JOURNAL OF DIFFERENTIAL EQUATIONS 96, 170–184 (1992)

Melnikov Method and Transversal Homoclinic Points in the Restricted Three-Body Problem

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Received March 28, 1990; revised October 5, 1990

In this paper we show, by Melnikov method, the existence of the transversal homoclinic orbits in the circular restricted three-body problem for all but some finite number of values of the mass ratio of the two primaries. This implies the existence of a family of oscillatory and capture motion. This also shows the non-existence of any real analytic integral in the circular restricted three-body problem besides the well-known Jacobi integral for all but possibly finite number of values of the mass ratio of the two primaries. This extends a classical theorem of Poincaré [10]. Because the resulting singularities in our equation are degenerate, a stable manifold theorem of McGehee [7] is used. © 1992 Academic Press, Inc.

1. INTRODUCTION

Sitnikov [13] proved the existence of a family of oscillatory and capture orbits for a special restricted three-body problem. Alekseev [1] generalized Sitnikov's idea and emphasized its relation with the transversal homoclinic orbits. He proved the same result for all non-zero masses. See Moser [9] for a detailed discussion on this example. For other examples of capture and oscillatory solutions, see [3, 8, 12].

In this paper we show the existence of the transversal homoclinic orbits in the restricted three-body problem to a periodic orbit at the infinity, for all but finite number of values of the mass ratio of the two primaries. By the symbolic dynamics, such transversal homoclinic orbits produce the Smale horseshoe and hence chaos in the restricted three-body problem. Moreover, it produces uncountably many oscillatory and capture orbits in the same way as in the Sitnikov problem. Another consequence of the presence of the transversal homoclinic orbits is the non-integrability of the restricted three-body problem. This extends a classical theorem of Poincaré

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0022-0396/92 $3.00

[10] which states that there is no real analytic integral, besides the known Jacobi integral, which is also analytic in the masses of the two primaries.

The periodic orbit at the infinity in this problem is degenerate, so the standard stable manifold theorem does not apply. Instead, we use a theorem due to McGehee [7] and generalized by Easton [2] and Robinson [11]. We use the perturbation technique of Melnikov to show the existence of transversal homoclinic orbits. The small perturbation parameter we are going to use is the mass ratio of the two primaries. If one primary has mass zero, the system reduces to a two-body problem and therefore is integrable. If the mass of one primary is small compared to the mass of the other primary, Melnikov method shows the existence of the transversal homoclinic orbits to the periodic orbit at infinity. Then, by the analyticity of the stable manifold, the result is extended to all other mass ratios with probably some exceptions at some finite number of points.

The author is grateful to both the referee and J. Llibre for pointing out that a similar work has been done by Llibre and Simó [5, 6]. In [5], Llibre and Simó showed that for sufficiently large values of Jacobi constant C and sufficiently small values of the mass ratio of the two primaries, there exists transversal homoclinic orbits to the periodic orbit at the infinity. In this work, we are able to show that this is also true for sufficiently small mass ratio with the Jacobi constant close to $\pm\sqrt{2}$. As compared to their method in [5, 6], ours is clearer and simpler. Furthermore, we are able to extend this result, by analyticity of the concerned manifolds and the symplectic nature of the problem, to almost all values of the mass ratio of the two primaries with sufficiently large Jacobi constant.

The author also thanks Professor K. Meyer for pointing out a crucial error in the original manuscript.

2. The Restricted Three-Body Problem

We consider three point masses P_1, P_2, P_3 moving in the Euclidean plan $\mathbb{R}^2$ under Newton's gravitation. Assume the center of mass is fixed at the origin and let the masses of P_1 and P_2 be μ and $1-\mu$, respectively, where $0 \leqslant \mu \leqslant 1$. Further assume that the particle P_3 has zero mass, then the resulting mechanical system is the so-called restricted three-body problem. Since P_3 has no influence on the motion of P_1 and P_2, this two particle system defines a two-body problem and the orbits of P_1 and P_2 can be completely understood: they either move in circular, elliptical, parabolic, or hyperbolic orbits. Assume that P_1 and P_2 move in a circular orbit, then the system is called the circular restricted three-body problem or simply, the restricted three-body problem.

