \vee ((v_1 = v_2 \vee \vec{z} = \vec{t}_2) \wedge \vec{z} \perp v_1 v_2)),$$

*where $v_1$, $v_2$, and $\vec{z}$ do not occur in $\vec{t}_1$ or $\vec{t}_2$ and where, as in [9], $\vec{z} \perp v_1 v_2$ is a shorthand for $\vec{z} \perp_{\emptyset} v_1 v_2$.*

**Proof.** Suppose that $M \models_X \vec{t}_1 \subseteq \vec{t}_2$. Then split the team $X' = X[M/v_1 v_2 \vec{z}]$ into three teams $Y$, $Z$ and $W$ as follows.

- $Y = \{s \in X' : s(\vec{z}) \neq \vec{t}_1\langle s\rangle$ and $s(\vec{z}) \neq \vec{t}_2\langle s\rangle\}$.
- $Z = \{s \in X' : s(v_1) \neq s(v_2)$ and $s(\vec{z}) \neq \vec{t}_2\langle s\rangle\}$.
- $W = X' \backslash (Y \cup Z) = \{s \in X' : s(\vec{z}) = \vec{t}_2\langle s\rangle$ or $(s(\vec{z}) = \vec{t}_1\langle s\rangle$ and $s(v_1) = s(v_2))\}$.

Clearly, $X' = Y \cup Z \cup W$, $M \models_Y z \neq t_1 \wedge z \neq t_2$, and $M \models_Z v_1 \neq v_2 \wedge z \neq t_2$; hence, if we can prove that

$$M \models_W ((v_1 = v_2 \vee \vec{z} = \vec{t}_2)) \wedge \vec{z} \perp v_1 v_2,$$

then we can conclude that $M \models_X \phi$, as required.

Now, suppose that $s \in W$ and $s(v_1) \neq s(v_2)$; then necessarily $s(\vec{z}) = \vec{t}_2$, since otherwise we would have that $s \in Z$ instead. Hence, the first conjunct $v_1 = v_2 \vee \vec{z} = \vec{t}_2$ is satisfied by $W$.

Consider two assignments $s$ and $s'$ in $W$; in order to conclude this direction of the proof, we need to show that there exists an $s'' \in W$ such that $s''(\vec{z}) = s(\vec{z})$ and $s''(v_1 v_2) = s'(v_1 v_2)$. There are two distinct cases to examine.

1. If $s(\vec{z}) = \vec{t}_2\langle s\rangle$, consider the assignment $s'' = s[s'(v_1)/v_1][s'(v_2)/v_2]$. By construction, $s'' \in X'$; and, furthermore, since $s''(\vec{z}) = \vec{t}_2\langle s\rangle = \vec{t}_2\langle s''\rangle$, $s''$ is neither in $Y$ nor in $Z$. Hence, it is in $W$, as required.
2. If $s(\vec{z}) \neq \vec{t}_2\langle s\rangle$ and $s \in W$, then necessarily $s(\vec{z}) = \vec{t}_1\langle s\rangle$ and $s(v_1) = s(v_2)$. Since $s \in W \subseteq X' = X[M/v_1 v_2 \vec{z}]$, there exists an assignment $o \in X$ such that $\vec{t}_1\langle o\rangle = \vec{t}_1\langle s\rangle = s(\vec{z})$; and since $M \models_X \vec{t}_1 \subseteq \vec{t}_2$, there also exists an assignment $o' \in X$ such that $\vec{t}_2\langle o'\rangle = \vec{t}_1\langle o\rangle = s(\vec{z})$. Now, consider the assignment $s'' = o'[s'(v_1)/v_1][s'(v_2)/v_2][s(\vec{z})/\vec{z}]$; by construction, $s'' \in X'$, and, since $s''(\vec{z}) = s(\vec{z}) = \vec{t}_2\langle o'\rangle = \vec{t}_2\langle s''\rangle$, we have that $s'' \in W$, that $s''(\vec{z}) = s(\vec{z})$, and that $s''(v_1 v_2) = s'(v_1 v_2)$, as required.

Conversely, suppose that $M \models_X \phi$, let 0 and 1 be two distinct elements of the domain of $M$, and let $s \in X$.

By the definition of $\phi$, the fact that $M \models_X \phi$ implies that the team $X[M/v_1 v_2 \vec{z}]$ can be split into three teams $Y$, $Z$, and $W$ such that

$$M \models_Y \vec{z} \neq \vec{t}_1 \wedge \vec{z} \neq \vec{t}_2;$$
$$M \models_Z v_1 \neq v_2 \wedge \vec{z} \neq \vec{t}_2;$$
$$M \models_W (v_1 = v_2 \vee \vec{z} = \vec{t}_2) \wedge \vec{z} \perp v_1 v_2.$$

Then consider the assignments $h = s[0/v_1][0/v_2][\vec{t}_1\langle s\rangle/\vec{z}]$ and $h' = s[0/v_1][1/v_2][\vec{t}_2\langle s\rangle/\vec{z}]$.

Clearly, $h$ and $h'$ are in $X[M/v_1 v_2 \vec{z}]$. However, neither of them is in $Y$, since $h(\vec{z}) = \vec{t}_1\langle h\rangle$ and $h'(\vec{z}) = \vec{t}_2\langle h'\rangle$, nor in $Z$, since $h(v_1) = h(v_2)$ and, again, since $h'(\vec{z}) = \vec{t}_2\langle h'\rangle$. Hence, both of them are in $W$.

But we know that $M \models_W \vec{z} \perp v_1 v_2$, and thus there exists an assignment $h'' \in W$ with $h''(\vec{z}) = h(\vec{z}) = \vec{t}_1\langle s\rangle$ and $h''(v_1 v_2) = h'(v_1 v_2) = 01$.

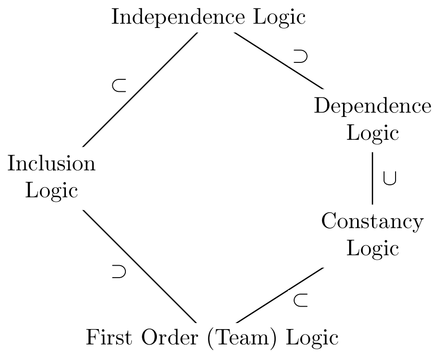


**Fig. 1.** Translatability relations between logics (with respect to formulas).

Now, since $h''(v_1) \neq h''(v_2)$, since $h'' \in W$, and since $M \models_W v_1 = v_2 \vee \vec{z} = \vec{t}_2$, it must be the case that $h''(\vec{z}) = \vec{t}_2\langle h''\rangle$. Finally, this $h''$ corresponds to some $s'' \in X$; and, for this $s''$,

$$\vec{t}_2\langle s''\rangle = \vec{t}_2\langle h''\rangle = h''(\vec{z}) = h(\vec{z}) = \vec{t}_1\langle s\rangle.$$

This concludes the proof. □

The relations between first order (team) logic, constancy logic, dependence logic, inclusion logic, and independence logic discovered so far are then represented by Fig. 1.

