Operator Valued Weights in von Neumann Algebras, I

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Communicated by A. Connes

Received November 1, 1977

An operator valued weight is a kind of generalized conditional expectation from a von Neumann algebra $M$ to a sub von Neumann algebra $N$. If $T$ is a n.f.s. (normal, faithful, semifinite) operator valued weight from $M$ to $N$, and $\phi$ is a n.f.s. weight on $N$, then $\phi \circ T$ defines a n.f.s. weight on $M$. Our main result is that $\varphi_{\text{ref}}$ extends $\sigma_{\phi}$, and that the map $\phi \rightarrow \phi \circ T$ preserves cocycle Radon Nikodym derivatives.

INTRODUCTION

The work on this paper started as an attempt to simplify or at least clarify the dual weight construction for crossed products due to Takesaki, Digerne and Sauvageot (cf. [18, 6, 16]). If $M = R(N, \alpha)$ is the crossed product of a von Neumann algebra $N$ with an action $\alpha$ of a discrete group $G$, then there is a natural conditional expectation $\varepsilon: M \rightarrow N$ (where $N$ is regarded as a subalgebra of $M$). It is not hard to prove, that in this case, the dual weight $\varphi$ of a given n.f.s. weight $\varphi$ on $N$ is given by $\bar{\varphi} = \varphi \circ \varepsilon$. Our goal was to prove, that when $M = R(N, \alpha)$ is the crossed product of $N$ with an arbitrary locally compact group, then there exists a kind of generalized conditional expectation $T$ from $M$ to $N$, such that $\phi = \phi \circ T$ for any n.f.s. weight on $N$. We found that “operator valued weights” as defined below (Definition 2.1) would suit the purpose. In this and a subsequent paper [10] we go through the general theory of operator valued weights, and in a third paper [11] we will give the applications to the dual weight construction. The main results of these three papers were announced in [9].

DEFINITIONS AND NOTATION

Let us recall the main properties of (ordinary) weights (cf. [2, 3, 6, 8]). A weight $\phi$ on a von Neumann algebra $M$ is a function $\phi$ from $M_+$ to $[0, \infty]$, that satisfies
\[ \phi(\lambda x) = \lambda \phi(x) \quad x \in M_+, \quad \lambda \geq 0 \] (homogeneity)

\[ \phi(x + y) = \phi(x) + \phi(y) \quad x, y \in M_+ \] (additivity).

Put \( n_\phi = \{ x \in M \mid \phi(x^* x) < \infty \} \) and \( m_\phi = n_\phi^* n_\phi \).

We say that \( \phi \) is normal if \( \phi(\sup x_i) = \sup \phi(x_i) \) for any uniformly bounded increasing net of positive operators \( (x_i)_{i \in I} \). \( \phi \) is faithful if \( \phi(x^* x) = 0 \Rightarrow x = 0 \) and semi-finite if \( n_\phi \) is \( \sigma \)-strongly dense in \( M \). The set of n.f.s. weights on \( M \) is denoted \( P(M) \). To any \( \phi \in P(M) \) is associated a \( \sigma \)-weakly continuous one parameter group \( (\sigma_t^\phi)_{t \in \mathbb{R}} \) of *-automorphisms on \( M \), called the modular automorphism group. \( \sigma_t^\phi \) is characterized by the K.M.S. conditions:

1. \( \phi \) is \( \sigma_t^\phi \)-invariant.
2. For any pair \( x, y \in n_\phi \cap n_\phi^* \) there exists a bounded, continuous function \( f \) on the strip \( 0 \leq \text{Im} \, z \leq 1 \), analytic in the interior, such that

\[ f(t) = \phi(\sigma_t^\phi(x) y) \quad \text{and} \quad f(t + i\eta) = \phi(y \sigma_t^\phi(x)), \quad t \in \mathbb{R}. \]

Let \( F_2 \) be the algebra of \( 2 \times 2 \)-matrices with natural basis \( (e_{ij})_{i,j=1,2} \). If \( \phi, \psi \in P(M) \) the formula

\[ \theta(\Sigma x_{ij} \otimes e_{ij}) = \phi(x_{11}) + \psi(x_{22}), \quad \Sigma x_{ij} \otimes e_{ij}(M \otimes F_2)_+ \]

defines a n.f.s. weight on \( M \otimes F_2 \).

There exists a \( \sigma \)-weakly continuous one parameter group \( (\sigma_t^\phi)_{t \in \mathbb{R}} \) of isometries on \( M \), such that

\[ \sigma_t^\phi(x \otimes e_{21}) = \sigma_t^\phi(x) \otimes e_{21}, \quad x \in M. \]

Put \( (D\psi: D\phi)_t = \sigma_t^\psi \sigma_t^\phi(1). \) For any \( t \in \mathbb{R} \), \( (D\psi: D\phi)_t \) is a unitary operator, and the function \( t \mapsto (D\psi: D\phi)_t \) is called the cocycle Radon Nikodym derivative for \( \psi \) with respect to \( \phi \). If \( \phi_0 \) is a fixed n.f.s. weight on \( M \), the map \( \phi \mapsto (D\phi: D\phi_0) \) is a bijection of \( P(M) \) onto the set of \( \sigma_0^\phi \)-cocycles on \( M \), i.e. the set of strongly continuous functions \( t \mapsto u_t \) of \( \mathbb{R} \) into the unitary group of \( M \), that satisfy

\[ u_{s+t} = u_s \sigma_s^\phi(u_t), \quad s, t \in \mathbb{R}. \]

In the following we list the main definitions and theorems of this paper. Recall that a conditional expectation is a linear map from a von Neumann algebra \( M \) to a subalgebra \( N \), that satisfies:

1. \( x \geq 0 \Rightarrow \epsilon(x) \geq 0 \).
2. \( \epsilon(axb) = a \epsilon(x) b \) \quad \( x \in M, \quad a, b \in N \).
3. \( \epsilon(1) = 1 \).
Recently “unbounded conditional expectations” have appeared in literature in different settings (cf. [14] and [5, Chap. II]). Instead of shrinking the domain as in [14] and [5], we let the operator valued weights be defined on the whole positive part of $M$, and allow the range to contain certain unbounded “objects” affiliated with $N$.

**DEFINITION 1.1.** The extended positive part $\hat{N}_+$ of the von Neumann algebra $N$, is the set of maps $m: N_+^* \rightarrow [0, \infty]$ that satisfy:

1. $m(\lambda \phi) = \lambda m(\phi), \quad \phi \in N_+^*, \quad \lambda \geq 0$
2. $m(\phi + \psi) = m(\phi) + m(\psi) \quad \phi, \psi \in N_+^*$.
3. $m$ is lower semicontinuous.

Clearly $N_+$ can be regarded as a subset of $\hat{N}_+$. Moreover $\hat{N}_+$ is closed under addition and increasing limits. For each $a \in N$, the map $x \rightarrow a^*xa$, $x \in N_+$, has a natural extension to $\hat{N}_+$ (cf. Definition 1.3). If $N$ is abelian, then $N \cong L^\infty(\Omega, \mu)$ for some measure space $(\Omega, \mu)$. In this case $\hat{N}_+$ corresponds to the set of equivalence classes of $\mu$-measurable functions on $\Omega$ with values in $[0, \infty]$.

**DEFINITION 2.1.** An operator valued weight $T$ from $M$ to $N$ is a map $T$ of $M_+$ into the extended positive part $N_+^*$ of $N$, that satisfies

1. $T(\lambda x) = \lambda T(x), \quad x \in M_+, \quad \lambda \geq 0$
2. $T(x + y) = T(x) + T(y) \quad x, y \in M_+$
3. $T(a^*xa) = a^*T(x)a \quad x \in M_+, \quad a \in N$.

If $T(1) = 1$, then $T$ is the restriction of a conditional expectation to $M_+$. Normality, faithfulness and semifiniteness can be defined as for ordinary weights. The set of n.f.s. operator valued weights from $M$ to $N$ is denoted $P(M, N)$. Any normal weight on $N$ has a unique normal extension to $\hat{N}_+$ (Proposition 1.10). Hence if $\phi$ is a normal weight on $N$ and $T$ is a normal operator valued weight from $M$ to $N$, then $\phi \circ T$ is a well defined normal weight on $M$.

Moreover if $\phi \in P(N)$ and $T \in P(M, N)$ then $\phi \circ T \in P(M)$ (cf. Proposition 2.3).

**THEOREM 2.7.** If $M$ and $N$ are semifinite von Neumann algebras, and $\tau_1$ and $\tau_2$ n.f.s. traces on $M$ and $N$ respectively, then there is a unique $T \in P(M, N)$ such that $\tau_1 \circ T = \tau_2$.

The proof is based on a slight generalization of the Radon Nikodym Theorem for weights in [15]. In Section 3 we give an algebraic characterization of the analytic generator $\sigma_{\phi_t}^\phi$ for the one parameter group $\sigma_t^\phi$ (cf. [1]). We let $G(\sigma_{\phi_t}^\phi)$ denote the graph of the operator.
Theorem 3.2. Let $N$ be a von Neumann algebra, and let $\phi, \psi$ be two n.s.f. weights on $N$. For $a, b \in N$ the following conditions are equivalent:

1. $(a, b) \in G(\sigma_{\phi, \psi}^{\phi})$
2. $a \cdot n_{\phi}^{*} \subseteq n_{\phi}^{*} \cdot n_{\phi} \subseteq n_{\phi}$ and $\psi(ax) = \phi(xb)$ for any $x \in n_{\phi}^{*}n_{\phi}$.

Theorem 3.2 is used to prove:

Theorem 4.7. Let $T \in P(M, N)$. Then

$$\sigma_{T}^{\phi}(x) = \sigma_{T}^{\psi}(x), \quad x \in N, \quad \phi \in P(N).$$

This result is well known for conditional expectations, cf. [17, Remark p. 309] and [3, Lemma 1.4.4]. However the methods used there cannot be generalized to our situation.

1. The Extended Positive Part of a von Neumann Algebra

Definition 1.1. Let $M$ be a von Neumann algebra, and $M_*$ its predual. A generalized positive operator affiliated with $M$ is a map $m: M_+ \to [0, \infty]$ satisfying

1. $m(\lambda \phi) = \lambda m(\phi), \quad \phi \in M_+, \quad \lambda \geq 0.$
2. $m(\phi + \psi) = m(\phi) + m(\psi), \quad \phi, \psi \in M_+.$
3. $m$ is lower semicontinuous.

The set of all such maps is called the extended positive part of $M$, and is denoted $M_+$. Note that each $x \in M_*$ defines an element in $M_+$ by $\phi \rightarrow \phi(x)$, $\phi \in M_+^*$. Hence we can regard $M_+$ as a subset of $M_+^*$.

Example 1.2. let $M$ be a von Neumann algebra on a Hilbert space $H$ and let $A$ be a positive selfadjoint (not necessarily bounded) operator affiliated with $M$. Let

$$A = \int_{0}^{\infty} \lambda \, d\lambda$$

be the spectral resolution of $A$. Put

$$m_A(\phi) = \int_{0}^{\infty} \lambda \, d(\phi(e_{\lambda})), \quad \phi \in M_+^*$$

Then $m_A$ satisfies (1) and (2) in Definition 1.1. To show (3) put $x_n = \int_{0}^{\infty} \lambda \, d\lambda$. Then $m_A(\phi) = \sup_{n \in N} \phi(x_n)$. Hence $m_A$ is lower semicontinuous on $M_+^*$. 