Let $\mathbf{q}=(q_1, q_2)$ be the position of P_3 and $\mathbf{p}=(p_1, p_2)=(q_1', q_2')$ be the velocity of P_3, then the motion of P_3 can be determined by the equations

$$\begin{aligned} \mathbf{q}' &= \mathbf{p} \\ \mathbf{p}' &= U_{\mathbf{q}}, \end{aligned} \tag{1}$$

where U is the potential function and $U_{\mathbf{q}}$ is the gradient of U with respect to $\mathbf{q}$.

$$U=\frac{1-\mu}{\sqrt{(q_1-\mu x_{12})^2+(q_2-\mu y_{12})^2}}+\frac{\mu}{\sqrt{(q_1+(1-\mu)\,x_{12})^2+(q_2+(1-\mu)\,y_{12})^2}},$$

where x_{12}, y_{12} are the components on the x and y axes of the distance vector from P_1 to P_2. For the circular restricted three-body problem, we have

$$x_{12}=\cos t, \qquad y_{12}=\sin t.$$

Note that if $\mu=0$, the problem reduces to a two-body problem with one particle having mass 1 at the origin and the other particle with zero mass moving around. This problem is completely integrable. We will treat the restricted three-body problem as a perturbation problem for small values of μ. To the first order in μ, U can be expressed as

$$\begin{aligned} U=\frac{1}{\sqrt{q_1^2+q_2^2}}+\mu\Bigg(&-\frac{1}{(q_1^2+q_2^2)^{1/2}}+\frac{q_1\cos t+q_2\sin t}{(q_1^2+q_2^2)^{3/2}} \\ &+\frac{1}{((q_1+\cos t)^2+(q_2+\sin t)^2)^{1/2}}\Bigg)+O(\mu^2). \end{aligned} \tag{2}$$

The particular solution we are especially interested in for this two-body problem with $\mu=0$ is its parabolic solution. This is the solution where the zero mass particle approaches infinity with limiting velocity zero. We will show, by proper coordinate change, that there is a periodic solution at infinity and it turns out that this periodic solution is a degenerate saddle and the parabolic orbits are precisely the orbits that approach this period orbit both as $t\to\infty$ and as $t\to-\infty$. We will show that for small values of μ, above periodic orbit persists and some of these parabolic orbits turn into the transversal homoclinic orbits to the periodic solution.

Some coordinate changes are in order for putting the equation in the desired form and for making the singularity at infinity finite. The best scaling seems to be $|\mathbf{q}|=x^{-2}$. Define the angular variable $\mathbf{s}$ in S^1 by $\mathbf{q}=x^{-2}\mathbf{s}$. Decompose the momentum into radial and angular components by

$$\mathbf{p}=y\mathbf{s}+x^2\rho i\mathbf{s},$$

where is is the complex notation for the unit vector perpendicular to unit vector $\mathbf{s}$ and ρ is the angular momentum of P_3. Under these new variables,

$$U = x^2 + \mu x^2 \left(-1 + x^2 \cos(t-\theta) + \frac{1}{(1+2x^2\cos(t-\theta)+x^4)^{3/2}} \right) + O(\mu^2) \quad (3)$$

and the resulting equations are

$$\begin{aligned} x' &= -\tfrac{1}{2}x^3 y \\ y' &= -x^4 + x^6\rho^2 + \mu g_1(x, t-\theta) + O(\mu^2) \\ \theta' &= x^4\rho \\ \rho' &= \mu g_2(x, t-\theta) + O(\mu^2), \end{aligned} \quad (4)$$

where $\theta \in S^1$ is the angular variable such that $\mathbf{s} = (\cos\theta, \sin\theta)$ and $g_1(x, t-\theta)$, $g_2(x, t-\theta)$ are the first order perturbation terms,

$$\begin{aligned} g_1(x, t-\theta) &= x^4 \left(1 - 2x^2\cos(t-\theta) - \frac{1 + x^2\cos(t-\theta)}{(1 + 2x^2\cos(t-\theta) + x^4)^{3/2}} \right) \\ g_2(x, t-\theta) &= x^4 \sin(t-\theta) \left(1 - \frac{1}{(1 + 2x^2\cos(t-\theta) + x^4)^{3/2}} \right). \end{aligned} \quad (5)$$

The above system is in a five-dimensional space. However, the system has a first integral, called the *Jacobi integral*,

$$\tfrac{1}{2}y^2 + \tfrac{1}{2}x^4\rho^2 - U - \rho = C,$$

where C is called the *Jacobi constant*. We can use the above integral to solve for ρ, therefore, the equation for ρ can be dropped.

$$\begin{aligned} \rho &= \rho_0 - \frac{\mu x^2}{(1 - x^4\rho_0)} \left(-1 + x^2\cos(t-\theta) + \frac{1}{(1+2x^2\cos(t-\theta)+x^4)^{3/2}} \right) + O(\mu^2) \\ \rho_0 &= \frac{1 \pm \sqrt{1 - x^4(y^2 - 2x^2 - 2C)}}{x^4}, \end{aligned} \quad (6)$$

where the $\pm$ sign depends on the relative angular velocity of P_3.