However, things change if we take in consideration the expressive power of these logics with respect to their sentences alone. Then, as we saw, first order logic and constancy logic have the same expressive power, in the sense that every constancy logic formula is equivalent to some first order formula and vice versa, and so do dependence and independence logic. What about inclusion logic sentences?

At the moment, relatively little is known by the author about this. In essence, all that we know is the following result.

**Proposition 4.12.** *Let $\psi(\vec{x}, \vec{y})$ be any first order formula, where $\vec{x}$ and $\vec{y}$ are tuples of disjoint variables of the same arity. Furthermore, let $\psi'(\vec{x}, \vec{y})$ be the result of writing $\neg\psi(\vec{x}, \vec{y})$ in negation normal form. Then, for all suitable models M and all suitable pairs $\vec{a}, \vec{b}$ of constant terms of the model,*

$$M \models_{\{\emptyset\}} \exists\vec{z}(\vec{a} \subseteq \vec{z} \wedge \vec{z} \neq \vec{b} \wedge \forall\vec{w}(\psi'(\vec{z}, \vec{w}) \vee \vec{w} \subseteq \vec{z}))$$

*if and only if $M \models \neg[TC_{\vec{x},\vec{y}}\ \psi](\vec{a}, \vec{b})$, that is, if and only if the pair of tuples of elements corresponding to $(\vec{a}, \vec{b})$ is not in the transitive closure of $\{(\vec{m}_1, \vec{m}_2) : M \models \psi(\vec{m}_1, \vec{m}_2)\}$.*

**Proof.** Suppose that $M \models_{\{\emptyset\}} \exists\vec{z}(\vec{a} \subseteq \vec{z} \wedge \vec{z} \neq \vec{b} \wedge \forall\vec{w}(\psi'(\vec{z}, \vec{w}) \vee \vec{w} \subseteq \vec{z}))$. Then, by definition, there exists a tuple of functions $\vec{H} = H_1 \ldots H_n$ such that

1. $M \models_{\{\emptyset\}[\vec{H}/\vec{z}]} \vec{a} \subseteq \vec{z}$, that is, $\vec{a} \in \vec{H}(\{\emptyset\})$;
2. $M \models_{\{\emptyset\}[\vec{H}/\vec{z}]} \vec{z} \neq \vec{b}$, and therefore $\vec{b} \notin \vec{H}(\{\emptyset\})$;
3. $M \models_{\{\emptyset\}[\vec{H}/\vec{z}][\vec{M}/\vec{w}]} \psi'(\vec{z}, \vec{w}) \vee \vec{w} \subseteq \vec{z}$.

Now, the third condition implies that, whenever $M \models \psi(\vec{m}_1, \vec{m}_2)$ and $\vec{m}_1$ is in $\vec{H}(\{\emptyset\})$, $\vec{m}_2$ is in $\vec{H}(\{\emptyset\})$ too. Indeed, let $Y = \{\emptyset\}[\vec{H}/\vec{z}][\vec{M}/\vec{w}]$; then, by the semantics of our logic, we know that $Y = Y_1 \cup Y_2$ for two subteams $Y_1$ and $Y_2$ such that $M \models_{Y_1} \psi'(\vec{z}, \vec{w})$ and $M \models_{Y_2} \vec{w} \subseteq \vec{z}$. But $\psi'$ is logically equivalent to the negation of $\psi$, and therefore we know that, for all $s \in Y_1$, $M \not\models \psi(s(\vec{z}), s(\vec{w}))$ in the usual Tarskian semantics.

Suppose now that $\vec{m}_1 \in \vec{H}(\{\emptyset\})$ and that $M \models \psi(\vec{m}_1, \vec{m}_2)$. Then $s = (\vec{z} := \vec{m}_1, \vec{w} := \vec{m}_2)$ is in $Y$; but it cannot be in $Y_1$, as we saw, and hence it must belong to $Y_2$. But $M \models_{Y_2} \vec{w} \subseteq \vec{z}$, and therefore there exists another assignment $s' \in Y_2$ such that $s'(\vec{z}) = s(\vec{w}) = \vec{m}_2$. But we necessarily have that $s'(\vec{z}) \in \vec{H}(\{\emptyset\})$, and therefore $\vec{m}_2 \in \vec{H}(\{\emptyset\})$, as required.

So, $\vec{H}(\{\emptyset\})$ is an set of tuples of elements of our models which contains the interpretation of $\vec{a}$ but not that of $\vec{b}$, and such that

$$\vec{m}_1 \in H(\{\emptyset\}), M \models \psi(\vec{m}_1), \vec{M}_2 \Rightarrow \vec{m}_2 \in H(\{\emptyset\}).$$

This implies that $M \models \neg[\mathrm{TC}_{\vec{x},\vec{y}}\ \psi](\vec{a}, \vec{b})$, as required.

Conversely, suppose that $M \models \neg[\mathrm{TC}_{\vec{x},\vec{y}}\ \psi](\vec{a}, \vec{b})$; then there exists a set $A$ of tuples of elements of the domain of $M$ which contains the interpretation of $\vec{a}$ but not that of $\vec{b}$, and such that it is closed by transitive closure for $\psi(\vec{x}, \vec{y})$. Then, by choosing the functions $\vec{H}$ so that $\vec{h}(\{\emptyset\}) = A$, it is easy to verify that $M$ satisfies our inclusion logic sentence. □

As a corollary, we have that inclusion logic is strictly more expressive than first order logic over sentences; for example, for all finite linear orders $M = (\text{Dom}(M), <, S, 0, e)$, where $S$ is the successor function, 0 is the first element of the linear order and $e$ is the last one, we have that

$$M \models \exists z(0 \subseteq z \wedge z \neq e \wedge \forall w(w \neq S(S(z)) \vee w \subseteq z))$$

if and only if $|M|$ is odd. It is not difficult to see, for example through the Ehrenfeucht-Fraïssé method [11], that this property is not expressible in first order logic.

### 4.3. Equiextension logic

Let us now consider *equiextension logic*, that is, the logic obtained by adding to first order logic (with lax team semantics) equiextension atoms $\vec{t}_1 \bowtie \vec{t}_2$ with the semantics of Definition 4.4.

It is easy to see that equiextension logic is contained in inclusion logic.

**Proposition 4.13.** *Let $\vec{t}_1$ and $\vec{t}_2$ be any two tuples of terms of the same length. Then, for all suitable models M and teams X,*

$$M \models_X \vec{t}_1 \bowtie \vec{t}_2 \Leftrightarrow M \models_X \vec{t}_1 \subseteq \vec{t}_2 \wedge \vec{t}_2 \subseteq \vec{t}_1.$$

**Proof.** Obvious. □

Translating in the other direction, however, requires a little more care.