From well known properties of spectral resolutions, it follows that for \( \xi \in H \):

\[
m_A(\omega \xi) = \int_0^\infty \lambda \, d(\omega \xi | \xi) = \begin{cases} \| A^{1/2} \xi \|^2 & \xi \in D(A^{1/2}) \\ \infty & \xi \notin D(A^{1/2}) \end{cases}
\]

Assume that \( A, B \) are two positive selfadjoint operators affiliated with \( M \), such that \( m_A = m_B \). Then \( D(A^{1/2}) = D(B^{1/2}) \) and \( \| A^{1/2} \xi \| = \| B^{1/2} \xi \| \) \( \forall \xi \in D(A^{1/2}) \). Hence from uniqueness of polar decomposition \( A^{1/2} = B^{1/2} \). Thus \( A = B \), which proves that the map \( A \rightarrow m_A \) is injective. Therefore, the set of positive selfadjoint operators affiliated with \( M \), can be identified with a subset of \( \hat{M}_+ \).

**Definition 1.3.** Let \( m, n \in \hat{M}_+ \), \( a \in M \) and \( \lambda \geq 0 \). We define \( \lambda m, m + n \), and \( a^* ma \) by:

\[
\begin{align*}
(\lambda m)(\phi) &= \lambda m(\phi) & \phi & \in M_+^+ \\
(m + n)(\phi) &= m(\phi) + n(\phi) & \phi & \in M_+^+ \\
(a^* ma)(\phi) &= m(a^* a) & \phi & \in M_+^+
\end{align*}
\]

where \( a^* a = \phi(a^* \cdot a) \). It is easy to see that \( \lambda m, m + n \) and \( a^* ma \) belong to \( \hat{M}_+ \). If \( m \) and \( n \) are bounded, these definitions coincide with the usual operations in \( M_+^+ \).

Note that if \( (m_i)_{i \in I} \) is an increasing net of elements in \( \hat{M}_+ \), then

\[
m(\phi) = \sup_{i \in I} m_i(\phi), \quad \phi \in M_+^+
\]

defines an element in \( \hat{M}_+ \). In particular, if \( (m_j)_{j \in J} \) is a family of elements in \( \hat{M}_+ \), then \( m = \sum_{j \in J} m_j \) is again in \( \hat{M}_+ \).

**Lemma 1.4.** Let \( M \) be a von Neumann algebra on a Hilbert space \( H \), and let \( m \in \hat{M}_+ \). There exist a closed subspace \( K \subseteq H \) and a positive selfadjoint (not necessarily bounded) operator \( A \) on \( K \) such that

\[
m(\omega \xi) = \begin{cases} \| A^{1/2} \xi \|^2 & \xi \in D(A^{1/2}) \\ \infty & \text{otherwise} \end{cases}
\]

The pair \((K, A)\) is unique; \( K \) and \( A \) are affiliated with \( M \).

**Proof.** Let \( m \in \hat{M}_+ \). Put \( s(\xi) = m(\omega \xi) \). The map \( s : H \rightarrow [0, \infty] \) has the properties:

1. \( s(\lambda \xi) = |\lambda|^2 s(\xi) \).
2. \( s(\xi + \eta) + s(\xi - \eta) = 2s(\xi) + 2s(\eta) \).
3. \( s \) is lower semicontinuous.
4. \( s(u\xi) = s(\xi) \) for any unitary operator in \( M' \).
Put $K_0 = \{ \xi \mid s(\xi) < \infty \}$ and $K = \overline{K}_0$ (closure in $H$). $K_0$ and $K$ are subspaces because $s(\xi + \eta) \leq 2s(\xi) + 2s(\eta)$. Since the map $s$ satisfies (1) and (2), there exists a positive sesquilinear form $(\xi | \eta)_s$ on $K_0$, such that

$$(\xi | \xi)_s = s(\xi) \quad \forall \xi \in K_0$$

Put $K_{00} = \{ \xi \mid s(\xi) = 0 \}$ and let $L$ be the Hilbert space obtained by completion of $K_0/K_{00}$ with respect to $(\xi | \xi)_s$.

Let $T$ be the quotient map

$$T : K_0 \rightarrow K_0/K_{00} \subseteq L.$$ 

Then

$$m(\omega_T) = s(\xi) = \begin{cases} \| T\xi \|^2 & \xi \in D(T) \\ \infty & \text{otherwise.} \end{cases}$$

We will prove that $T$ is closed, regarded as an operator from $K$ to $L$.

Let $(\xi_n)$ be a sequence in $K_0$ such that

$$\xi_n \rightarrow \xi \in K$$

$$T\xi_n \rightarrow \eta \in L.$$ 

Since $\sup_{n \in \mathbb{N}} s(\xi_n) = \sup_{n \in \mathbb{N}} \| T\xi_n \|^2 < \infty$ and $s$ is lower semi-continuous, we get $s(\xi) < \infty$. Hence $\xi \in D(T)$. For a given $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\| T\xi_n - T\xi_m \| \leq \epsilon \quad \text{for } n, m \geq n_0 \quad \text{or equivalently}$$

$$s(\xi_n - \xi_m) \leq \epsilon^2 \quad \text{for } n, m \geq n_0.$$ 

In the limit $m \rightarrow \infty$ we get $s(\xi_n - \xi) \leq \epsilon^2$, because $s$ is lower semicontinuous. Therefore $\| T\xi_n - T\xi \| \leq \epsilon$ for $n \geq n_0$. This proves that $T\xi_n \rightarrow T\xi$. Hence $T$ is closed. Put $A = T^*T$. $A$ is a selfadjoint operator on the subspace $K$. Moreover

$$\| A^{1/2} \xi \| = \| T\xi \| \quad \text{for } \xi \in D(A^{1/2}) = D(T).$$

Hence the existence of the pair $(K, A)$ is proved. The uniqueness of $K$ is trivial and the uniqueness of $A$ follows from the uniqueness of polar decomposition of a closed densely defined operator. $K$ and $A$ are affiliated with $M$, because $s$ is invariant under the unitary group in $M'$. 

Remark. An immediate consequence of Lemma 1.4 is that if $m \in \mathcal{M}_+$ and $m(\phi) < \infty$ for any $\phi \in M^+_\infty$, then there exists a positive bounded operator $A \in M^+_\infty$, such that $m(\phi) = \phi(A)$, $\phi \in M^+_\infty$. 


THEOREM 1.5. Let $M$ be a von Neumann algebra. Each $m \in M_*$ has a unique spectral resolution of the form

$$m(\phi) = \int_0^\infty \lambda \, d\phi(e_\lambda) + \infty \phi(p), \quad \phi \in M_+^*$$

where $(e_\lambda)_{\lambda \in [0, \infty]}$ is an increasing family of projections in $M$, such that $\lambda \to e_\lambda$ is strongly continuous from right, and $p = 1 - \lim_{\lambda \to \infty} e_\lambda$. Moreover

$$e_0 = 0 \quad \text{iff} \quad m(\phi) > 0 \quad \text{for any} \quad \phi \in M_+^* \setminus \{0\} \quad (*)$$

$$p = 0 \quad \text{iff} \quad \{\phi \in M_+^* \mid m(\phi) < \infty\} \text{ is dense in } M_+^*. \quad (**)$$

Proof. Let $(K, A)$ be as in Lemma 1.5, let $A = \int_0^\infty \lambda \, de_\lambda$ be the spectral resolution of $A$, and let $p$ be the projection on $K^\perp$. Obviously $e_\lambda \in M$ for $\lambda \in [0, \infty]$, and $e_\lambda \not\succ 1 - p$ for $\lambda \to \infty$. From well known properties of spectral resolutions we have:

$$A^{1/2} = \int_0^\infty \lambda^{1/2} \, de_\lambda$$

$$D(A^{1/2}) = \left\{ \xi \in K \mid \int_0^\infty \lambda \, d(e_\lambda \xi \mid \xi) < \infty \right\}$$

and

$$\| A^{1/2} \xi \|^2 = \int_0^\infty \lambda \, d(e_\lambda \xi \mid \xi) \quad \text{for} \quad \xi \in D(A^{1/2}).$$

Hence

$$m(\omega_\xi) = \int_0^\infty \lambda \, d(e_\lambda \xi \mid \xi) \quad \text{for} \quad \xi \in K$$

Moreover if $p \xi \neq 0$ then $m(\omega_\xi) = +\infty$, which proves that the formula

$$m(\omega_\xi) = \int_0^\infty \lambda \, d(e_\lambda \xi \mid \xi) + \infty (p \xi \mid \xi)$$

is valid for any $\xi \in H$. Since any $\phi \in M_*^+$ is a finite or infinite sum of vector functionals, we get

$$m(\phi) = \int_0^\infty \lambda \, d\phi(e_\lambda) + \infty \phi(p) \quad \forall \phi \in M_*^+. \quad \forall \xi \in H \setminus \{0\}.$$
Hence (**) is proved. If $p = 0$, then $K = H$. Therefore $\{e | m(\omega e) < \infty\} = D(A^{1/2})$ is dense in $H$. Let $\phi \in M_+^*$, $\phi = \omega e$ for some $e \in H$. Choose $\xi_n \in D(A^{1/2})$ such that $\|\xi_n - \xi\| \to 0$. Then $\phi_n = \omega e_n \to \phi$ and $m(\phi_n) < \infty$. This proves that $p = 0 \Rightarrow \{\phi \in M_+^* | m(\phi) < \infty\}$ is dense in $M_+^*$. Since

$$\{\phi \in M_+^* | m(\phi) < \infty\} \subseteq \{\phi \in M_+^* | \phi(p) = 0\},$$

$p \neq 0$ implies that $\{\phi \in M_+^* | m(\phi) < \infty\}$ is not dense in $M_+^*$. Hence (**) is proved.

**Corollary 1.6.** Any $m \in M_+$ is the pointwise limit of an increasing sequence of bounded operators in $M_+^*$.

Proof. Let $m = \int_0^\infty \lambda \, d\lambda + \infty \rho$ be the spectral resolution of $m$. Put $x_n = \int_0^\infty \lambda \, d\lambda + np$, $n \in \mathbb{N}$, then $(x_n)_{n \in \mathbb{N}}$ is an increasing sequence in $M_+$ and $m(\phi) = \lim_{n \to \infty} \phi(x_n)$, $\phi \in M_+^*$.

**Example 1.7.** Let $A$ be a commutative von Neumann algebra. Then $A \simeq L^\infty(\Omega, \mu)$, where $\Omega$ is a locally compact space with a Radon measure $\mu$. The predual $M_+$ is in this case $L^1(\Omega, \mu)$.

Let $f: \Omega \to [0, \infty]$ be a measurable function. Define

$$m_f(\phi) = \int_\Omega f \cdot \phi \, d\mu \quad \phi \in L^1(\Omega, \mu)_+ \simeq A_+^*.$$

Then $m_f \in A_+^*$. (Use Fatou's lemma). Moreover $m_f = m_g$ implies $f = g$ a.e. Conversely, let $m \in A_+^*$, then by corollary 1.6, there exists $f_n \in L^\infty(\Omega, \mu)$ such that $f_n$ is an increasing sequence, and $f_n \to m(\phi)$ for any $\phi \in A_+^*$.