Note that in the above equations, the time variable t always appears in the form of $t - \theta$. This fact can be used to further reduce the dimension of the system. Define a new angular variable s by

$$s = t - \theta, \qquad s \in S^1.$$

The system of equations now becomes

$$\begin{aligned} x' &= -\tfrac{1}{2}x^3y \\ y' &= -x^4 + x^6\rho^2 + \mu g_1(x, s) + O(\mu^2) \\ s' &= 1 - x^4\rho, \end{aligned} \tag{7}$$

where ρ is a function of x, y, s, and C, given by Eq. (6) above.

First we consider the case with $\mu = 0$, i.e., the unperturbed two-body system. The equations of motion reduce to

$$\begin{aligned} x' &= -\tfrac{1}{2}x^3y \\ y' &= -x^4 + x^6\rho^2 \\ s' &= 1 - x^4\rho, \end{aligned} \tag{8}$$

where ρ is now a constant (recall that ρ is the angular momentum of P_3). The equations for x, y are independent of s and can be solved explicitly. Let H be a function of x and y defined by

$$H(x, y, \rho) = \tfrac{1}{2}y^2 + \tfrac{1}{2}x^4\rho^2 - x^2. \tag{9}$$

It is easy to see that $H(x, y, \rho)$ is a constant of motion (in fact, H is the energy of P_3). Figure 1 shows the flow in x, y coordinates (Note that $x > 0$ corresponds to the physical space). The flow lines correspond to the level curves of H. The origin $(0, 0)$ is a singularity of the flow. This singularity is degenerate and after rescaling the time by a factor of x^{-3}, we see it is a hyperbolic saddle point and there is a homoclinic loop to this saddle point.

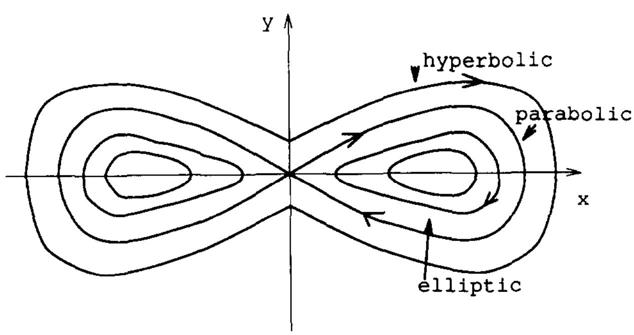

FIG. 1. The flow of the unperturbed problem.

Our later analysis will be based on this homoclinic loop. The orbit of the homoclinic loop can be found explicity to be the following:

$$x(t)=\xi(t, C)=\frac{\sqrt{2}}{\sqrt{(3t+\sqrt{9t^2+C^6})^{2/3}+(3t-\sqrt{9t^2+C^6})^{2/3}-C^2}}$$
$$y(t)=\eta(t, C)=\begin{cases} \pm\sqrt{2\xi^2(t)-\xi^4(t)\,C^2} & \text{for } x\geqslant 0 \\ \mp\sqrt{2\xi^2(t)-\xi^4(t)\,C^2} & \text{for } x\leqslant 0 \end{cases} \tag{10}$$

where the $\pm$ depends on the sign of C. Note that $\xi(t, C)$ is an even function of t and $\eta(t, C)$ is an odd function of t.

Because of the time independence of the equations, for any $t_0\in\mathbb{R}$, $(\xi(t-t_0, C),\ \eta(t-t_0, C))$ is also a homoclinic loop to the degenerate saddle at the origin. It would be nice if $\xi(t, C)$ and $\eta(t, C)$ can be expressed explicitly in terms of s, because eventually we will use s as the independent time variable. It seems that the equations are very complicated and it is not likely that the solutions can be put into closed forms. We will let $x_0(s)$ and $y_0(s)$ be the homoclinic loop expressed in terms of s, i.e., $x_0(s)=\xi[t(s), C]$ and $y_0(s)=\eta[t(s), C]$. We choose the initial value of θ to be 0. In doing so, we have that $x_0(s)$ is an odd function and $y_0(s)$ is an even function in s.