**Proposition 4.14.** *Let $\vec{t}_1$ and $\vec{t}_2$ be any two tuples of terms of the same length. Then, for all suitable models M and teams X, $M \models_X \vec{t}_1 \subseteq \vec{t}_2$ if and only if*

$$M \models_X \forall u_1 u_2 \exists \vec{z}(\vec{t}_2 \bowtie \vec{z} \wedge (u_1 \neq u_2 \vee \vec{z} = \vec{t}_1)),$$

*where $u_1$, $u_2$, and $\vec{z}$ do not occur in $\vec{t}_1$ and $\vec{t}_2$.*

**Proof.** Suppose that $M \models_X \vec{t}_1 \subseteq \vec{t}_2$. Then, let $X' = X[M/u_1u_2]$, and pick the tuple of functions $\vec{H}$ used to choose $\vec{z}$ so that

$$\vec{H}(s) = \begin{cases} \{\vec{t}_1\langle s\rangle\}, & \text{if } s(\vec{u}_1) = s(\vec{u}_2); \\ \{\vec{t}_2\langle s\rangle\}, & \text{otherwise} \end{cases}$$

for all $s \in X'$.

Then, for $Y = X'[\vec{H}/\vec{z}]$, by definition we have that $M \models_Y u_1 \neq u_2 \vee \vec{z} = \vec{t}_1$, and it only remains to verify that $M \models_Y \vec{t}_2 \bowtie \vec{z}$, that is, that $Y(\vec{t}_2) = Y(\vec{z})$.

- $Y(\vec{t}_2) \subseteq Y(\vec{z})$:
  Let $h \in Y$. Then there exists an assignment $s \in X$ with $\vec{t}_2\langle s\rangle = \vec{t}_2\langle h\rangle$. Now, let 0 and 1 be two distinct elements of $M$, and consider the assignment $h' = s[0/u_1][1/u_2][\vec{H}/\vec{z}]$. By construction, $h' \in Y$; and furthermore, by the definition of $\vec{H}$ we have that $h'(\vec{z}) = \vec{t}_2\langle s\rangle = \vec{t}_2\langle h\rangle$, as required.
- $Y(\vec{z}) \subseteq Y(\vec{t}_2)$:
  Let $h \in Y$. Then, by construction, $h(\vec{z})$ is $\vec{t}_1\langle h\rangle$ or $\vec{t}_2\langle h\rangle$. But, since $X(\vec{t}_1) \subseteq X(\vec{t}_2)$, in either case there exists an assignment $s \in X$ such $\vec{t}_2\langle s\rangle = h(\vec{z})$. Now, consider $h' = s[0/u_1][1/u_2][\vec{H}/\vec{z}]$; again, $h' \in Y$ and $\vec{t}_2\langle h'\rangle = \vec{t}_2\langle s\rangle = h(\vec{z})$, as required.

Conversely, suppose that $M \models_X \forall u_1 u_2 \exists \vec{z}(\vec{t}_2 \bowtie \vec{z} \wedge (u_1 \neq u_2 \vee \vec{z} = \vec{t}_1))$, and that therefore there exists a tuple of functions $\vec{H}$ such that, for $Y = X[M/u_1u_2][\vec{H}/\vec{z}]$, $M \models_Y \vec{t}_2 \bowtie \vec{z} \wedge (u_1 \neq u_2 \vee \vec{z} = \vec{t}_1)$. Then, consider any assignment $s \in X$, and let $h = s[0/u_1][0/u_2][\vec{H}/\vec{z}]$. Now, $h \in Y$ and $h(\vec{z}) = \vec{t}_1\langle s\rangle$; but, since $M \models_Y \vec{t}_2 \bowtie \vec{z}$, this implies that there exists an assignment $h' \in Y$ such that $\vec{t}_2\langle h'\rangle = h(\vec{z}) = \vec{t}_1\langle s\rangle$. Finally, $h'$ derives from some assignment $s' \in X$, and for this assignment we have that $\vec{t}_2\langle s\rangle = \vec{t}_2\langle h'\rangle = \vec{t}_1\langle s\rangle$, as required. □

As a consequence, inclusion logic is precisely as expressive as equiextension logic.

**Corollary 4.15.** *Any formula of inclusion logic is equivalent to some formula of equiextension logic, and vice versa.*

### 4.4. Exclusion logic

With the name of *exclusion logic* we refer to (lax, team) first order logic supplemented with the *exclusion atoms* $\vec{t}_1 \mid \vec{t}_2$, with the satisfaction condition given in Definition 4.3.

As the following results show, exclusion logic is, in a very strong sense, equivalent to dependence logic.

**Theorem 4.16.** *For all tuples of terms $\vec{t}_1$ and $\vec{t}_2$, of the same length, there exists a dependence logic formula $\phi$ such that*

$$M \models_X \phi \Leftrightarrow M \models_X \vec{t}_1 \mid \vec{t}_2$$

*for all suitable models M and teams X.*

**Proof.** This follows immediately from Theorem 2.5, since the satisfaction condition for the exclusion atom is downwards monotone and expressible in $\Sigma_1^1$. $\square$

An exclusion atom $\vec{t}_1 \mid \vec{t}_2$ can also be translated explicitly in dependence logic as $\forall \vec{z} \exists u_1 u_2 (=(\vec{z}, u_1) \wedge =(\vec{z}, u_2) \wedge ((u_1 = u_2 \wedge \vec{z} \neq \vec{t}_1) \vee (u_1 \neq u_2 \wedge \vec{z} \neq \vec{t}_2)))$, and interested readers should be able to verify that the interpretation of this expression is indeed the required one.

**Theorem 4.17.** *Let $t_1 \dots t_n$ be terms, and let $z$ be a variable not occurring in any of them. Then the dependence atom $=(t_1 \dots t_n)$ is equivalent to the exclusion logic expression*

$$\phi = \forall z (z = t_n \vee (t_1 \dots t_{n-1} z \mid t_1 \dots t_{n-1} t_n)),$$

*for all suitable models $M$ and teams $X$.*

**Proof.** Suppose that $M \models_X =(t_1 \dots t_n)$, and consider the team $X[M/z]$. Now, let $Y = \{s \in X[M/z] : s(z) = t_n\langle s \rangle\}$, and let $Z = X[M/z] \backslash Y$.

Clearly, $Y \cup Z = X[M/x]$ and $M \models_Y z = t_n$; hence, if we show that $Z \models t_1 \dots t_{n-1} z \mid t_1 \dots t_{n-1} t_n$, we can conclude that $M \models_X \phi$, as required.

Now, consider any two $s, s' \in Z$, and suppose that $t_i\langle s \rangle = t_i\langle s' \rangle$ for all $i = 1 \dots n-1$. But then $s(z) \neq t_n\langle s' \rangle$; indeed, since $M \models_X =(t_1 \dots t_n)$, by the locality of dependence logic and by the downwards closure property, we have that $M \models_Z =(t_1 \dots t_n)$, and hence that $t_n\langle s \rangle = t_n\langle s' \rangle$.

Therefore, if we had that $s(z) = t_n\langle s' \rangle$, it would follow that $s(z) = t_n\langle s \rangle$, and $s$ would be in $Y$ instead.

So, $s(z) \neq t_n\langle s' \rangle$, and, since this holds for all $s$ and $s'$ in $Z$ which coincide over $t_1 \dots t_{n-1}$, we have that

$$M \models_Z t_1 \dots t_{n-1} z \mid t_1 \dots t_{n-1} t_n,$$

as required.