Put $f = \sup_{n \in \mathbb{N}} f_n$. Then $m(\phi) = \int_\Omega f \cdot \phi \, d\mu$, $\forall \phi \in A_+^*$. This proves that $A_+^*$ can be identified with the set of (equivalence classes) of measurable functions $\Omega \to [0, \infty]$. (Compare with [7, p. 243]).

**Definition 1.8.** Let $M$ be a von Neumann algebra on a Hilbert space $H$, and let $m \in B(H)_+^\wedge$. We say that $m$ is affiliated with $M$ if $u^* mu = m$ for any unitary in $M'$.

**Proposition 1.9.** There is a 1-1 correspondence between $M_\wedge$ and the set of $m \in B(H)_+^\wedge$ affiliated with $M$.

Proof. Let $n \in M_\wedge$, and put $\tilde{n}(\phi) = n(\phi \mid_M)$, $\phi \in B(H)_+^\wedge$. Obviously $\tilde{n} \in B(H)_+^\wedge$, and $\tilde{n}$ is affiliated with $M$. Note that $n$ and $\tilde{n}$ have the same spectral resolution. Conversely, let $m \in B(H)_+^\wedge$ be affiliated with $M$. By uniqueness of the spectral resolution

$$m = \int_0^\infty \lambda \, d\lambda + \infty \rho$$
we have $e_i \in M$ and $p \in M$. Let $n \in \hat{M}_+$ be given by the same spectral resolution. Then obviously $m = \hat{n}$.

**Proposition 1.10.** Any normal weight $\phi$ on $M$ has a unique extension (also denoted $\hat{\phi}$) to $\hat{M}_+$, such that

1. $\phi(\lambda m) = \lambda \phi(m)$ for $m \in \hat{M}_+$, $\lambda \geq 0$
2. $\phi(m + n) = \phi(m) + \phi(n)$ for $m, n \in \hat{M}_+$
3. $m \not\succ m \Rightarrow \phi(m_i) \not\succ \phi(m)$ for $m_i, m \in \hat{M}_+$

where $m \not\succ m$ means that $\omega(m_i) \nless \omega(m)$ for all $\omega \in M_+^*$.

**Proof.** Let $m \in \hat{M}_+$ with spectral resolution $\int^\infty_0 \lambda \, d\lambda + np$. Put $\phi(m) = \lim_{n \to \infty} \phi(x_n)$, where $x_n = \int^\infty_0 \lambda \, d\lambda + np$. By [15], $\phi$ has the form $\phi = \sum_{i \in I} \phi_i$ where $(\phi_i)_{i \in I}$ is a family of normal functionals. Hence

$$\phi(m) = \lim_{n \to \infty} \sum_{i \in I} \phi_i(x_n) = \sum_{i \in I} \phi_i(m).$$

Using this formula (1), (2) and (3) are easily verified. Assume that $\psi$ is another extension of $\phi$ to $\hat{M}_+$, satisfying (1), (2) and (3). Then by (3)

$$\psi(m) = \lim_{n \to \infty} \psi(x_n) = \lim_{n \to \infty} \phi(x_n) = \phi(m) \quad m \in \hat{M}_+.$$

This proves the uniqueness.

Let $M$ be a semifinite von Neumann algebra, and let $\tau$ be a normal, semifinite, faithful trace on $M$. In [15] the weight $\phi = \tau(x \cdot)$ is defined for a positive self-adjoint operator $x$ affiliated with $M$. We generalize this notation to $x \in \hat{M}_+$ and prove that $x \rightarrow \tau(x \cdot)$ is a bijection of $\hat{M}_+$ onto the set of normal (not necessarily semifinite) weights on $M$.

Let $\tau$ be a n.f.s. trace on a semifinite von Neumann algebra $M$. For $x, y \in M_+$ one defines

$$\tau(x \cdot y) = \tau(x^{1/2} y x^{1/2}) \in [0, \infty].$$

Obviously $\tau(x \cdot y) = \tau(y \cdot x)$.

**Proposition 1.11.** The map $(x, y) \rightarrow \tau(x \cdot y)$ has a unique extension to $\hat{M}_+ \times \hat{M}_+$, such that

1. $\tau(x \cdot y) = \tau(y \cdot x)$ for $x, y \in \hat{M}_+$
2. $\tau$ is homogeneous and additive in both variables.
3. If $x_\alpha \not\succ x$ and $y_\beta \not\succ y$ then $\tau(x_\alpha \cdot y_\beta) \not\succ \tau(x \cdot y)$ where $x_\alpha$ and $y_\beta$ are increasing nets on $\hat{M}_+$. The extension satisfies:
4. $\tau((ax^a) \cdot y) = \tau(x \cdot (a^* y a))$ for $x, y \in \hat{M}_+, a \in M$. 

Proof. For \(x \in \hat{M}_+\) we put \(x_n = \int_0^x \lambda \, dx_\lambda + \infty p\), where \(x = \int_0^\infty \lambda \, dx_\lambda + \infty p\) is the spectral resolution of \(x\). For \(x, y \in \hat{M}_+\) \(\tau(x_n, y_m)\) is increasing in both \(n\) and \(m\). Put \(\tau(x \cdot y) = \sup_{n,m} \tau(x_n \cdot y_m)\) \(x, y \in \hat{M}_+\). Obviously \(\tau\) satisfies (1). Let \(x \in \hat{M}_+\) and put

\[
\phi_x(a) = \tau(x \cdot a), \quad a \in M_+.
\]

Since \(\phi_x(a) = \sup_n \tau(x_n \cdot ax_n^{1/2})\), \(a \in M_+\), we get that \(\phi_x\) is a normal weight on \(M\).

The unique extension of \(\phi_x\) to \(\hat{M}_+\) is by Proposition 1.11 given by

\[
\phi_x(a) = \sup_m \phi_x(a_m) = \sup_{n,m} \tau(x_n \cdot a_m) = \tau(x \cdot a), \quad a \in M_+.
\]

Hence by Proposition 1.11 \(y \rightarrow \tau(x \cdot y)\) is homogeneous, additive and normal. Since \(\tau(x \cdot y) = \tau(y \cdot x)\) (2) and (3) are proved. The uniqueness of the extension follows as in the proof of Proposition 1.11. To prove (4) we note that if \(x, y\) are Hilbert–Schmidt operators with respect to \(\tau\), then

\[
\tau((ax^a) \cdot y) = \tau(x \cdot (a^* y a)).
\]

Since any \(x \in M\) is the supremum of an increasing net of positive Hilbert–Schmidt operators (cf. [7, Chap. I, Section 6]) the formula is also valid for \(x, y \in \hat{M}_+\). Then using corollary 1.6 we get (4).

Remark. If \(x\) is a positive selfadjoint operator affiliated with \(M\), and \(y\) is bounded, then our definition of \(\tau(x \cdot y)\) coincides with that of [15].

Theorem 1.12. Let \(\tau\) be a n.f.s. trace on a semifinite von Neumann algebra \(M\).

1. The map \(x \rightarrow \phi_x = \tau(x \cdot \cdot)\) is a homogeneous and additive bijection of \(\hat{M}_+\) onto the set of normal weights on \(M\). Moreover

\[
x \leq y \iff \phi_x \leq \phi_y, \quad x, y \in \hat{M}_+.
\]

\[
x_i \leq x \iff \phi_{x_i} \leq \phi_x, \quad x_i, x \in \hat{M}_+.
\]

2. Let \(x \in \hat{M}_+\) with spectral resolution \(x = \int_0^\infty \lambda \, dx_\lambda + \infty p\). Then:

\[
\phi_x \text{ is faithful} \iff e_0 = 0 \quad (\ast)
\]

\[
\phi_x \text{ is semifinite} \iff p = 0. \quad (\ast\ast)
\]

Proof. It follows from Proposition 1.11 that the map \(x \rightarrow \phi_x\) is homogeneous, additive and

\[
x \leq y \Rightarrow \phi_x \leq \phi_y, \quad x, y \in \hat{M}_+.
\]

\[
x_i \leq x \Rightarrow \phi_{x_i} \leq \phi_x, \quad x_i, x \in \hat{M}_+.
\]

\(x \rightarrow \phi_x\) is surjective.
If $\omega \in M_+^*$, then by [15, Theorem 5.4] $\omega$ is of the form $\omega = \tau(\cdot x)$ for a positive selfadjoint operator affiliated with $M$. By the preceding remark, we may regard $x$ as an element of $M_+$.

Let $\phi$ be a normal weight on $M$. Since $\phi$ is the sum of bounded normal functionals $\phi = \sum_{i \in I} \omega_i$, and each $\omega_i$ is of the form $\omega_i = \tau(x_i)$, we have $\phi = \tau(\cdot x)$ where $x = \sum_{i \in I} x_i \in \hat{M}_+$. Hence $x \rightarrow \phi_x$ is surjective. $x \rightarrow \phi_x$ is injective:

$$\phi_x = \phi_y \Rightarrow \phi_x(a) = \phi_y(a) \quad \forall a \in \hat{M}_+$$

$$\Rightarrow \phi_x(x) = \phi_y(y) \quad \forall a \in \hat{M}_+$$

$$\Rightarrow \omega(x) = \omega(y) \quad \forall \omega \in M_+^*$$

$$\Rightarrow x = y$$

Similar calculations give

$$\phi_x \leq \phi_y \Rightarrow x \leq y$$

and

$$\phi_{x_i} \vee \phi_x \Rightarrow x_i \vee x.$$ 

Hence (1) is proved. (2): Let $x \in \hat{M}_+$ with spectral resolution $x = \int_0^\infty \lambda d e \lambda + \infty p$. Put $x_n = \int_0^\infty \lambda d e \lambda + n \cdot p$, then $\phi_{x_n} \vee \phi_x$ for $n \rightarrow \infty$.

For any projection $q \in M$ we get:

$$\phi_x(q) = 0 \Leftrightarrow \phi_{x_n}(q) = 0 \quad \forall n \in \mathbb{N}$$

$$\Leftrightarrow \tau(q x_n q) = 0 \quad \forall n \in \mathbb{N}$$

$$\Leftrightarrow \text{supp}(x_n) \perp q \quad \forall n \in \mathbb{N}$$

$$\Leftrightarrow q \ll e_0$$

Hence $\text{supp}(\phi_x) = 1 - e_0$. In particular $\phi_x$ is faithful iff $e_0 = 0$. This proves $\ast$.

By [15, Proposition 4.2 and Theorem 5.12] the map $x \rightarrow \tau(\cdot x)$ is a bijection of the set of positive selfadjoint operators affiliated with $M$ onto the set of semifinite weights on $M$.

Since the set of positive selfadjoint operators corresponds to the set of $x \in \hat{M}_+$ for which $p = 0$ in their spectral resolution, we have proved $\ast \ast$.