Now we consider the case where $0<\mu\ll 1$. First we eliminate the equation for s by rescaling the equations for x and y. The resulting equations are then periodically time dependent.

$$\frac{dx}{ds}=\frac{-\frac{1}{2}x^3y}{1-x^4\rho}$$
$$\frac{dy}{ds}=\frac{-x^4+x^6\rho^2}{1-x^4\rho}+\frac{\mu g_1(x, s)}{1-x^4\rho}+O(\mu^2), \tag{11}$$

where again, ρ is a function of x, y, s, and the Jacobi constant C determined by the Jacobi Integral (6).

Note that we need $1-x^4\rho\neq 0$ for using s as the independent time variable. This is always true for x small or for $\rho<0$.

For $\mu\neq 0$, the original fixed point at the origin now becomes a periodic point with a period 2π in s. The orbits which asymptotically approach this periodic point as $t\to\infty$ are called *ω-parabolic* and similarly, the orbits approach this periodic orbit asymptotically as $t\to-\infty$ are called *α-parabolic*. The periodic orbit is again degenerate. Because the equations are time dependent, one cannot rescale the time variable by the factor x^{-3} to remove the degeneracy. Hence, the standard stable manifold theorem does not apply to this periodic orbit and it is not obvious whether these ω-parabolic orbits and α-parabolic orbits form a smooth manifold.

However, a stable manifold theorem of McGehee [7] for the degenerate fixed point does assure the smooth manifold structure for these ω-parabolic and α-parabolic orbits.

THEOREM 1. *The stable and unstable manifolds of the periodic orbit* $\gamma: x=0,\ y=0,\ s\in S^1$ *in* $\{x\geqslant 0\}$ *for Eq.* (11) *are real analytic manifold for* $x>0$. *That is,* $W^s(\gamma)\cap\{x>0\}$ *and* $W^u(\gamma)\cap\{x>0\}$ *are real analytic. In other words, for fixed Jacobi constant C, the* ω*-parabolic orbits and* α*-parabolic orbits form real analytic submanifolds of the phase space.*

McGehee [7] proved a stable manifold theorem for a class of degenerate fixed points. This theorem was proved by McGehee as an application to his stable manifold theorem. Recently, Robinson [11] generalized McGehee's theorem and proved a stable manifold theorem which applies to the more general degenerate invariant sets. See Robinson [11] for more detailed discussion.

We remark that, due to the binary collisions of the third particle with the two primaries, there are some singularities for Eq. (11). However, as is well known, these binary collisions can be regularized, i.e., these singularities can be removed by appropriate change of cordinates and the time variable. The analyticity of the stable and unstable manifolds in Theorem 1 are in the sense that they are real analytic after the binary collisions have be regularized.

We also point out that from the proof of Theorem 1, the stable manifold and the unstable manifold are also real analytic with respect to the Jacobi costant C and the mass ratio μ for $\mu\neq 0$ (for $\mu=0$, the regularization of the binary collision is singular).

3. Melnikov Function

Now we can use the perturbation techniques of Melnikov to show the existence of transversal homoclinic orbits. Note that, for $\mu=0$, $W^s(\gamma)$ and $W^u(\gamma)$ are exactly the same manifold. For $\mu\neq 0$, these two manifolds split. Intuitively, these two manifolds always intersect one the other, as can be seen from the following:

Let P_3 be placed at x axis at $t=0$, we can adjust its velocity and its distance to the origin in such a way that it has the desired Jacobi constant and escapes to infinity with limiting velocity zero. So it is ω-parabolic. By symmetry, it must also be α-parabolic, which implies it is a homoclinic orbit of γ.

We will show that this indeed gives us a homoclinic orbit in the phase space, and moreover, it is transversal for some Jacobi constants.