Conversely, suppose that $M \models_X \phi$, and let $s, s' \in X$ assign the same values to $t_1 \dots t_{n-1}$. Now, by the definition of $\phi$, $X[M/z]$ can be split into two subteams $Y$ and $Z$ such that $M \models_Y z = t_n$, and such that $M \models_Z (t_1 \dots t_{n-1} z \mid t_1 \dots t_{n-1} t_n)$.

Suppose that $t_n\langle s \rangle = m$ and $t_n\langle s' \rangle = m'$, and that $m \neq m'$; then $s[m'/z]$ and $s'[m/z]$ are in $s[M/z]$ but not in $Y$, and hence they are both in $Z$. But then, since $\vec{t}_i\langle s \rangle = \vec{t}_i\langle s' \rangle$ for all $i = 1 \dots n-1$,

$$t_n\langle s' \rangle = m' = s[m'/z](z) \neq t_n\langle s'[m/z] \rangle = t_n\langle s' \rangle,$$

which is a contradiction. Therefore, $m = m'$, as required. $\square$

**Corollary 4.18.** *Dependence logic is precisely as expressive as exclusion logic, both with respect to definability of sets of teams and with respect to sentences.*

*4.5. Inclusion/exclusion logic*

Now that we have some information about inclusion logic and about exclusion logic, let us study *inclusion/exclusion logic* (I/E logic for short), that is, the formalism obtained by adding both inclusion and exclusion atoms to the language of first order logic.

By the results of the previous sections, we already know that inclusion atoms are expressible in independence logic and that exclusion atoms are expressible in dependence logic; furthermore, by Theorem 2.7, dependence atoms are expressible in independence logic.

Then it follows at once that I/E logic is contained in independence logic.

**Corollary 4.19.** *For every inclusion/exclusion logic formula $\phi$ there exists an independence logic formula $\phi^*$ such that*

$$M \models_X \phi \Leftrightarrow M \models_X \phi^*$$

*for all suitable models $M$ and teams $X$.*

Now, is I/E logic properly contained in independence logic?

As the following result illustrates, this is not the case.

**Theorem 4.20.** *Let $\vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3$ be an independence atom, and let $\phi$ be the formula*

$$\forall \vec{p}\vec{q}\vec{r}\, \exists u_1 u_2 u_3 u_4 (\bigwedge_{i=1}^{4} =(\vec{p}\vec{q}\vec{r}, u_i) \wedge ((u_1 \neq u_2 \wedge (\vec{p}\vec{q} \mid \vec{t}_1\vec{t}_2)) \vee$$
$$\vee (u_1 = u_2 \wedge u_3 \neq u_4 \wedge (\vec{p}\vec{r} \mid \vec{t}_1\vec{t}_3)) \vee (u_1 = u_2 \wedge u_3 = u_4 \wedge (\vec{p}\vec{q}\vec{r} \subseteq \vec{t}_1\vec{t}_2\vec{t}_3)))),$$

*where the dependence atoms are used as shorthand for the corresponding exclusion logic expressions, which exist because of Theorem 4.17, and where all the quantified variables are new.*

*Then, for all suitable models $M$ and teams $X$,*

$$M \models_X \vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3 \Leftrightarrow M \models_X \phi.$$

**Proof.** Suppose that $M \models_X \vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3$, and consider the team $X' = X[M/\vec{p}\vec{q}\vec{r}]$.

Now, let 0 and 1 be two distinct elements of the domain of $M$, and let the functions $F_1 \dots F_4$ be defined as follows.

- For all $s \in X'$, $F_1(s) = 0$.
- For all $s \in X'[F_1/u_1]$,

$$F_2(s) = \begin{cases} 0 & \text{if there exists a } s' \in X \text{ such that } \vec{t}_1\langle s'\rangle\vec{t}_2\langle s'\rangle = s(\vec{p})s(\vec{q}); \\ 1 & \text{otherwise.} \end{cases}$$

- For all $s \in X'[F_1/u_1][F_2/u_2]$, $F_3(s) = 0$.
- For all $s \in X'[F_1/u_1][F_2/u_2][F_3/u_3]$,

$$F_4(s) = \begin{cases} 0 & \text{if there exists a } s' \in X \text{ such that } \vec{t}_1\langle s'\rangle\vec{t}_3\langle s'\rangle = s(\vec{p})s(\vec{r}); \\ 1 & \text{otherwise.} \end{cases}$$

Now, let $Y = X'[F_1/u_1][F_2/u_2][F_3/u_3][F_4/u_4]$; by the definitions of $F_1 \dots F_4$, it holds that all dependencies are respected. Then, let $Y$ be split into $Y_1$, $Y_2$, and $Y_3$, according to: the following:

- $Y_1 = \{s \in Y : s(u_1) \neq s(u_2)\}$;
- $Y_2 = \{s \in Y : s(u_3) \neq s(u_4)\} \backslash Y_1$;
- $Y_3 = Y \backslash (Y_1 \cup Y_2)$.

Now, let $s$ be any assignment of $Y_1$; then, since $s(u_1) \neq s(u_2)$, by the definitions of $F_1$ and $F_2$, we have that, for all $s' \in Y$, $s(\vec{p})s(\vec{q}) \neq \vec{t}_1\langle s'\rangle\vec{t}_2\langle s'\rangle$, and, in particular, that the same holds for all the $s' \in Y_1$. Hence, $M \models_{Y_1} u_1 \neq u_2 \wedge (\vec{p}\vec{q} \mid \vec{t}_1\vec{t}_2)$, as required.

Analogously, let $s$ be any assignment of $Y_2$; then $s(u_1) = s(u_2)$, since otherwise $s$ would be in $Y_1$. Moreover, $s(u_3) \neq s(u_4)$, and therefore, for all $s' \in Y$, $s(\vec{p})s(\vec{r}) \neq \vec{t}_1\langle s'\rangle\vec{t}_3\langle s'\rangle$, and thus $M \models_{Y_2} u_1 = u_2 \wedge u_3 \neq u_4 \wedge (\vec{p}\vec{r} \mid \vec{t}_1\vec{t}_3)$.

Finally, suppose that $s \in Y_3$; then, by definition, $s(u_1) = s(u_2)$ and $s(u_3) = s(u_4)$. Therefore, there exist two assignments $s'$ and $s''$ in $X$ such that $\vec{t}_1\langle s'\rangle\vec{t}_2\langle s'\rangle = s(\vec{p})s(\vec{q})$ and $\vec{t}_1\langle s''\rangle\vec{t}_3\langle s''\rangle = s(\vec{p})s(\vec{r})$.

But, by hypothesis, $M \models_X \vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3$, and $s'$ and $s''$ coincide over $\vec{t}_1$; thus, there exists a new assignment $h \in X$ such that $\vec{t}_1\langle h\rangle\vec{t}_2\langle h\rangle\vec{t}_3\langle h\rangle = s(\vec{p})s(\vec{q})s(\vec{r})$. Now, let $o$ be the assignment of $Y$ given by

$$o = h[\vec{t}_1\langle h\rangle\vec{t}_2\langle h\rangle\vec{t}_3\langle h\rangle/\vec{p}\vec{q}\vec{r}][F_1 \dots F_4/u_1 \dots u_4];$$

then, by the definitions of $F_1 \dots F_4$ and by the construction of $o$, we get that $o(u_1) = o(u_2) = o(u_3) = o(u_4) = 0$, and therefore that $o \in Y_3$.