Remark on the definition of $\hat{M}_+$. The extended positive part $\hat{M}_+$ of a von Neumann algebra $M$ can be considered as the “set of weights” on the predual $M_+$. In [8] we proved that for a weight $\phi$ on a von Neumann algebra the following conditions are equivalent

(1) $\phi$ is completely additive

(2) $\phi$ is $\sigma$-weakly lower semicontinuous.
Analogously we can prove that if $m$ is a map $M^+_* \rightarrow [0, \infty]$, such that

(a) $m(\lambda \phi) = \lambda m(\phi)$ \quad $\phi \in M^+_*$, \quad $\lambda \geq 0$

(b) $m \left( \sum_{n=1}^\infty \phi_n \right) = \sum_{n=1}^\infty m(\phi_n)$, \quad $\phi_n \in M^+_*$, \quad $\sum_{n=1}^\infty \phi_n \in M^+_*$

then $m$ is lower semicontinuous (i.e. $m \in \tilde{M}_+$). The proof is based on a reduction to the corresponding problem for weights:

We may assume that $M$ acts on a Hilbert space $H$. Put $\tilde{m}(\phi) = m(\phi \mid M)$, $\phi \in B(H)^+_*$. Then $\tilde{m}$ is a map $B(H)^+_* \rightarrow [0, \infty]$ satisfying (a) and (b). By Proposition 1.9 $m$ is lower semicontinuous iff $m$ is lower semicontinuous. Hence it is enough to treat the case $M = B(H)$. It is no loss of generality to assume that $m(\phi) \geq \| \phi \|$, $\phi \in B(H)^+_*$, because if this is not the case, we can instead consider $m'(\phi) = m(\phi) + \phi(1)$. For any positive trace class operator $x$ on $H$ we let $\phi_x$ be the positive normal functional defined by $\phi_x = \text{tr}(x \cdot)$. Define a map $\omega : B(H)^+_* \rightarrow [0, \infty]$ by

$$
\omega(x) = \begin{cases} 
m(\phi_x) & \text{if } x \text{ is trace class} \\
\infty & \text{otherwise}.
\end{cases}
$$

Clearly $\omega$ is a weight on $B(H)$, and $\omega \geq \text{tr}$. Let $(x_i)_{i \in I}$ be a family of positive operators, such that $x = \sum_{i \in I} x_i$ is bounded. If $\text{tr}(x) = \infty$, then $\sum_{i \in I} \text{tr}(x_i) = +\infty$. Therefore

$$
\sum_{i \in I} \omega(x_i) = \sum_{i \in I} \text{tr}(x_i) = +\infty = \omega(x).
$$

If $\text{tr}(x) < \infty$, then $\sum_{i \in I} \text{tr}(x_i) < \infty$. Therefore $x_i \neq 0$ for at most countably many $i \in I$. Hence

$$
\sum_{i \in I} \omega(x_i) = \sum_{i \in I} m(\phi_{x_i}) = m \left( \sum_{i \in I} \phi_{x_i} \right) = m(\phi_x) = \omega(x).
$$

This proves that $\omega$ is completely additive and hence $\sigma$-weakly lower semicontinuous by [8]. Let $(\psi_n)_{n \in \mathbb{N}}$ be a sequence of positive normal functionals which converges to $\psi \in B(H)^*_*$ and let $(x_n)_{n \in \mathbb{N}}$ and $x$ be the corresponding trace class operators. Then $\| x_n - x \|_1 \rightarrow 0$ ($\| y \|_1 = \text{tr}(| y |)$). In particular $x_n \rightarrow x$ $\sigma$-weakly. Therefore

$$
m(\psi) = \omega(x) \leq \liminf_{n \in \mathbb{N}} \omega(x_n) = \liminf_{n \in \mathbb{N}} m(\psi_n)
$$

which proves that $m$ is lower semicontinuous.
Let $M$ and $N$ be von Neumann algebras, $N \subseteq M$. A conditional expectation from $M$ to $N$ is a positive linear map $\epsilon: M \to N$ satisfying

1. $\epsilon(1) = 1$
2. $\epsilon(axb) = a\epsilon(x)b \quad x \in M, \quad a, b \in N$

Throughout the paper we shall only consider normal conditional expectations i.e.

3. $x_i \not\approx x \Rightarrow \epsilon(x_i) \not\approx \epsilon(x), \quad x_i, x$ selfadjoint.

Using the identity $a^*xb = \frac{1}{2} \sum_{k=0}^{3} i^{-k}(a + i^k b)^* x(a + i^k b)$ it is easily seen that (2) is equivalent to

$\epsilon(a^*xa) = a^*\epsilon(x)a \quad x \in M, \quad a \in N.$

**DEFINITION 2.1.** Let $M$ and $N$ be von Neumann algebras $N \subseteq M$. An operator valued weight from $M^+$ to $N$ is a map $T: M^+ \to N^+$ which satisfies the following conditions:

1. $T(hx) = AT(x)h > 0, \quad x \in M^+$
2. $T(x + y) = T(x) + T(y) \quad x, y \in M^+$
3. $T(a^*xa) = a^*T(x)a \quad x \in M^+, \quad a \in N$

Moreover we say that $T$ is *normal* if

4. $x_i \not\approx x \Rightarrow T(x_i) \not\approx T(x), \quad x_i, x \in M^+$.

In analogy with ordinary weights we put

$$n_T = \{x \in M \mid \|T(x^*x)\| < \infty\}$$
$$m_T = n_T^*n_T = \text{span}\{x^*y \mid x, y \in n_T\}$$

It is easy to verify that

(i) $m_T = \text{span}\{x \in M_+ \mid \|T(x)\| < \infty\}$
(ii) $m_T$ and $n_T$ are twosided modules over $N$
(iii) $T$ has a unique linear extension $T: m_T \to N$ which satisfies

$$T(axb) = aT(x)b \quad x \in m_T, \quad a, b \in N$$

In particular if $T(1) = 1$, then $T$ is a conditional expectation from $M$ to $N$. 
DEFINITION 2.2. We say that $T$ is **faithful** if $T(x^*x) = 0 \Rightarrow x = 0$ and **semifinite** if $n_T$ is $\sigma$-weakly dense in $M$. We let $P(M)$ (resp. $P(N)$) denote the set of n.f.s. weights on $M$ (resp. $N$), and the set of n.f.s. operator valued weights from $M$ to $N$ is denoted $P(M, N)$.

**PROPOSITION 2.3.** Let $T$ be a normal operator valued weight from $M$ to $N$. If $\phi$ is a normal weight on $N$, then $\phi \circ T$ is a normal weight on $M$ (here $\phi$ is extended to $\tilde{N}_\sigma$ as in Proposition 1.11). If $\phi \in P(N)$ and $T \in P(M, N)$ then $\phi \circ T \in P(M)$.

**Proof.** The only non-trivial statement is that $\phi$ semifinite implies $\phi \circ T$ semifinite. Let $x \in n_T$, then $T(x^*x) \in N_\sigma$. Since $\phi$ is semifinite there exists a net $(a_\lambda)_{\lambda \in \Lambda}$ in $n_\phi$ such that $a_\lambda \rightarrow 1$ strongly. Since

$$\phi \circ T(a_\lambda x^*xa_\lambda) = \phi(a_\lambda^* T(x^*x) a_\lambda) \leq \| T(x^*x) \| \| a_\lambda^* a_\lambda \| < \infty$$

we have that $xa_\lambda \in n_{\phi \circ T}$ is strongly dense in $n_T$, which is dense in $N$.

**Remark 2.4.** Any normal operator valued weight $T: M_+ \rightarrow \tilde{N}_\sigma$ has a unique normal extension to a map $\tilde{M}_+ \rightarrow \tilde{N}_\sigma$ satisfying (1)-(4) in Definition 2.1. (Same proof as in Proposition 1.11). Hence if $M, N$ and $R$ are von Neumann algebras $R \subseteq N \subseteq M$ and $S: M_+ \rightarrow \tilde{N}_\sigma$, $T: N_+ \rightarrow \tilde{R}_\sigma$ are normal operator valued weights, then $T \circ S$ is a well defined normal operator valued weight from $M$ to $R$. By the arguments of the preceding proof one gets that $S \in P(M, N)$ and $T \in P(N, R)$ implies that $T \circ S \in P(M, N)$.

**PROPOSITION 2.5.** Let $M$ and $N$ be von Neumann algebras, $N \subseteq M$, and let $T \in P(M, N)$. Then

1. $T(m_T)$ is a $\sigma$-weakly dense twosided ideal in $N$.
2. If $T$ is extended to a map from $\tilde{M}_+$ to $\tilde{N}_\sigma$ as in the preceding remark, then

$$T(\tilde{M}_+) = \tilde{N}_\sigma.$$ 

**Proof.** (1) $T(m_T)$ is a twosided ideal in $N$ because

$$aT(x)h = T(axh) \quad x \in m_T, \quad a, b \in N.$$

Let $p$ be the maximal projection in the $\sigma$-weak closure of $T(m_T)$. Note that $p \in Z(N)$ (center of $N$). Assume now that $p \neq 1$.

Since $n_T$ is $\sigma$-weakly dense in $M$, there exists $x \in n_T$ such that $x(1 - p) \neq 0$. Hence

$$(1 - p)x^*x(1 - p) \in m_T^{-\epsilon}\{0\}$$
and since $T$ is faithful, it follows that:

$$T((1 - p) x^*x(1 - p)) = (1 - p) T(x^*x)(1 - p) \in (1 - p) N \setminus \{0\}.$$ 

This contradicts that $T(m_r) \subseteq p \cdot N$. Hence $p = 1$.

(2) By [7, Chap. 1, Section 3 Corollary 5 de Theorem 2] any $y \in N_+$ has the form $y = \sum_{i \in I} T x_i$ where $(x_i)_{i \in I}$ is a family of elements in $M_+$. Let $z \in \bar{N}_+$. By Corollary 1.6 there exists a sequence $y_n \in N_+$ such that $y_n \not\to z$. Put $z_1 = y_1$, and $z_n = y_n - y_{n-1}$ for $n \geq 2$. Then $y = \sum_{n=1}^{\infty} y_n$. Each $y_n$ has the form $\sum_{i \in I_n} T x_{ni}$, $x_{ni} \in M_+$. Therefore $y = T(\sum_{n=1}^{\infty} \sum_{i \in I_n} x_{ni})$, where $T$ is extended to $\bar{M}_+$ as in Remark 2.4. Hence $T$ maps $\bar{M}_+$ to $\bar{N}_+$.

**Lemma 2.6.** Let $M, N$ be von Neumann algebras, $N \subseteq M$ and let $T$ be a normal operator valued weight on $N$. If there exists a n.f.s. weight $\psi$ on $N$ such that $\psi \circ T$ is a n.f.s. weight on $M$, then $T$ is faithful and semifinite.

**Proof.** If $x \geq 0$ then $Tx = 0$ implies $\psi(Tx) = 0$, hence $x = 0$. Hence $T$ is faithful. The set $m_{\psi}^+ = \{x \in M_+ | \psi(x) < \infty\}$ is $\sigma$-weakly dense in $M_+$. Let $x \in m_{\psi}^+$ and let

$$Tx = \int_0^\infty \lambda d e_\lambda + \infty p$$

be the spectral resolution of $Tx$. Since $\psi(Tx) < \infty$ we have $p = 0$.

Therefore $e_{a^*}x e_a \to x$ $\sigma$-weakly for $a \to \infty$ and $T(e_{a^*}xe_a) = e_a T(x) e_a = \int_0^\infty \lambda d e_\lambda$ is bounded. Hence $m_r$ is $\sigma$-weakly dense in $M$, which proves that $T$ is semifinite.

**Theorem 2.7.** Let $M$ and $N$ be semifinite von Neumann algebras, $N \subseteq M$, and let $\tau_1$ and $\tau_2$ be n.f.s. traces on $M$ and $N$ respectively. There is a unique n.f.s. operator valued weight $\Phi$ from $M$ to $N$ such that $\tau_1 = \tau_2 \circ \Phi$.