Let $\sum^{s_0}$ be the cross section $s = s_0$ in the phase space and let $(x_\mu^u(s, s_0), y_\mu^u(s, s_0))$ be the orbits lying in $W^u(\gamma)$ such that $((x_\mu^u(0, s_0), y_\mu^u(0, s_0)) \in \sum^{s_0}$. Similarly let $(x_\mu^s(s, s_0), y_\mu^s(s, s_0))$ be the orbits lying in $W^s(\gamma)$ such that $(x_\mu^s(0, s_0), y_\mu^s(0, s_0)) \in \sum^{s_0}$, then we have the following lemma:

LEMMA 1. *The following approximations hold with uniform validity in the indicated time intervals for any fixed C such that* $|C| > \sqrt{2}$,

$$\begin{aligned}
x_\mu^s(s, s_0) &= x_0(s - s_0) + \mu x_1^s(s, s_0) + O(\mu^2), && s \in [s_0, \infty) \\
y_\mu^s(s, s_0) &= y_0(s - s_0) + \mu y_1^s(s, s_0) + O(\mu^2), && s \in [s_0, \infty) \\
x_\mu^u(s, s_0) &= x_0(s - s_0) + \mu x_1^u(s, s_0) + O(\mu^2), && s \in (-\infty, s_0] \\
y_\mu^u(s, s_0) &= y_0(s - s_0) + \mu y_1^u(s, s_0) + O(\mu^2), && s \in (-\infty, s_0],
\end{aligned} \tag{12}$$

where The functions $x_1^s(s, s_0)$, $y_1^s(s, s_0)$ *and* $x_1^u(s, s_0)$, $y_1^u(s, s_0)$ *above are determined by the first variational equation of Eq.* (11) *along the unperturbed orbit* (x_0, y_0).

The proof of this lemma follows easily from the following two observations: (1) the standard Gronwall estimate shows that the perturbed orbits starting with $O(\mu)$ of $(x_0(0), y_0(0))$ remain within $O(\mu)$ of $(x_0(s), y_0(s))$ in arbitrary but finite times; (2) from Theorem 1, $W^u(\gamma)$, $W^s(\gamma)$ are C^∞ manifolds and the perturbed manifold is C^∞ close to the unperturbed manifold.

We remark that the condition for $|C| > \sqrt{2}$ is nessaccery due to the fact that the perturbation for $\mu > 0$ is singular at the binary collisions and for $|C| \leqslant \sqrt{2}$, it may be possible that the orbits in the stable manifold and the unstable manifold experiences binary collisions.

Let $H(x, y) = \frac{1}{2} y^2 + \frac{1}{2} x^4 \rho^2 - x^2$ be the energy function defined in Eq. (9), where ρ is given by (6). For $\mu \neq 0$, H is no longer a constant of motion and

$$\frac{dH}{ds} = \frac{\mu(y g_1(x, s) + x^4 \rho g_2(x, s))}{1 - x^4 \rho} + O(\mu^2). \tag{13}$$

Let $\mathbf{d}(s_0)$ be the following vector:

$$\mathbf{d}(s_0) = (x_\mu^s(s_0, s_0), y_\mu^s(s_0, s_0)) - (x_\mu^u(s_0, s_0), y_\mu^u(s_0, s_0)).$$

This vector measures how far that $W^s(\gamma)$ and $W^u(\gamma)$ split at $s = s_0$ on $\sum^{s_0}$. It is difficult to find $\mathbf{d}(s_0)$ by integrating the first variational equation. Here we use the technique of Melnikov to find out how $W^s(\gamma)$ and $W^u(\gamma)$ intersect one another. Let $H^N(x_0(0), y_0(0))$ be the normal of $H(x, y)$ at the point $(x_0(0), y_0(0))$, i.e.,

$$H^N(x_0(0), y_0(0)) = (H_x(x_0(0), y_0(0)), H_y(x_0(0), y_0(0))).$$

Further let $d(s_0)$ be the projection of the vector $\mathbf{d}(s_0)$ into the normal direction of the function $H(x, y)$ at the point $(x_0(0), y_0(0))$. Then to the first order approximation,

$$\begin{aligned} d(s_0) &= \frac{H^N(x_0(0), y_0(0))}{|H^N(x_0(0), y_0(0))|} \\ &\quad \cdot (x^s_\mu(s_0, s_0) - x^u_\mu(s_0, s_0), y^s_\mu(s_0, s_0) - y^u_\mu(s_0, s_0)) \\ &= \mu \frac{H^N(x_0(0), y_0(0))}{|H^N(x_0(0), y_0(0))|} \\ &\quad \cdot (x^s_1(s_0, s_0) - x^u_1(s_0, s_0), y^s_1(s_0, s_0) - y^u_1(s_0, s_0)) + O(\mu^2), \end{aligned}$$

where $x^s_1(s, s_0)$, $y^s_1(s, s_0)$, $x^u_1(s, s_0)$, $y^u_1(s, s_0)$ are given by Lemma 1.