But, by construction, $\vec{t}_1\langle o\rangle\vec{t}_2\langle o\rangle\vec{t}_3\langle o\rangle = \vec{t}_1\langle h\rangle\vec{t}_2\langle h\rangle\vec{t}_3\langle h\rangle = s(\vec{p})s(\vec{q})s(\vec{r})$, and hence $M \models_{Y_3} \vec{p}\vec{q}\vec{r} \subseteq \vec{t}_1\vec{t}_2\vec{t}_3$, as required.

Conversely, suppose that $M \models_X \phi$, and let $s, s' \in X$ be such that $\vec{t}_1\langle s\rangle = \vec{t}_1\langle s'\rangle$. Now, consider the two assignments $h, h' \in X' = X[M/\vec{p}\vec{q}\vec{r}]$ given by $h = s[\vec{t}_1\langle s\rangle/\vec{p}][\vec{t}_2\langle s\rangle/\vec{q}][\vec{t}_3\langle s'\rangle/\vec{r}]$ and $h' = s'[\vec{t}_1\langle s\rangle/\vec{p}][\vec{t}_2\langle s\rangle/\vec{q}][\vec{t}_3\langle s'\rangle/\vec{r}]$.

Now, since $M \models_X \phi$, there exist functions $F_1 \dots F_4$, depending only on $\vec{p}$, $\vec{q}$, and $\vec{r}$, such that $Y = X'[F_1/u_1][F_2/u_2][F_3/u_3][F_4/u_4]$ can be split into three subteams $Y_1$, $Y_2$, and $Y_3$, and

$$M \models_{Y_1} (u_1 \neq u_2 \wedge (\vec{p}\vec{q} \mid \vec{t}_1\vec{t}_2));$$
$$M \models_{Y_2} (u_1 = u_2 \wedge u_3 \neq u_4 \wedge (\vec{p}\vec{r} \mid \vec{t}_1\vec{t}_3));$$
$$M \models_{Y_3} (u_1 = u_2 \wedge u_3 = u_4 \wedge (\vec{p}\vec{q}\vec{r} \subseteq \vec{t}_1\vec{t}_2\vec{t}_3)).$$

Now, let $o = h[F_1/u_1][F_2/u_2][F_3/u_3][F_4/u_4]$, and let $o' = h'[F_1/u_1][F_2/u_2][F_3/u_3][F_4/u_4]$; since the $F_i$ depend only on $\vec{p}\vec{q}\vec{r}$, and the values of these variables are the same for $h$ and for $h'$, we have that $o$ and $o'$ have the same values for $u_1 \dots u_4$, and therefore that they belong to the same $Y_i$. But they cannot be in $Y_1$ or in $Y_2$, since

$$o(\vec{p})o(\vec{q}) = o'(\vec{p})o'(\vec{q}) = \vec{t}_1\langle s\rangle\vec{t}_2\langle s\rangle = \vec{t}_1\langle o\rangle\vec{t}_2\langle o\rangle$$

and

$$o(\vec{p})o(\vec{r}) = o'(\vec{p})o'(\vec{r}) = \vec{t}_1\langle s'\rangle\vec{t}_3\langle s'\rangle = \vec{t}_1\langle o'\rangle\vec{t}_3\langle o'\rangle;$$

therefore, $o$ and $o'$ are in $Y_3$, and there exists an assignment $o'' \in Y_3$ with

$$\vec{t}_1\langle o''\rangle\vec{t}_2\langle o''\rangle\vec{t}_3\langle o''\rangle = o(\vec{p})o(\vec{q})o(\vec{r}) = \vec{t}_1\langle s\rangle\vec{t}_2\langle s\rangle\vec{t}_3\langle s'\rangle;$$

and, finally, there exists an $s'' \in X$ such that $\vec{t}_1\langle s''\rangle\vec{t}_2\langle s''\rangle\vec{t}_3\langle s''\rangle = \vec{t}_1\langle s\rangle\vec{t}_2\langle s\rangle\vec{t}_3\langle s'\rangle$, as required. □

Independence logic and inclusion/exclusion logic are therefore equivalent.

**Corollary 4.21.** *Any independence logic formula is equivalent to some inclusion/exclusion logic formula, and any inclusion/exclusion logic formula is equivalent to some independence logic formula.*

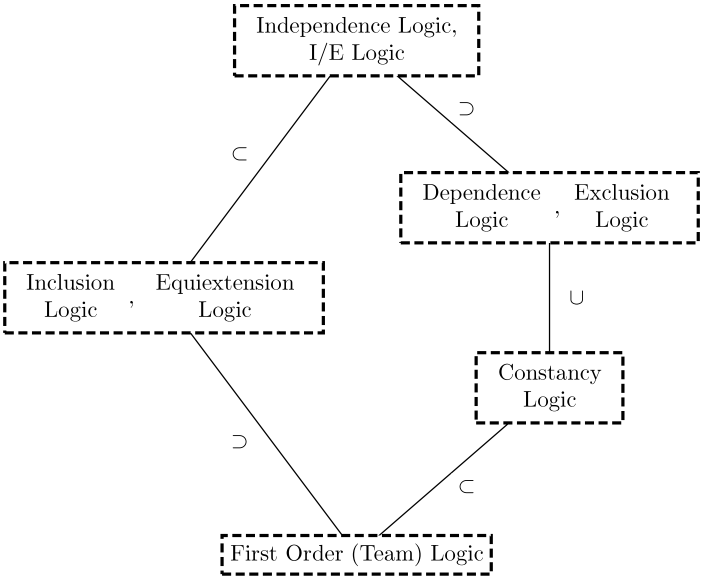


**Fig. 2.** Relations between logics of imperfect information (with respect to formulas).

Fig. 2 summarizes the translatability[8] relations between the logics of imperfect information which have been considered in this work.

Let us finish this section by verifying that I/E logic (and, as a consequence, also inclusion logic, equiextension logic, and independence logic) with lax semantics is local.

**Theorem 4.22** (*Inclusion/Exclusion Logic with Lax Semantics is Local*)**.** *Let $M$ be a first order model, let $\phi$ be any I/E logic formula, and let $V$ be a set of variables such that $Free(\phi) \subseteq V$. Then, for all suitable teams $X$,*

$$M \models_X \phi \Leftrightarrow M \models_{X_{\upharpoonright V}} \phi.$$

**Proof.** The proof is by structural induction on $\phi$.

1. If $\phi$ is a first order literal, an inclusion atom, or an exclusion atom then the statement follows trivially from the corresponding semantic rule.
2. Let $\phi$ be of the form $\psi \vee \theta$, and suppose that $M \models_X \psi \vee \theta$. Then, by definition, $X = Y \cup Z$ for two subteams $Y$ and $Z$ such that $M \models_Y \psi$ and $M \models_Z \theta$. Then, by the induction hypothesis, $M \models_{Y_{\upharpoonright V}} \psi$ and $M \models_{Z_{\upharpoonright V}} \theta$. But $X_{\upharpoonright V} = Y_{\upharpoonright V} \cup Z_{\upharpoonright V}$. Hence, $M \models_{X_{\upharpoonright V}} \psi \vee \theta$, as required.