**Proof.** For $x \in M_+$ the map $y \to \tau_1(x \cdot y)$, $y \in N_+$ is a normal weight on $N$. Hence by Theorem 1.12 there exists a unique $\Phi(x) \in \bar{N}_+$ such that

$$\tau_1(x \cdot y) = \tau_2(\Phi(x) \cdot y) \quad \forall y \in N_+ \quad \ldots \quad (\ast)$$

By Theorem 1.13 the map $x \to \Phi(x)$ is homogeneous, additive and normal. Let $a \in N$, then by Proposition 1.11 (4) for any $a \in N$:

$$\tau_1((a x a^*) \cdot y) = \tau_1(x \cdot (a^* y a))$$

$$= \tau_2(\Phi(x) \cdot (a^* y a)) = \tau_2(a \Phi(x) a^*) \cdot y$$

Hence $\Phi(axa^*) = a \Phi(x) a^*$, which proves that $\Phi$ is an operator valued weight from $M$ to $N$. 

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As a special case of (\ast) we get
\[ \tau_1(x) = \tau_2 \circ \Phi(x) \quad \forall x \in M_+ . \]

By Lemma 2.6 \( \Phi \) is semifinite and faithful.

Assume that \( T \) is another operator valued weight such that \( \tau_1 = \tau_2 \circ T \), then for \( x \in M_+ \) and \( a \in N_+ \):
\[ \tau_2(a^{1/2}(Tx)a^{1/2}) = \tau_2 \circ T(a^{1/2}xa^{1/2}) = \tau_1(a^{1/2}xa^{1/2}) \]
or equivalently
\[ \tau_2(a \cdot (Tx)) = \tau_1(a \cdot x) \]

Hence by the definition of \( \Phi \) it follows that \( \Phi = T \).

Theorem 2.7 will be generalized in [10, Theorem 5.1].

Remark. Let \( M \) be a semifinite von Neumann algebra, and \( Z(M) \) its center. Choose n.f.s. traces \( \tau_1 \) and \( \tau_2 \) on \( M \) and \( Z(M) \) respectively. Then the operator valued weight \( \Phi \) in Theorem 2.7 is a central valued trace on \( M \) in the sense of [7, Chap. III].

Definition 2.8. Let \( T \) be a normal operator valued weight from \( M \) to \( N \). Since \( T \) is normal, there is a biggest projection \( q \in M \) such that \( T(q) = 0 \). We call \( 1 - q \) the support of \( T \), and denote it \([T] \). Since \( T(u^*xu) = u^*T(x)u \) for any unitary \( u \in N \) the support of \( T \) is a projection in the relative commutant \( N_\cap M \).

Let \( (T_i)_{i \in I} \) be a family of operator valued weights. We say that \( (T_i)_{i \in I} \) is faithful if \( T_i(x^*x) = 0 \) for any \( i \in I \) implies that \( x = 0 \).

Proposition 2.9. Let \( M \) and \( N \) be von Neumann algebras, \( N \subseteq M \). If there exists a faithful family of normal conditional expectations from \( M \) to \( N \), then \( P(M, N) \) is not empty.

Proof. Choose a maximal family \( (T_i)_{i \in I} \) of non-zero bounded normal operator valued weights from \( M \) to \( N \), with mutually orthogonal supports. Assume that
\[ q = 1 - \sum_{i \in I} [T_i] \neq 0 \]
then by the assumptions there exists a normal conditional expectation \( \epsilon: M \to N \) such that \( \epsilon(q) \neq 0 \). Since \( q \in N_\cap N_\epsilon \) the map \( x \to \epsilon(qxq) \) is a non-zero bounded normal operator valued weight from \( M \) to \( N \) with support less than \( q \). This contradicts the maximality of \( (T_i)_{i \in I} \). Hence \( \sum_{i \in I} [T_i] = 1 \). Put \( T(x) = \)
\[ \sum_{i \in I} T_i(x), \] then \( T \) is a normal faithful operator valued weight from \( M \) to \( N \). 

\( T \) is also semifinite, because \( T([T_i]) = T_i(1) \) is bounded for any \( i \in I \).

Remark. The operator valued weight \( T \) constructed in the preceding proof is semifinite on \( N^e = N' \cap M \).

Corollary 2.10. Let \( M \) and \( N \) be von Neumann algebras, \( N \subseteq M \), then \( P(M, N) \) is non-empty in the following cases:

1. \( N \) is the center of \( M \)
2. \( M \cong N \otimes R \), where \( R \) is a subalgebra of the relative commutant \( N^e = N' \cap M \).
3. \( N \) is a direct sum of type I factors.

Proof. We check that the conditions of Proposition 2.9 are satisfied. (1) If the center is \( \sigma \)-finite, then it follows from [13, Theorem 1] that the general case can easily be reduced to this case by writing \( M \) in the form \( \sum_{i \in I} M_i \) where each \( M_i \) has \( \sigma \)-finite center. (2) Follows from [12, Lemma 2.5]. (3) Let \( (p_i)_{i \in I} \) be the minimal projections in the center of \( N \). Since \( p_i N \) is a type I factor, there is a normal conditional expectation \( e_i \) from \( p_i M p_i \) to \( p_i N \), for each \( i \in I \).

Put
\[ \epsilon(x) = \sum_{i \in I} e_i(p_i x p_i) \]

Then \( \epsilon \) is a normal conditional expectation from \( M \) to \( N \). Moreover it is not hard to show that there is a faithful family of conditional expectations of this form.

Recently some examples of "generalized conditional expectations" have occurred in literature. (cf. [5, Chap. II] and [14]). It is natural to ask: under which conditions can such maps be extended to normal operator valued weights. For instance, can the following be used?

Problem 2.11. Let \( M, N \) be von Neumann algebras \( N \subseteq M \), and let \( m \subseteq M \) be a \( \sigma \)-weakly dense *subalgebra of \( M \), such that \( m = \text{span}(m_+) \), \( N \cdot m \subseteq m \) and \( m \cdot N \subseteq m \). Let \( S \) be a linear map from \( m \) to \( N \) with the following properties:

1. \( S(axb) = aS(x)b, \quad x \in m, \quad a, b \in N \).
2. \( \text{If } (x_i)_{i \in I} \text{ is an increasing net in } m_+, \text{ that satisfies } \sup \|x_i\| < \infty \) and \( \sup \|Sx_i\| < \infty \) then \( \sup x_i \in m_+ \) and \( S(\sup x_i) = \sup(Sx_i) \).

Does a normal operator valued weight \( T \) exist from \( M_+ \) to \( N_+ \), such that \( S = T \)? If yes, is \( T \) unique?
3. Analytic Extensions of the One Parameter Groups $\sigma_t^{\phi, \psi}$

**Definition 3.1.** Let $t \to \sigma_t$ be a $\sigma$-weakly continuous one parameter group of isometries of a von Neumann algebra $M$, and let $\alpha \in \mathbb{C}$. If $\Im \alpha > 0$ we define $\sigma_\alpha$ to be an operator on the set $D(\sigma_\alpha)$ of those $x \in M$ for which the map $t \to \sigma_t x$ can be extended to a $\sigma$-weakly continuous function $f_\alpha(t) \in \mathcal{A}$ on the strip $0 \leq \Im t \leq \Im \alpha$, analytic in the interior of the strip. Such an extension is unique, and we put $\sigma_\alpha x = f_\alpha(t)x$, $x \in D(\sigma_\alpha)$. For $\Im \alpha < 0$ we define $\sigma_\alpha$ analogously.

From [1, Theorem 2.4] it follows that $\sigma_\alpha$, $\alpha \in \mathbb{C}$ are closed densely defined operator on $M$ (equipped with $\sigma$-weak topology). Moreover

\[
\sigma_\alpha \sigma_\beta \subseteq \sigma_{\alpha + \beta}, \quad \alpha, \beta \in \mathbb{C} \\
\sigma_\alpha \sigma_\beta = \sigma_{\alpha + \beta}, \quad (\Im \alpha)(\Im \beta) \geq 0. \\
\sigma_{-\alpha} = \sigma^{-1}_\alpha, \quad \alpha \in \mathbb{C}
\]

The operator $\sigma_{-1}$ is called the analytic generator of the one parameter group. Two one parameter groups coincide iff they have the same analytic generator (cf. [1, Theorem 4.4]).

Let $\phi, \psi$ be two n.f.s. weights on a von Neumann algebra $M$, and let $F_2$ be the factor of type $I_2$. On $M \otimes F_2$ one considers the weight

\[
\theta(\Sigma x_{ij} \otimes e_{ij}) = \phi(x_{11}) + \psi(x_{22}),
\]

where $(e_{ij})_{i,j} = 1, 2$ is the natural basis for the $2 \times 2$-matrices. According to [6] there exists a $\sigma$-weakly continuous one parameter group $t \to \sigma_t^{\phi, \psi}$ of isometries on $M$, such that

\[
\sigma_t^{\phi}(x \otimes e_{21}) = \sigma_t^{\phi}(x) \otimes e_{21}.
\]

Moreover

\[
\sigma_t^{\phi, \psi}(x) = (D\psi : D\phi), \sigma_t^{\phi}(x) = \sigma_t^{\phi}(x)(D\psi : D\phi) \]

where $(D\psi : D\phi)$ is Connes cocycle Radon Nikodym derivative. In particular

\[
\sigma_t^{\phi, \psi} = \sigma_t^{\psi}.
\]

The following theorem provides an algebraic characterization of the analytic generators of the one parameter groups $\sigma_t^{\phi, \psi}$, $\phi, \psi \in P(M)$.

**Theorem 3.2.** Let $M$ be a von Neumann algebra, and let $\phi, \psi \in P(M)$. For $a, b \in M$ the following conditions are equivalent:

1. $(a, b) \in G(\sigma_t^{a, b})$
(2) \( an^* \subseteq n^* \), \( n^* b \subseteq n^* \) and
\[
\psi(ax) = \phi(xb) \quad \text{for any } x \in n^* n^*.
\]
\((G(\cdot) \text{ means the graph of the operator}).\)

For the proof we need a lemma essentially due to A. Connes [4].

**Lemma 3.3.** Let \( \phi, \psi \) be n.f.s. weights on a von Neumann algebra \( M \). Let \( a \in M \) and \( k \in \mathbb{R}_+ \). The following conditions are equivalent:

1. \( \#(a - a^*) \leq k^2 \phi \)
2. \( n^*_a \subseteq n^* \) and \( \| A_\psi(xa^*)\| \leq k \| A_\phi(x)\|, \ x \in n^* \)
3. \( a \in D(\psi^{\phi,\phi}_1) \) and \( \| \psi^{\phi,\phi}_1(a)\| \leq k \).