$d(s_0)$ approximates the separation of the manifold $W^u(\gamma)$, $W^s(\gamma)$ on the section $\sum_{s_0}$ at the point $(x_0(0), y_0(0))$. To find its value, observe that, by Lemma 1

$$\begin{aligned} &H(x^s_\mu(s_0, s_0), y^s_\mu(s_0, s_0)) \\ &\qquad = H(x_0(0), y_0(0)) + \mu H^N(x_0(0), y_0(0)) \cdot (x^s_1(s_0, s_0), y^s_1(s_0, s_0)) + O(\mu^2) \\ &H(x^u_\mu(s_0, s_0), y^u_\mu(s_0, s_0)) \\ &\qquad = H(x_0(0)\, y_0(0)) + \mu H^N(x_0(0), y_0(0)) \cdot (x^u_1(s_0, s_0), y^u_1(s_0, s_0)) + O(\mu^2). \end{aligned}$$

Therefore,

$$\begin{aligned} &d(s_0)\, |H^N(x_0(0), y_0(0))| \\ &\qquad = H(x^s_\mu(s_0, s_0), y^s_\mu(s_0, s_0)) - H(x^u_\mu(s_0, s_0), y^u_\mu(s_0, s_0)) + O(\mu^2) \\ &\qquad = \int_{-\infty}^{\infty} \frac{dH(x, y)}{ds}\, ds + O(\mu^2) \qquad (14) \\ &\qquad = \mu M(s_0) + O(\mu^2), \end{aligned}$$

where $M(s_0)$ is the *Melnikov function* and it is given by

$$\begin{aligned} M(s_0) &= \int_{-\infty}^{\infty} \mu^{-1} \frac{dH}{ds}(x_0(s - s_0), y_0(s - s_0))\, ds \\ &= \int_{-\infty}^{\infty} \frac{y g_1(x_0(s - s_0), s) + x_0^4(s - s_0)\, \rho g_2(x_0(s - s_0), s))}{1 - x_0^4(s - s_0)\, \rho}\, ds. \qquad (15) \end{aligned}$$

Note that $\rho = -C$ along the unperturbed homoclinic loop. In terms of the Melnikov function, we have

$$\begin{aligned} d(s_0) &= \mu \frac{M(s_0)}{|H^N(x_0(0), y_0(0))|} + O(\mu^2) \\ &= \mu \frac{M(s_0)}{|2x_0^3(0)\, C^2 - 2x_0(0)|} + O(\mu^2) \\ &= \mu \frac{C}{2\sqrt{2}} M(s_0) + O(\mu^2). \end{aligned} \tag{16}$$

The equation for the Melnikov function is too complicated and its value is hard to compute. Some simplifications are necessary. One easily checks that

$$\begin{aligned} \frac{g_1(x, s) + x^4 \rho g_2(x, s)}{1 - x^4 \rho} = \frac{g_2(x, s)}{1 - x^4 \rho} + \frac{d}{ds}\Bigg(x^2 \Bigg(-1 + x^2 \cos(t-\theta) \\ + \frac{1}{(1 + 2x^2 \cos(t-\theta) + x^4)^{3/2}} \Bigg)\Bigg) + O(\mu), \end{aligned}$$

where ρ and ρ_0 are in terms of $x(s)$, $y(s)$, and s. Using the above equation, the Melnikov integral reduces to the following simple form:

$$\begin{aligned} M(s_0) &= \int_{-\infty}^{\infty} \frac{g_2(x(s - s_0), s)}{1 - x_0^4(s - s_0)\, \rho_0} \, ds \\ &= \int_{-\infty}^{\infty} \frac{g_2(x(s), s + s_0)}{1 - x_0^4(s)\, \rho_0} \, ds \\ &= \int_{-\infty}^{\infty} g_2(x_0(s), s + s_0) \, dt \\ &= \int_{-\infty}^{\infty} x_0^4(s) \sin(s + s_0) \left(1 - \frac{1}{(1 + 2x_0^2(s) \cos(s + s_0) + x_0^4)^{3/2}} \right) dt. \end{aligned} \tag{17}$$