   Conversely, suppose that $M \models_{X_{\upharpoonright V}} \psi \vee \theta$, that is, that $X_{\upharpoonright V} = Y' \cup Z'$ for two subteams $Y'$ and $Z'$ such that $M \models_{Y'} \psi$ and $M \models_{Z'} \theta$. Then, define $Y = \{s \in X : s_{\upharpoonright V} \in Y'\}$ and $Z = \{s \in X : s_{\upharpoonright V} \in Z'\}$. Now, $X = Y \cup Z$ and, furthermore, $Y_{\upharpoonright V} = Y'$ and $Z_{\upharpoonright V} = Z'$, and hence, by the induction hypothesis, $M \models_Y \psi$ and $M \models_Z \theta$, and finally $M \models_X \psi \vee \theta$.
3. Let $\phi$ be of the form $\psi \wedge \theta$. Then $M \models_X \psi \wedge \theta$ if and only if $M \models_X \psi$ and $M \models_X \theta$, that is, by the induction hypothesis, if and only if $M \models_{X_{\upharpoonright V}} \psi$ and $M \models_{X_{\upharpoonright V}} \theta$. But this is the case if and only if $M \models_{X_{\upharpoonright V}} \psi \wedge \theta$, as required.
4. Let $\phi$ be of the form $\exists x \psi$, and suppose that $M \models_X \exists x \psi$. Then there exists a function $H : X \to \mathcal{P}(\text{Dom}(M)) \backslash \{\emptyset\}$ such that $M \models_{X[H/x]} \psi$. Then, by the induction hypothesis, $M \models_{(X[H/x])_{\upharpoonright V \cup \{x\}}} \psi$.

   Now, consider the function $H' : X_{\upharpoonright V} \to \mathcal{P}(\text{Dom}(M)) \backslash \emptyset$, which assigns to every $s' \in X_{\upharpoonright V}$ the set

   $$H'(s') = \bigcup \{H(s) : s \in X, s' = s_{\upharpoonright V}\}.$$

   Then $H'$ assigns a nonempty set to every $s' \in X_{\upharpoonright V}$, as required; and furthermore, $X_{\upharpoonright V}[H'/x]$ is precisely $(X[H/x])_{\upharpoonright V \cup \{x\}}$. Therefore, $M \models_{X_{\upharpoonright V}} \exists x \psi$, as required.

   Conversely, suppose that $M \models_{X_{\upharpoonright V}} \exists x \psi$, that is, that $M \models_{X_{\upharpoonright V}[H'/x]} \psi$ for some $H'$. Then, define the function $H : X \to \mathcal{P}(\text{Dom}(M)) \backslash \{x\}$ so that $H(s) = H'(s_{\upharpoonright V})$ for all $s \in X$; now, $X_{\upharpoonright V}[H'/x] = (X[H/x])_{\upharpoonright V \cup \{x\}}$, and hence, by the induction hypothesis, $M \models_X \exists x \psi$.
5. For all suitable teams $X$, $X[M/x]_{\upharpoonright V \cup \{x\}} = X_{\upharpoonright V}[M/x]$; and hence, $M \models_{X_{\upharpoonright V}} \forall x \psi \Leftrightarrow M \models_{X[M/x]_{\upharpoonright V \cup \{x\}}} \psi \Leftrightarrow M \models_{X[M/x]} \psi \Leftrightarrow M \models_X \forall x \psi$, as required. □

[8] To be more accurate, Fig. 2 represents the translatability relations between the logics which we considered, *with respect to all formulas*. Considering sentences only would lead to a different graph.

## 5. Definability in I/E logic (and in independence logic)

In [16], Kontinen and Väänänen characterized the expressive power of dependence logic formulas (Theorem 2.5 here), and, in [15], Kontinen and Nurmi used a similar technique to prove that a class of teams is definable in team logic [21] if and only if it is expressible in full second order logic.

In this section, I will attempt to find an analogous result for I/E logic (and hence, through Corollary 4.21, for independence logic). One direction of the intended result is straightforward.

**Theorem 5.1.** *Let $\phi(\vec{v})$ be a formula of I/E logic with free variables in $\vec{v}$. Then there exists an existential second order logic formula $\Phi(A)$, where $A$ is a second order variable with arity $|\vec{v}|$, such that*

$$M \models_X \phi(\vec{v}) \Leftrightarrow M \models \Phi(Rel_{\vec{v}}(X))$$

*for all suitable models $M$ and teams $X$.*

**Proof.** The proof is an unproblematic induction over the formula $\phi$, and follows closely the proof of the analogous results for dependence logic [20] or independence logic [9]. □

The other direction, instead, requires some care.[9]

**Theorem 5.2.** *Let $\Phi(A)$ be a formula in $\Sigma_1^1$ such that $Free(\Phi) = \{A\}$, and let $\vec{v}$ be a tuple of distinct variables with $|\vec{v}| = Arity(A)$. Then there exists an I/E logic formula $\phi(\vec{v})$ such that*

$$M \models_X \phi(\vec{v}) \Leftrightarrow M \models \Phi(Rel_{\vec{v}}(X))$$

*for all suitable models $M$ and nonempty teams $X$.*

**Proof.** It is easy to see that any $\Phi(A)$ as in our hypothesis is equivalent to the formula $\Phi^*(A) = \exists B(\forall \vec{x}(A\vec{x} \leftrightarrow B\vec{x}) \wedge \Phi(B))$, in which the variable $A$ occurs only in the conjunct $\forall \vec{x}(A\vec{x} \leftrightarrow B\vec{x})$. Then, as in [16], it is possible to write $\Phi^*(A)$ in the form $\exists \vec{f}\, \forall \vec{x}\vec{y}((A\vec{x} \leftrightarrow f_1(\vec{x}) = f_2(\vec{x})) \wedge \psi(\vec{x}, \vec{y}, \vec{f}))$, where $\vec{f} = f_1 f_2 \ldots f_n$, $\psi(\vec{f}, x, y)$ is a quantifier-free formula in which $A$ does not appear, and each $f_i$ occurs only as $f(\vec{w}_i)$ for some fixed tuple of variables $\vec{w}_i \subseteq \vec{x}\vec{y}$. Now, define the formula $\phi(\vec{v})$ as

$$\forall \vec{x}\vec{y}\, \exists \vec{z} \left( \bigwedge_i =\!(\vec{w}_i, z_i) \wedge (((\vec{v} \subseteq \vec{x} \wedge z_1 = z_2) \vee (\vec{v} \mid \vec{x} \wedge z_1 \neq z_2)) \wedge \psi'(\vec{x}, \vec{y}, \vec{z})) \right),$$

where $\psi'(\vec{x}, \vec{y}, \vec{z})$ is obtained from $\psi(\vec{x}, \vec{y}, \vec{f})$ by substituting each $f_i(\vec{w}_i)$ with $z_i$, and the dependence atoms are used as shorthand for the corresponding expressions of I/E logic.