If \( \phi = \psi \) and (1)–(3) are satisfied, then
\[ A_\phi(xa^*) = J_\phi \pi_\phi(\sigma_{-\frac{i}{2}}^\phi(a)) J_\phi A_\phi(x). \]

**Proof.** (1) \( \Rightarrow \) (2) is trivial.

(1) \( \Rightarrow \) (3) If \( \phi = \psi \) this equivalence follows from [4, Lemma 7 (a)] with trivial modifications. For the general case consider as usual the weight \( \theta \) on \( M \otimes F_2 \):
\[ \theta(\Sigma x_{ij} \otimes e_{ij}) = \phi(x_{11}) + \psi(x_{22}). \]

Put \( \tilde{a} = a \otimes e_{21} \) and let \( \tilde{x} = \Sigma x_{ij} \otimes e_{ij} \) be a positive element in \( M \otimes F_2 \).
An elementary calculation gives
\[ \theta(\tilde{a} \tilde{x}^*) = \psi(axa^*). \]
Hence
\[
\psi(axa^*) \leq k^2 \phi(x) \quad \forall x \in M_+
\]
\[
\Rightarrow \theta(\tilde{a} \tilde{x}^*) \leq k^2 \theta(\tilde{x}) \quad \forall \tilde{x} \in (M \otimes F_2)_+.
\]
\[
\Rightarrow \tilde{a} \in D(\sigma_{-\frac{i}{2}}^\phi) \quad \text{and} \quad \| \sigma_{-\frac{i}{2}}^\phi(\tilde{a})\| \leq k
\]
\[
\Rightarrow a \in D(\sigma_{-\frac{i}{2}}^\psi) \quad \text{and} \quad \| \sigma_{-\frac{i}{2}}^\psi(a)\| \leq k.
\]

For the last implication we have used that
\[ \sigma_t^\phi(\tilde{a}) = \sigma_t^\psi(a) \otimes e_{21}, \quad t \in \mathbb{R}. \]

This completes the proof of (1) \( \Rightarrow \) (3).
The equation
\[ A_\phi(xa^*) = J_\phi^* \sigma_{-1/2}(a) J_\phi A_\phi(x) \]
is proved in [4, Lemma 7(b)].

**Proof of Theorem 3.2.** Assume first that \( \phi = \psi \).

(1) \( \Rightarrow \) (2) Put \( \sigma_t = \sigma_t^\phi \). Assume that \((a, b) \in G(\sigma_-)\). Then \( a \in D(\sigma_{-1/2}) \) and \( b \in D(\sigma_{+1/2}) \) and \( \sigma_{-t/2}(a) = \sigma_{+t/2}(b) \).

Since \( \sigma_t(x^*) = \sigma_t(x)^* \) \( t \in \mathbb{R}, x \in M \) we get
\[ x^* \in D(\sigma_\phi) \Leftrightarrow x \in D(\sigma_\phi). \]

In particular \( b^* \in D(\sigma_{-1/2}) \). Hence by Lemma 3.3 \( n_\phi a^* \subseteq n_\phi \) (or equivalently \( an_\phi^* \subseteq n_\phi^* \)) and \( n_\phi b \subseteq n_\phi^* \).

Note that this implies that \( a \) (resp. \( b \)) is a left (resp. right) multiplier on \( n_\phi \), such that both \( \phi(ax) \) and \( \phi(xb) \) make sense for any \( x \in n_\phi \). For \( x \in m_\phi^+ \) we get using Lemma 3.3 twice:
\[ \phi(ax) = (A_\phi(x^{1/2}) | A_\phi(x^{1/2}a^*)) \]
\[ = (A_\phi(x^{1/2}) | J_\phi \sigma_{-1/2}(a) J_\phi A_\phi(x^{1/2})) \]
\[ = (A_\phi(x^{1/2}) | J_\phi \sigma_{+1/2}(b) J_\phi A_\phi(x^{1/2})) \]
\[ = (J_\phi \sigma_{-1/2}(b^*) J_\phi A_\phi(x^{1/2}) | A_\phi(x^{1/2})) = (A_\phi(x^{1/2}b) | A_\phi(x^{1/2})) = \phi(xb). \]

Since \( m_\phi = \text{span}(m_\phi^+) \) we get (2).

(2) \( \Rightarrow \) (1) Let \((\pi_\phi, H_\phi)\) be the representation associated with \( \phi \). It is well known that \( \mathcal{A}_\phi = A_\phi(n_\phi \cap n_\phi^*) \) is an achieved left Hilbert algebra. Let \( \mathcal{A}_0 \subseteq \mathcal{A}_\phi \) be the maximal Tomita algebra equivalent to \( \mathcal{A}_\phi \) (cf. [2, Lemma 2.7]). As usual we let \( S_\phi \) be the closure of the involution \( A_\phi(x) \to A_\phi(x^*) \), \( x \in n_\phi \cap n_\phi^* \), and \( F_\phi = S_\phi^* \).

Let \( \xi, \eta \in \mathcal{A}_0 \), and put \( x = \pi_\phi(\eta^* \xi) = \pi_\phi(\eta)^* \pi_\phi(\xi) \). Obviously \( x \in m_\phi \).

Let \( a \in M \) be a left multiplier on \( n_\phi^* \), then
\[ \phi(ax) = (A_\phi(\pi_\phi(\xi)) | A_\phi(\pi_\phi(\eta)^* a^*)) \]
\[ = (\xi | A_\phi(\pi_\phi(\eta)^* a^*)) \]
\[ = (S_\phi A_\phi(\pi_\phi(\eta)^* a^*) | F_\phi \xi) \]
\[ = (A_\phi(a\pi_\phi(\eta)^*) | F_\phi \xi) \]
\[ = (a\eta^* | \xi^*). \]

A similar calculation gives for \( b \in M \), \( b \) right multiplier on \( n_\phi \):
\[ \phi(xb) = (b\eta^* | \xi^*). \]
Hence if \((a, b)\) satisfies (2) then
\[
(a \eta^* \mid \xi^b) = (b \eta^* \mid \xi^a) \quad \forall \xi, \eta \in \mathcal{O}_0,
\]
By the substitution \(\xi_1 = \eta^b, \xi_2 = \xi^a\) one has equivalently
\[
(a \Delta_0^1 \xi_1 \mid \Delta_0^{-1} \xi_2) = (b \xi_1 \mid \xi_2) \quad \forall \xi_1, \xi_2 \in \mathcal{O}_0.
\]
For \(\alpha \in \mathbb{C}\) we put
\[
M_\alpha(\xi_1, \xi_2) = (a \Delta^{-i \alpha} \xi_1 \mid \Delta^{+i \alpha} \xi_2) \quad \xi_1, \xi_2 \in \mathcal{O}_0.
\]
\(M_\alpha\) is a sesquilinear form on \(\mathcal{O}_0 \times \mathcal{O}_0\). For \(t \in \mathbb{R}\):
\[
M_t(\xi_1, \xi_2) = (\sigma_t^\phi(a) \xi_1 \mid \xi_2)
\]
\[
M_{t-}(\xi_1, \xi_2) = (\sigma_t^\phi(b) \xi_1 \mid \xi_2).
\]
Using the Phragmen–Lindelöf principle (cf. [20], p. 93) one gets
\[-1 \leq \text{Im} \alpha \leq 0 \Rightarrow |M_\alpha(\xi_1, \xi_2)| \leq \max(\|a\|, \|b\|) \|\xi_1\| \|\xi_2\|.
\]
Hence there exists \(x(\alpha) \in B(H)\) such that
\[
x(\alpha) \xi_1 \mid \xi_2) = (a \Delta^{-i \alpha} \xi_1 \mid \Delta^{+i \alpha} \xi_2) \quad \text{Im} \alpha \in [-1, 0].
\]
It is easily seen that \(\alpha \to x(\alpha)\) is \(\sigma\)-weakly continuous on the strip \(\text{Im} \alpha \in [-1, 0]\) and analytic in the interior. Moreover \(x(t) = \sigma_t^\phi(a) \in M\) and \(x(t - i) = \sigma_t^\phi(b) \in M\) for \(t \in \mathbb{R}\). This implies that \(x(\alpha) \in M\) for any \(\alpha\) in the strip, and thus \((a, b) \in G(\sigma^\phi)\) by the definition of \(\sigma^\phi\).

Now we return to the general case. Define \(\theta\) on \(M \otimes F_2\) as in the preceding lemma. Let \(a, b \in M\) and put \(\tilde{a} = a \otimes e_{21}, \tilde{b} = b \otimes e_{21}\). Obviously \((a, b) \in G(\sigma^\phi)\) iff \((\tilde{a}, \tilde{b}) \in G(\sigma^\phi_2)\). Hence by the first part of the proof
\[
(a, b) \in G(\sigma^\phi_2) \iff \{\begin{array}{l}
d_\theta^* \subseteq n_\theta^* \quad \text{and} \\
\theta(\tilde{a}x) = \theta(\tilde{x}b) \end{array} \quad \forall x \in m_\theta.
\]
Using that \(n_\theta = n_\theta \otimes e_{11} + n_\theta \otimes e_{12} + n_\theta \otimes e_{21} + n_\theta \otimes e_{22}\) an easy calculation gives:
\[
d_\theta^* \subseteq n_\theta^* \iff d_\theta^* \subseteq n_\theta^*
\]
and
\[
n_\theta \subseteq n_\theta \iff n_\theta \subseteq n_\theta.
\]
For $\tilde{x} \in m_\theta = m_\phi \otimes e_{11} + n_\phi^* n_\phi \otimes e_{12} + n_\phi^* n_\phi \otimes e_{21} + m_\phi \otimes e_{22}$, $\tilde{x} = \sum_{i,j} \xi_{ij} e_{ij}$ one gets:

$$\theta(\tilde{a}\tilde{x}) = \psi(ax_{12})$$

and

$$\theta(\tilde{x}b) = \phi(x_{12}b)$$

Hence $\theta(\tilde{a}\tilde{x}) = \theta(\tilde{x}b) \forall \tilde{x} \in m_\theta$ is equivalent to:

$$\psi(ax) = \phi(xb) \forall x \in n_\phi^* n_\phi$$

This completes the proof.

As a special case of Theorem 3.2 one has:

**Corollary 3.4.** Let $\phi, \psi$ be positive, normal, faithful functionals on a von Neumann algebra $M$. Then $(a, b) \in G(\sigma^{\phi, \psi})$ iff $\psi(ax) = \phi(xb)$ for any $x \in M$.

### 4. Modular Properties of Operator Valued Weights

Let $\epsilon$ be a normal, faithful, conditional expectation of a von Neumann algebra $M$ onto a sub-von Neumann algebra $N$. In [17] and [3] M. Takesaki and A. Connes have proved:

1. $\sigma^{\phi, \psi}(x) = \sigma^\phi(x)$, $x \in N$, $\phi \in P(N)$
2. $(D\psi \circ \epsilon; D\phi \circ \epsilon)_t = (D\psi; D\phi)_t$, $\phi, \psi \in P(N)$

Using the identity

$$\sigma^{\psi, \phi}_t(x) = (D\psi; D\phi)_t \sigma^\phi_t(x)$$

(1) and (2) can be expressed in the single equation:

3. $\sigma^{\phi, \psi, \phi, \psi}_t(x) = \sigma^{\psi, \phi}_t(x)$, $x \in N$, $\phi, \psi \in P(N)$

In this section we will generalize (1) and (2) to normal, faithful, semifinite operator valued weights. This will be done by proving that the analytic generator of $\sigma^{\phi, \psi, \phi, \psi}_t$ coincides with the analytic generator of $\sigma^\phi_t$ on $D(\sigma^\phi_t)$.

**Definition 4.1.** Let $t \mapsto \sigma_t$ be a $\sigma$-weakly continuous one parameter group of isometries of a von Neumann algebra $M$.

(a) An element $x \in M$ is **analytic** (with respect to $\sigma_t$) if the map $t \mapsto \sigma_t x$ has an analytic extension to an entire function from $\mathbb{C}$ to $M$. 