It is easy to see that, by symmetry, $M(s_0) = 0$ for $s_0 = \pi$. To show that $s_0 = \pi$ is a simple zero for the Melnikov function $M(s_0)$, we need to show that $M'(s_0)|_{s_0 = \pi} \neq 0$. Direct computation shows that

$$\begin{aligned} M'(\pi) = \int_{-\infty}^{\infty} & x_0^4(s) \cos(s + \pi) \, dt \\ & - \int_{-\infty}^{\infty} \frac{x_0^4(s) \cos(s + \pi)}{(1 + 2x_0^2(s) \cos(s + \pi) + x_0^4(s))^{3/2}} \, dt \\ & - \int_{-\infty}^{\infty} \frac{3x_0^6 \sin^2(s + \pi)}{(1 + 2x_0^2(s) \cos(s + \pi) + x_0^4(s))^{5/2}} \, dt. \end{aligned} \tag{18}$$

In the above integral, $x_0(s) = \xi(t, C)$ is given by (10). And s can be found by the integral

$$s = t - \theta = t - \int_0^t x^4 \rho \, dt = t + \int_0^t \xi^4(t, C)\, C \, dt.$$

Observe that for $C = \pm\sqrt{2}$, the Melnikov integral $M(s_0)$ and $M'(\pi)$ has a singularity at $t = 0$. This is because $1 + 2\xi^2 \cos(s + \pi) + \xi^4 = 0$ for $t = 0$ and $C = \pm\sqrt{2}$. And since $1 + 2\xi^2 \cos(s + \pi) + \xi^4$ is real analytic at $t = 0$, we see that $M'(0) \to -\infty$ as $C \to \pm\sqrt{2}$ with $|C| > \sqrt{2}$. Therefore we conclude that $M'(\pi) \neq 0$ as long as $|C| > \sqrt{2}$ and $|C| - \sqrt{2}$ sufficiently small. This implies that, for these values of C, and for μ sufficiently small, the stable manifold and the unstable manifold of γ, $W^s(\gamma)$, and $W^u(\gamma)$, intersect transversely. This shows the existence of the transversal homoclinic orbits for the restricted three-body problem for small values of μ.

Next, we use the analyticity of $W^s(\gamma)$ and $W^u(\gamma)$ to extend above results to show the existence of transversal homoclinic orbits for all but some finite number of values of μ.

Let us fix the cross section $\sum^{s0}$ with $s_0 = \pi$. By symmetry, $W^s(\gamma)$ is precisely the reflection of $W^u(\gamma)$ along the x axis. It is easy to see that $W^u(\gamma)$ and $W^s(\gamma)$ intersect at the x axis for sufficiently large values of $|C|$ for any fixed μ (note that for large values of $|C|$, the system is close to a two-body problem). The restricted three-body system is a conservative system, therefore, the Poincaré map is area-preserving. By the Lagrangian intersection theory (see [14]), $W^u(\gamma)$ and $W^s(\gamma)$ must intersect one another at the x axis for all values of μ and C. Let $p(\mu, C)$ be the point of intersection and let $k(\mu, C)$ be the slope of the tangent line of $W^s(\gamma)$ at $p(\mu, C)$. Then by symmetry, the slope of the tangent line of $W^u(\gamma)$ at $p(\mu, C)$ is -$k(\mu, C)$. If $k(\mu, C) \neq \infty$ or 0, then the intersection of $W^u(\gamma)$ and $W^s(\gamma)$ at $p(\mu, C)$ is transversal.

It follows from Theorem 1 that $W^u(\gamma)$ and $W^s(\gamma)$ are real analytic submanifolds and they are also real analytic with respect to parameters C and μ. Therefore, $k(\mu, C)$ is a real analytic function whenever $k(\mu, C) \neq 0$ (i.e., whenever $W^u(\gamma)$ intersects the x axis transversally). For $|C|$ sufficiently large or for μ sufficiently small, it is obvious that $k(\mu, C) \neq 0$, therefore, $k(\mu, C)$ is a real analytic function in C and μ for $|C|$ sufficiently large or for μ sufficiently small.