Now, we have that $M \models_X \phi(\vec{v}) \Leftrightarrow M \models \Phi^*(\mathrm{Rel}_{\vec{v}}(X))$. Indeed, suppose that $M \models_X \phi(\vec{v})$. Then, by construction, for each $i = 1 \ldots n$ there exists a function $F_i$, depending only on $\vec{w}_i$, such that, for $Y = X[M/\vec{x}\vec{y}][\vec{F}/\vec{z}]$,

$$M \models_Y ((\vec{v} \subseteq \vec{x} \wedge z_1 = z_2) \vee (\vec{v} \mid \vec{x} \wedge z_1 \neq z_2)) \wedge \psi'(\vec{x}, \vec{y}, \vec{z}).$$

Therefore, we can split $Y$ into two subteams $Y_1$ and $Y_2$ such that $M \models_{Y_1} \vec{v} \subseteq \vec{x} \wedge z_1 = z_2$ and $M \models_{Y_2} \vec{v} \mid \vec{x} \wedge z_1 \neq z_2$.

Now, for each $i$, define the function $f_i$ so that, for every tuple $\vec{m}$ of the required arity, $f_i(\vec{m})$ corresponds to $F_i(s)$ for an arbitrary $s \in X[M/\vec{x}\vec{y}]$ with $s(\vec{w}_i) = \vec{m}$, and let $o$ be any assignment with domain $\vec{x}\vec{y}$. Thus, if we can prove that $M \models_o ((\mathrm{Rel}_{\vec{v}}(X))\vec{x} \leftrightarrow f_1(\vec{x}) = f_2(\vec{x})) \wedge \psi(\vec{x}, \vec{y}, \vec{f})$, then the left-to-right direction of our proof is done.

First of all, suppose that $M \models_o (\mathrm{Rel}_{\vec{v}}(X))\vec{x}$, that is, that $o(\vec{x}) = \vec{m} = s(\vec{v})$ for some $s \in X$. Then, choose an arbitrary tuple of elements $\vec{r}$, and consider the assignment $h = s[\vec{m}/\vec{x}][\vec{r}/\vec{y}][\vec{F}/\vec{z}] \in Y$. This $h$ cannot belong to $Y_2$, since $h(\vec{v}) = s(\vec{v}) = \vec{m} = h(\vec{x})$, and therefore it is in $Y_1$, and $h(z_1) = h(z_2)$. By the definition of the $f_i$, this implies that $f_1(\vec{m}) = f_2(\vec{m})$, as required.

Analogously, suppose that $M \not\models_o (\mathrm{Rel}_{\vec{v}}(X))\vec{x}$, that is, that $o(\vec{x}) = \vec{m} \neq s(\vec{v})$ for all $s \in X$. Then, pick an arbitrary such $s \in X$ and an arbitrary tuple of elements $\vec{r}$, and consider the assignment $h = s[\vec{m}/\vec{x}][\vec{r}/\vec{y}][\vec{F}/\vec{z}] \in Y$.

If $h$ were in $Y_1$, there would exist an assignment $h' \in Y_1$ such that $h'(\vec{v}) = h(\vec{x}) = \vec{m}$; but this is impossible, and therefore $h \in Y_2$. Hence $h(z_1) \neq h(z_2)$, and therefore $f_1(\vec{m}) \neq f_2(\vec{m})$.

Putting everything together, we have just proved that $M \models_o R\vec{x} \Leftrightarrow f_1(\vec{x}) = f_2(\vec{x})$ for all assignments $o$ with domain $\vec{x}\vec{y}$, and we still need to verify that $M \models_o \psi(\vec{x}, \vec{y}, f)$ for all such $o$.

But this is immediate: indeed, let $s$ be an arbitrary assignment of $X$, and construct the assignment $h = s[o(\vec{x}\vec{y})/\vec{x}\vec{y}][\vec{F}/\vec{z}] \in X[M/\vec{x}\vec{y}][\vec{F}/\vec{z}]$.

Then, since $M \models_{X[M/\vec{x}\vec{y}][\vec{F}/\vec{z}]} \psi'(\vec{x}, \vec{y}, \vec{z})$ and $\psi'(\vec{x}, \vec{y}, \vec{z})$ is first order, $M \models_{\{h\}} \psi'(\vec{x}, \vec{y}, \vec{z})$; but $\psi'(\vec{x}, \vec{y}, \vec{f}(\vec{x}\vec{y}))$ is equivalent to $\psi(\vec{x}, \vec{y}, \vec{f})$ and $h(z_i) = f(h(\vec{w}_i)) = f(o(\vec{w}_i))$, and therefore $M \models_o \psi(\vec{x}, \vec{y}, \vec{f})$, as required.

9 The details of this proof are similar to those of [16,15].

Conversely, suppose that $M \models_s (\mathrm{Rel}_{\vec{v}}(X))\vec{x} \leftrightarrow (f_1(\vec{x}) = f_2(\vec{x})) \wedge \psi(\vec{x}, \vec{y}, \vec{f})$ for all assignments $s$ with domain $\vec{x}\vec{y}$ and for some fixed choice of the tuple of functions $\vec{f}$. Then, let $\vec{F}$ be such that, for all assignments $h$, and for all $i$, $F_i(h) = f_i(h(\vec{w}_i))$, and consider $Y = X[M/\vec{x}\vec{y}][F/\vec{z}]$.

Clearly, $Y$ satisfies the dependency conditions; furthermore, it satisfies $\psi'(\vec{x}, \vec{y}, \vec{z})$, because, for every assignment $h \in Y$ and every $i \in 1 \dots n$, we have that $h(z_i) = F_i(h) = f_i(h(\vec{w}_i))$. Finally, we can split $Y$ into two subteams $Y_1$ and $Y_2$ as follows:

$$Y_1 = \{o \in Y : o(\vec{z}_1) = o(\vec{z}_2)\};$$
$$Y_2 = \{o \in Y : o(\vec{z}_1) \neq o(\vec{z}_2)\}.$$

It is then trivially true that $M \models_{Y_1} z_1 = z_2$ and $M \models_{Y_2} z_1 \neq z_2$, and all that is left to do is to prove that $M \models_{Y_1} \vec{v} \subseteq \vec{x}$ and $M \models_{Y_2} \vec{v} \mid \vec{x}$.

As for the former, let $o \in Y_1$; then, since $o(z_1) = o(z_2)$, $f_1(o(\vec{x})) = f_2(o(\vec{x}))$. This implies that $o(\vec{x}) \in \mathrm{Rel}_{\vec{v}}(X)$, and hence that there exists an assignment $s' \in X$ with $s'(\vec{v}) = o(\vec{x})$.