(b) \(x \in M\) is of exponential type if \(x\) is analytic, and there exists a constant \(c > 0\) such that

\[
\sup_{\alpha \in \mathbb{C}} \| \sigma_\alpha x \| e^{-c|\text{Im}|} < \infty
\]

**Lemma 4.2.** The set \(M_{\text{exp}}\) of elements of exponential type is \(\sigma\)-weakly dense in \(M\).

**Proof.** For \(a > 0\) we let \(F_a\) denote the analytic function

\[
F_a(z) = \frac{1 - \cos az}{\pi az^2}, \quad z \in \mathbb{C}, \quad a > 0.
\]

Regarded as a real function \(F_a\) is the Fejer-kernel used in the theory of Fourier integrals. In particular \(F_a(s), s \in \mathbb{R}\) is an approximating unit in the sense that \(F_a(s) \geq 0,\)

\[
\lim_{a \to \infty} \int_{-\infty}^{\infty} F_a(s) \phi(s) \, ds = \phi(0)
\]

for any continuous bounded function \(\phi\) on \(\mathbb{R}\).

We shall need the following estimate:

\[
\int_{-\infty}^{\infty} |F_a(s + it)| \, ds \leq e^{a|t|}, \quad t \in \mathbb{R}.
\]

To prove (***), put

\[
\phi_a(x) = \begin{cases} \frac{1}{\sqrt{a}} & |x| \leq \frac{1}{2}a \\ 0 & |x| > \frac{1}{2}a \end{cases}
\]

The Fourier–Plancherel transformed of \(\phi_a\) can be extended to an entire function

\[
\phi_a(z) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \phi_a(x) e^{-ixz} \, dx = \frac{2}{\pi a} \sin(\frac{1}{4}ax) \, z^{-1}, \quad z \in \mathbb{C}.
\]

Moreover \(\phi_a(z)^2\) is equal to \(F_a(z)\). Since

\[
\phi_a(s + it) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \phi_a(x) e^{ixc - izx} \, dx
\]
it follows from Plancherel's theorem that

\[ \int_{-\infty}^{\infty} |F_a(s + it)|^2 ds = \int_{-\infty}^{\infty} |\phi_a(s + it)|^2 ds = \|\phi_a(x) e^{itx}\|_2^2 \]

\[ = \frac{1}{a} \int_{-a/2}^{a/2} e^{itx} dx \leq e^{a|t|} \]

Hence (*** is proved.

For \(x \in M\) put

\[ x_a = \int_{-\infty}^{\infty} F_a(s) \sigma_s x ds, \quad a > 0. \]

We will show that \(x_a\) is of exponential type, and that \(x_a \to x\) \(\sigma\)-weakly for \(a \to \infty\).

For \(t \in \mathbb{R}\):

\[ \sigma_t x_a = \int_{-\infty}^{\infty} F_a(s) \sigma_{s+t} x ds = \int_{-\infty}^{\infty} F_a(s - t) \sigma_s x ds. \]

One can easily check that the function

\[ \alpha \to \int_{-\infty}^{\infty} F_a(s - \alpha) \sigma_s x ds \]

is analytic from \(\mathbb{C}\) to \(M\). Hence \(x_a\) is an analytic element and

\[ \sigma_a x_a = \int_{-\infty}^{\infty} F_a(s - a) \sigma_s x ds \quad \forall \alpha \in \mathbb{C}. \]

By (*** we get

\[ \|\sigma_a x_a\| \leq e^{a|\text{Im}\alpha|} \|x\| \]

which proves that \(x_a\) is of exponential type for any \(a > 0\). For \(\phi \in M_*\) we have by (*):

\[ \phi(x_a - x) = \int_{-\infty}^{\infty} F_a(s) \phi(s) \phi(\sigma_s x - x) ds. \]

Since \(s \to \phi(\sigma_s x - x)\) is continuous and bounded we get by (**) that \(\phi(x_a - x) \to 0\) for \(a \to \infty\). Therefore \(x_a \to x\) \(\sigma\)-weakly for \(a \to \infty\).

**Lemma 4.3.** Let \(t \to \sigma_t\) be a \(\sigma\)-weak continuous group of isometries of a von Neumann algebra \(M\), and let \(B = \sigma_{-1}\) be the analytic generator. For \(x \in M\):

1. \(x\) is analytic iff \(x \in \bigcap_{n \in \mathbb{Z}} D(B^n)\)
(2) \( x \) is of exponential type iff \( x \in \bigcap_{n \in \mathbb{Z}} D(B^n) \) and there exists a constant \( c > 0 \) such that \( \sup_{n \in \mathbb{Z}} \| B^n x \| e^{-c|n|} < \infty \).

**Proof.** (1) is trivial because \( B^n = \sigma_{-in}, \ n \in \mathbb{Z} \).

(2) Assume that \( x \in \bigcap_{n \in \mathbb{Z}} D(B^n) \) and \( K = \sup_{n \in \mathbb{Z}} \| B^n x \| e^{-c|n|} < \infty \).

By a sharpened form of the Phragmen–Lindelöf principle known as the “three line theorem” (cf. [20, p. 93]) it follows that the function

\[
t \mapsto \| \sigma_{it} x \| = \sup_{s \in \mathbb{R}} \| \sigma_{s+it} x \|
\]

has convex logarithm.

Therefore

\[
\| \sigma_{in} x \| \leq Ke^{c|n|} \quad n \in \mathbb{Z}
\]

implies that

\[
\| \sigma_{it} x \| \leq Ke^{c|t|} \quad t \in \mathbb{R}
\]

Hence \( \| \sigma_{s+it} x \| = \| \sigma_{it} x \| \leq Ke^{c|t|}, s + it \in \mathbb{C} \), which proves that \( x \) is of exponential type. The converse implication is trivial.

**Lemma 4.4.** Let \( M \) and \( N \) be von Neumann algebras \( N \subseteq M \), and let \( \sigma_t^1 \) and \( \sigma_t^2 \) be \( \sigma \)-weakly continuous one parameter groups of isometries of \( M \) and \( N \) respectively.

If \( \sigma_t^2 \subseteq \sigma_t^1 \) then \( \sigma_t^1(x) = \sigma_t^2(x) \ \forall x \in N \ \forall t \in \mathbb{R} \).

**Proof.** Let \( B_1 \) and \( B_2 \) be the analytic generators of \( \sigma_t^1 \) and \( \sigma_t^2 \) respectively. Assume that \( B_2 \subseteq B_1 \). By Lemma 4.3 it is easily seen that if \( x \in N \) is of exponential type with respect to \( \sigma_t^2 \) then it is also of exponential type with respect to \( \sigma_t^1 \). Moreover

\[
\sigma_{-in}^1 x = \sigma_{-in}^2 x, \quad x \in N_{\text{exp}}, \ n \in \mathbb{Z}.
\]

For a given \( x \in N_{\text{exp}} \subseteq M_{\text{exp}} \) the difference \( \sigma_a^1 x - \sigma_a^2 x, \ \alpha \in \mathbb{C} \) satisfies an inequality of the form

\[
\| \sigma_a^1 x - \sigma_a^2 x \| \leq c_1 \exp(c_2 | \text{Im} \alpha |) \quad c_1, c_2 > 0.
\]

Hence by Carlson’s lemma (cf. [1, Proposition 1.3])

\[
\sigma_a^1 x = \sigma_a^2 x, \quad x \in N_{\text{exp}}, \ \alpha \in \mathbb{C}.
\]

In particular \( \sigma_t^1 x = \sigma_t^2 x, x \in N_{\text{exp}}, \ t \in \mathbb{R} \). Since \( N_{\text{exp}} \) is \( \sigma \)-weakly dense in \( N \) the lemma is proved.
**Remark.** The method used in Lemma 4.2-4.4 would provide a simple proof for the fact that a uniformly bounded continuous one parameter group of operators on a Banach space is determined by its analytic generator (cf. [1, Theorem 4.4]).

**Lemma 4.5.** Let $M$, $N$ be von Neumann algebras, $N \subseteq M$. Let $T$ be an n.f.s. operator valued weight from $M$ to $N$, and let $\mathcal{T}$ be the associated linear map $m_T \to N$. We define a linear map $R$ from $m_T \otimes F_2$ to $N \otimes F_2$ by

$$R(\Sigma x_{ij} \otimes e_{ij}) = \Sigma T(x_{ij}) \otimes F_2, \quad x_{ij} \in m_T$$

($F_2 = 2 \times 2$-matrices over $\mathbb{C}$). Then

1. $m_T \otimes F_2$ is a two sided module over $N \otimes F_2$ and

$$R(abx) = aR(x)b, \quad x \in m_T \otimes F_2, \quad a, b \in N \otimes F_2$$

2. $m_T \otimes F_2$ is spanned by its positive part. For

$$x \in m_T \otimes F_2 : x \geq 0 \Rightarrow R(x) \geq 0.$$

**Proof.** (1) follows from an elementary calculation.

(2) It follows from the polarization identity and the formula $m_T \otimes F_2 = (n_T \otimes F_2)^* (n_T \otimes F_2)$ that $m_T \otimes F_2$ is spanned by its positive part.

For $\phi \in N_+$ we put $\theta = \phi \otimes \text{tr}$ and $\overline{\theta} = (\phi \circ T) \otimes \text{tr}$, where $\text{tr}$ is the trace on $F_2$. Note that

$$\overline{\theta}(x) < \infty \quad \text{for any } x \in (m_T \otimes F_2)_+. $$

Hence $m_\theta$ contains span$(m_T \otimes F_2)_+ = m_T \otimes F_2$.

For $x \in m_T \otimes F_2$, $x = \Sigma x_{ij} \otimes e_{ij}$ we get

$$\overline{\theta}(x) = \phi \circ T(x_{11}) + \phi \circ T(x_{22}) = \theta \circ R(x).$$

Assume now that $x \in m_T \otimes F_2$ and $x \geq 0$. Let $(\pi_\theta, H_\theta, \xi_\theta)$ be the representation induced by the functional $\theta$ on $N \otimes F_2$. For any $a \in N \otimes F_2$:

$$(\pi_\theta(R(x)) a \xi_\theta \mid a \xi_\theta) = \theta(a^* R(x) a) = \theta \circ R(a^* xa) = \overline{\theta}(a^* xa) \geq 0.$$
Hence $\pi_\phi(R(x)) \geq 0$. Since $\{\phi \otimes \text{tr} : \phi \in \mathcal{N}_+\}$ is a separating family of states on $N \otimes F_2$ we conclude that $R(x) \geq 0$.

**Remark.** In [10, Section 5] we will be able to define the tensor product of two n.f.s. operator valued weights. Using this, it can be proved that the map $R$ in Lemma 4.5 has the form $R = \hat{S}$ where $S$ is the operator valued weight $T \otimes i$ from $M \otimes F_2$ to $N \otimes F_2$ ($i$ is identity on $F_2$).

**Lemma 4.6.** Let $M$ and $N$ be von Neumann algebras, $N \subseteq M$, $\phi, \psi \in P(N)$ and $T \in P(M, N)$. Put $\tilde{\phi} = \phi \circ T$, $\tilde{\psi} = \psi \circ T$. If $x \in (n_\phi \cap n_T)^*(n_\psi \cap n_T)$ then $Tx \in n_\phi^* n_\psi$.

**Proof.** Note first that $Tx$ makes sense because $x \in n_\phi^* n_T = m_T$. It is enough to consider the case where $x$ has the form $x = y^* z$, $y \in n_\phi \cap n_T$, $z \in n_\psi \cap n_T$.

Put $x_{11} = y^* y$, $x_{12} = y^* z = x$, $x_{21} = z^* y$, $x_{22} = z^* z$. Then $	ilde{x} = \sum x_{ij} \otimes e_{ij} \in m_T \otimes F_2$. Moreover $\tilde{x} \geq 0$ because $\tilde{x} = (y \otimes e_{11} + z \otimes e_{12})(y \otimes e_{11} + z \otimes e_{12})$. Put $R(\tilde{x}) = \Sigma T(x_{ij}) \otimes e_{ij}$. By the Lemma 4.5 (2) $R(\tilde{x}) \geq 0$. Define a weight $\theta$ on $N \otimes F_2$ by

$$\theta(\Sigma y_{ij} \otimes e_{ij}) = \phi(y_{11}) + \psi(y_{22}),$$

then $\theta(R(\tilde{x})) = \phi(Tx_{11}) + \psi(Tx_{22}) = \tilde{\phi}(y^* y) + \tilde{\psi}(z^* z) < \infty$. Hence $R(\tilde{x}) \in m_\theta$.

However, $m_\theta$ has the form

$$m_\theta = m_\phi \otimes e_{11} + n_\phi^* n_\psi \otimes e_{12} + n_\psi^* n_\phi \otimes e_{21} + m_\psi \otimes e_{22}.$$

Considering the $e_{12}$-komponent of $R(\tilde{x})$ it follows that

$$Tx = Tx_{12} \in n_\phi^* n_\psi.$$

**Theorem 4.7.** Let $M$ and $N$ be von Neumann algebras, $N \subseteq M$, and let $T \in P(M, N)$, then

1. $\sigma_\phi^{\phi \circ T}(x) = \sigma_\phi^{\phi}(x)$ \quad $\phi \in P(N)$, \quad $x \in N$

2. $(D\phi \circ T : D\phi \circ T)_t = (D\phi : D\phi)_t$ \quad $\phi, \psi \in P(N)$, \quad $t \in \mathbb{R}$.

**Proof.** Since $(D\phi : D\phi)_t = \sigma_\phi^{\phi \circ T}(1)$ it is enough to prove that

$$\sigma_t^{\phi \circ T, \phi \circ T}(x) = \sigma_t^{\phi \circ T}(x)$$ \quad $\phi, \psi \in P(N)$, \quad $x \in N$, \quad $t \in \mathbb{R}$.

By Lemma 4.4 this can be done by proving that

$$\sigma_t^{\phi \circ T} \subseteq \sigma_t^{\phi \circ T, \phi \circ T}$$ \quad $\phi, \psi \in P(N)$.
It might be instructive first to treat the case where $\phi$, $\psi$ and $T$ are bounded:

Let $a, b \in N$. Then by corollary 3.4:

$$(a, b) \in G(\sigma_{i-T}^{b})$$

$$\Rightarrow \psi(ax) = \phi(xb) \quad \forall x \in N$$

$$\Rightarrow \psi(a(Tx)) = \phi((Tx)b) \quad \forall x \in M$$

$$\Rightarrow \psi \circ T(ax) = \phi \circ T(xb) \quad \forall x \in M$$

$$\Rightarrow (a, b) \in G(\sigma_{i-T}^{b})$$.

Let us return to the general case, $T \in P(M, N)$, $\psi \in P(N)$. Put $\tilde{\phi} = \phi \circ T$, $\tilde{\psi} = \psi \circ T$ and let $(a, b) \in G(\sigma^{\phi, \psi})$. Since $a \in D(\sigma^{\phi, \psi})$ and $b \in D(\sigma^{\phi, \psi})$ or equivalently $b^{*} \in D(\sigma^{\psi, \phi})$ since $\sigma^{\phi}(b^{*}) = \sigma^{\psi}(b)^{*}, \, t \in \mathbb{R}$. It follows from Lemma 3.3 that for some $k > 0$:

$$\phi(axa^{*}) \leq k^{2}\psi(x) \quad \text{and} \quad \psi(b^{*}xb) \leq k^{2}\phi(x) \quad x \in N_{+}.$$

Taking increasing limits we obtain that these inequalities are also valid for $x \in \hat{N}_{+}$. Hence

$$\tilde{\phi}(axa^{*}) \leq k^{2}\tilde{\psi}(x) \quad \text{and} \quad \tilde{\psi}(b^{*}xb) \leq k^{2}\tilde{\phi}(x) \quad x \in M_{+}.$$

Again by Lemma 3.3 $n_{g}a^{*} \subseteq n_{g}$ ($\Rightarrow a_{n^{*}} \subseteq n_{g}^{*}$) and $n_{g}b \subseteq n_{g}$. Moreover

$$\| A_{g}(ya^{*}) \| \leq k \| A_{g}(y) \| \quad y \in n_{g}$$  \hspace{1cm} (*)

$$\| A_{g}(xb) \| \leq k \| A_{g}(x) \| \quad x \in n_{g}$$  \hspace{1cm} (**) \hspace{1cm} \text{(*)}

We shall prove that $(a, b) \in g_{i-T}^{\phi, \psi}$. By Theorem 3.2 it is sufficient to prove that

$$\tilde{\phi}(ax) = \tilde{\phi}(xb) \quad x \in n_{g}^{*} n_{g}.$$  

Consider first an element $x_{0}$ of the form $x_{0} = y^{*}z_{0}$ where $y_{0} \in n_{g} \cap n_{T}$ and $z_{0} \in n_{g} \cap n_{T}$. Using that $n_{g}a^{*} \subseteq n_{g}$ and that $n_{T}$ is a right module over $N$ we get

$$ax_{0} = (ya^{*})^{*}z_{0} \in (n_{g} \cap n_{T})^{*}(n_{g} \cap n_{T}) \subseteq \text{span}(m_{T}^{+} \cap m_{g}^{+}).$$

Similarly

$$x_{0}b = y_{0}^{*}(z_{0}b) \in \text{span}(m_{T}^{+} \cap m_{g}^{+})$$

It follows from Lemma 4.6 that $x_{0} \in m_{T}$ and $T_{x_{0}} \in n_{g}^{*} n_{g}$.
Since \((a, b) \in G(\sigma^b, \phi)\) we get by Lemma 3.3 that

\[\psi \circ T(ax_0) = \psi(a(Tx_0)) = \phi((Tx_0) b) = \phi \circ T(x_0 b)\]

However, for any normal weight \(\omega\) on \(N\), \(\omega \circ T\) coincides with the linear extension of the weight \(\omega \circ T\) on span \((m^+ \cap m^\circ_{\omega, T})\). Therefore

\[\tilde{\psi}(ax_0) = \tilde{\phi}(x_0 b)\]  

(***)

Let now \(x = y^*z, y \in n^\delta, z \in n^\delta\). Since \(\phi \circ T(y^*y) < \infty\). We have \(p = 0\) in the spectral resolution of \(T(y^*y)\). Hence

\[T(y^*y) = \int_0^\infty \lambda \, d\lambda,\]

For any \(s > 0\) we have \(ye_s \in n^\delta \cap n_T\) because

\[T(e_s y^*ye_s) = e_s T(y^*y) e_s = \int_0^s \lambda \, d\lambda,\]

is bounded and

\[\phi \circ T(e_s y^*ye_s) = \phi \left(\int_0^s \lambda \, d\lambda\right) \leq \phi \left(\int_0^\infty \lambda \, d\lambda\right) < \infty.\]

Moreover

\[
\| A_\delta( ye_s - y) \|^2 = \phi \circ T(( y e_s - y)^*( y e_s - y)) = \phi((1 - e_s) T(y^*y)(1 - e_s)) = \phi \left(\int_0^\infty \lambda \, d\lambda\right) \to 0 \quad \text{for} \quad s \to \infty
\]

Similarly \(T(z^*z)\) has a spectral resolution of the form

\[T(z^*z) = \int_0^\infty \lambda \, df\]

where \(zf_s \in n^\delta \cap n_T\) for \(s > 0\) and

\[
\| A_\delta(zf_s - z) \|^2 \to 0 \quad \text{for} \quad s \to \infty.
\]

Using (*) we get \(A_\delta(ye_s a^*) \to A_\delta(ya^*)\) for \(s \to \infty\). Hence

\[
\tilde{\psi}(ax) = (A_\delta(ya^*) | A_\delta(z)) = \lim_{s \to \infty} (A_\delta(ye_s a^*) | A_\delta(zf_s)) = \lim_{s \to \infty} \tilde{\psi}(a(e_s zf_s)).
\]
Similarly using (**):
\[ \hat{\phi}(xb) = \lim_{s \to \infty} \hat{\phi}((e_s x s) b) \]

From (***) with \( x_0 = e_s x s = (ye_s)^*(x s) \), it follows that
\[ \hat{\phi}(ax) = \hat{\phi}(xb). \]

By linearity the formula is valid for any \( x \in n^*_+ n^*_+ \). Hence \( (a, b) \in G(\sigma_{iT}^{\phi \to T}, \phi) \). This completes the proof.

**Lemma 4.8.** Let \( M, N \) be von Neumann algebras, \( N \subseteq M \), and let \( T_1, T_2 \in P(M, N) \). If \( \phi \circ T_1 = \phi \circ T_2 \) for some n.f.s. weight \( \phi \) on \( N \), then \( T_1 = T_2 \).

**Proof.** Let \( \psi \in P(N) \) then by Theorem 4.7
\[ (D\phi \circ T_1 : D\phi \circ T_1)_t = (D\phi : D\phi)_t = (D\phi \circ T_2 : D\phi \circ T_2)_t \]
Hence by [3, Theorem 1.2.4] \( \psi \circ T_1 = \psi \circ T_2 \) for any n.f.s. weight on \( N \). Let \( \omega \in M^*_+ \). Choose a normal semifinite weight \( X \) with support \( 1 - [\omega] \) where \([\omega]\) is the support of \( \omega \). The weight \( \psi = X + \omega \) is a n.f.s. weight on \( N \). Moreover
\[ \omega(y) = \psi([\omega] y[\omega]) \quad \forall y \in N_+. \]
Taking increasing limits we get
\[ \omega(y) = \psi([\omega] y[\omega]) \quad \forall y \in \bar{N}_+. \]
Hence for \( x \in M_+ \):
\[ \omega \circ T_1(x) = \psi([\omega] T_1(x)[\omega]) = \psi \circ T_1([\omega] x[\omega]) = \omega \circ T_2(x). \]
Thus \( T_1(x) \) and \( T_2(x) \) define the same function on \( M^*_+ \) i.e. \( T_1(x) = T_2(x) \), \( \forall x \in M^*_+ \).

**Proposition 4.9.** Let \( M \) and \( N \) be von Neumann algebras, \( N \subseteq M \), let \( \phi \in P(N) \) and \( T \in P(M, N) \). Then
\[ T \circ \sigma_{i\phi}^{T}(x) = \sigma_{i\phi}^{T}(T(x)), \quad x \in M_+. \]

In particular the set \( m_T \) is \( \sigma_{i\phi}^{T} \)-invariant.

**Proof.** Note first that \( \sigma_{i\phi}^{T} \) has a natural extension to \( \bar{N}_+ \) given by \( \langle \omega, \sigma_{i\phi}^{T} x \rangle = \)
Theorem 4.8. \( T_1 = T \).

Hence by Lemma 4.8 \( T_1 = T \). Thus

\[
\sigma_T(x) = \sigma_T(x) \quad \forall x \in M_+.
\]

REFERENCES

2. F. Combes, Poids associé à une algèbre Hilbertienne à gauche, Compositio Math. 23 (1971), 47–77.