Now, consider the assignment $o' = s'[o(\vec{x}\vec{y})/\vec{x}\vec{y}][\vec{F}/\vec{z}]$; since in $Y$ the values of $\vec{z}$ depend only on the values of $\vec{x}\vec{y}$, and, since $o(z_1) = o(z_2)$, we have that $o'(z_1) = o'(z_2)$, and hence $o' \in Y_1$ too. But $o'(\vec{v}) = s'(\vec{v}) = o(\vec{x})$, and, since $o$ was an arbitrary assignment of $Y_1$, this implies that $M \models_{Y_1} \vec{v} \subseteq \vec{x}$.

Finally, suppose that $o \in Y_2$. Then, since $o(z_1) \neq o(z_2)$, we have that $f_1(o(\vec{x})) \neq f_2(o(\vec{x}))$, and therefore, $o(\vec{x}) \notin \mathrm{Rel}_{\vec{v}}(X)$; that is, for all assignments $s \in X$ it holds that $s(\vec{v}) \neq o(\vec{x})$. Then, the same holds for all $o' \in Y_2$.

This concludes the proof. □

Since, by Corollary 4.21, we already know that independence logic and I/E logic have the same expressive power, this has the following corollary.

**Corollary 5.3.** *Let $\Phi(A)$ be an existential second order formula with Free$(\Phi) = A$, and let $\vec{v}$ be any set of variables such that $|\vec{v}| = Arity(A)$. Then there exists an independence logic formula $\phi(\vec{v})$ such that*

$$M \models_X \phi(\vec{v}) \Leftrightarrow M \models \Phi(Rel_{\vec{v}}(X))$$

*for all suitable models $M$ and teams $X$.*

Finally, by Fagin's Theorem [7], this gives an answer to Grädel and Väänänen's question.

**Corollary 5.4.** *All NP properties of teams are expressible in independence logic.*

This result has far-reaching consequences. First of all, it implies that independence logic (or, equivalently, I/E logic) is the most expressive logic of imperfect information which only deals with existential second order properties. Extensions of independence logic can of course be defined; but unless they are capable of expressing some property which is not existential second order (as, for example, is the case for the intuitionistic dependence logic of [22], or for the logics discussed in [1]), they will be expressively equivalent to independence logic proper. As Jouko Väänänen pointed out, in a private communication, this means that independence logic is *maximal* among the logics of imperfect information which always generate existential second order properties of teams. In particular, *any* dependency condition which is expressible as an existential second order property[10] over teams can be expressed in independence logic.

As was to be expected, this vast expressive power comes at a very high computational cost; but nonetheless, it is the hope of the author that these results may provide some justification for the study of variants of dependence logic of the sort that was discussed in this work.

## Acknowledgements

The author wishes to thank Jouko Väänänen for many valuable insights and suggestions. Furthermore, he thanks Erich Grädel for having suggested a better notation for inclusion and exclusion dependencies. Moreover, the author thankfully acknowledges the support of the EUROCORES LogICCC LINT programme. Finally, we thank an anonymous referee for a number of useful suggestions.

## References

[1] Samson Abramsky, Jouko Väänänen, From IF to BI, a Tale of Dependence and Separation, ILLC Publications, 2008, PP–2008–27.
[2] Marco A. Casanova, Ronald Fagin, Christos H. Papadimitriou, Inclusion dependencies and their interaction with functional dependencies, in: Proceedings of the 1st ACM SIGACT-SIGMOD Symposium on Principles of Database Systems, PODS'82, ACM, New York, NY, USA, 1982, pp. 171–176.
[3] Marco A. Casanova, Vânia M.P. Vidal, Towards a sound view integration methodology, in: Proceedings of the 2nd ACM SIGACT-SIGMOD Symposium on Principles of Database Systems, PODS'83, ACM, New York, NY, USA, 1983, pp. 36–47.
[4] Edgar F. Codd, Further normalization of the data base relational model, in: R. Rustin (Ed.), Data Base Systems, Prentice-Hall, 1972, pp. 33–64.
[5] Arnaud Durand, Juha Kontinen, Hierarchies in dependence logic, CoRR (2011) abs/1105.3324.
[6] Fredrik Engström, Generalized Quantifiers in Dependence Logic. Draft, 2010.
[7] Ronald Fagin, 1974. Generalized first-order spectra and polynomial-time recognizable sets. In: Complexity of Computation, SIAM-AMS Proceedings, vol. 7, pp. 43–73.
[8] Ronald Fagin, A normal form for relational databases that is based on domains and keys, ACM Transactions on Database Systems 6 (September 1981) 387–415.

10 Such as, for example, *tuple generating* and *equality generating* dependencies, two of the most general classes of dependencies studied in database theory.

[9] Erich Grädel, Jouko Väänänen, 2010. Dependence and Independence. Studia Logica (in press).
[10] Jaakko Hintikka, The Principles of Mathematics Revisited, Cambridge University Press, 1996.
[11] Wilfrid Hodges, A Shorter Model Theory, Cambridge University Press, 1997.
[12] W. Hodges, Compositional semantics for a language of imperfect information, Logic Journal of IGPL 5 (4) (1997) 539–563. doi:10.1093/jigpal/5.4.539.
[13] Wilfrid Hodges, Logics of imperfect information: why sets of assignments? in: D. van Benthem, J. Gabbay, B. Löwe (Eds.), Interactive Logic, in: Texts in Logic and Games, Amsterdam University Press, 2007, pp. 117–133.
[14] Theo M.V. Janssen, Francien Dechesne, Signaling in if-games: a tricky business, in: J. van Benthem, G. Heinzmann, M. Rebuschi, H. Visser (Eds.), The Age of Alternative Logics, Springer, 2006, pp. 221–241.
[15] Juha Kontinen, Ville Nurmi, Team logic and second-order logic, in: Hiroakira Ono, Makoto Kanazawa, Ruy de Queiroz (Eds.), Logic, Language, Information and Computation, in: Lecture Notes in Computer Science, vol. 5514, Springer, Berlin, Heidelberg, 2009, pp. 230–241.
[16] Juha Kontinen, Jouko Väänänen, On definability in dependence logic, Journal of Logic, Language and Information 3 (18) (2009) 317–332.
[17] Per Lindström, First order predicate logic with generalized quantifiers, Theoria 32 (3) (1966) 186–195.
[18] Andrzej Mostowski, On a generalization of quantifiers, Fundamenta Mathematicae 44 (1957) 12–36.
[19] Tero Tulenheimo, Independence friendly logic, Stanford Encyclopedia of Philosophy (2009).
[20] Jouko Väänänen, Dependence Logic, Cambridge University Press, 2007.
[21] Jouko Väänänen, Team Logic, in: J. van Benthem, D. Gabbay, B. Löwe (Eds.), Interactive Logic. Selected Papers from the 7th Augustus de Morgan Workshop, Amsterdam University Press, 2007, pp. 281–302.
[22] Fan Yang, 2010. Expressing second-order sentences in intuitionistic dependence logic. In: Juha Kontinen and Jouko Väänänen, editors, Proceedings of Dependence and Independence in Logic, pp. 118–132. ESSLLI 2010.