We have shown, by the Melnikov function, that for μ sufficiently small and for $|C| > \sqrt{2}$ small, $W^u(\gamma)$ and $W^s(\gamma)$ intersects transversally, i.e., $k(\mu, C) \neq 0$ and $k(\mu, C) \neq \infty$ for $|C| > \sqrt{2}$ small and μ sufficiently small. Therefore, for almost all values of μ and C, $k(\mu, C) \neq 0$ and $k(\mu, C) \neq \infty$, in the region where $|C|$ is sufficiently large or μ is sufficiently small. Therefore, in this region, for almost all values of C and μ, $W^u(\gamma)$, and $W^s(\gamma)$ intersects transversally.

Now, let us fix $C = C^*$ large. It follows from above that we can find some sufficienly large C^* such that there is some μ with $k(\mu, C^*) \neq \infty$ and $k(\mu, C^*) \neq 0$. By analyticity of $k(\mu, C)$, we see that $k(\mu, C^*) \neq \infty$ and $k(\mu, C^*) \neq 0$ for all $\mu \in [0, 1]$ except for at most some finite number of points.

We conclude this section by stating the following theorem:

THEOREM 2. *For all but some finite number of values of mass ratio of the primaries, μ, there is some Jacobi constant such that the stable and unstable manifolds of the periodic orbit γ, $W^s(\gamma)$, and $W^u(\gamma)$ intersect transversely.*

In what follows, we describe some of the consequences of the existence of transversal homoclinic orbits in the retricted three-body problem.

4. CAPTURE AND OSCILLATORY ORBITS

By the Smale–Birkhoff homoclinic theorem, the existence of the transversal homoclinic orbits of a periodic solution implies the presence of horseshoe maps and hence chaos. Among other things, this implies the existence of infinitely many periodic orbits with some very large period (we remark that the periodic points we find here may not correspond to the periodic solutions of the original system of the restricted three-body problem. This is because here we have used the angular variable s as the new time variable to reduce the dimension of the system. The periodic points thus found usually correspond to the quasi-periodic solutions of the original restricted three-body problem). We will show that the presence of the transversal homoclinic orbits here also implies the existence of infinitely many capture and oscillatory orbits. Recalling that in the restricted three-body problem, an orbit is said to be *oscillatory* if $\limsup r = \infty$ and $\liminf r < \infty$, as $t \to \pm\infty$, where r is the distance of P_3 from the origin: $r = \sqrt{q_1^2 + q_2^2} = x^{-2}$. And an orbit is said to be a *capture* orbit if $r \to \infty$ as $t \to \pm\infty$ and $\limsup r < \infty$ as $t \to \mp\infty$.

We follow Moser [9] to establish the necessary symbolic dynamics near the homoclinic points to show the existence of oscillatory motion.

Take a cross section $\Gamma = \sum^{s_0}$ for a fixed s_0. For large values of C, the first return map on Γ is well defined. Let p be the transversal homoclinic point from above theorem and let R be a small rectangle with two boundaries being parts of $W^s(p)$ and $W^u(p)$ (see Fig. 2). Let ϕ be the first return map wherever defined.

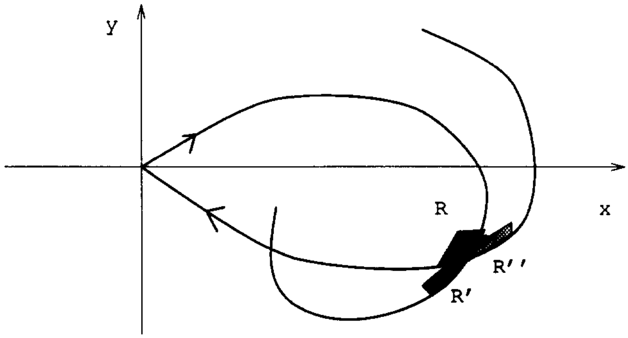

FIG. 2. The homoclinic points.

For a point $q \in R$, let $k = k(q)$ be the smallest positive integer for which $\phi^k(q) \in R$, if it exists. Let the set of all $q \in R$ for which such a $k > 0$ exists by D. We set

$$\tilde{\phi}(q) = \phi^k(q) \qquad \text{for all} \quad q \in D.$$

Moser calls this map $\tilde{\phi}$ the *transverse* map of ϕ for R. It is easy to see that D is a nonempty set and moreover we can embed a shift homeomorphism with infinitely many symbols in D.

THEOREM 3. *There is an invariant subset $I \subset D$ for the transverse map $\tilde{\phi}$, homeomorphic to the set $S = N^Z$ (the space of bi-infinite sequences on infinitely many symbols) and the transverse map $\tilde{\phi}$ on I is topologically conjugate to the shift map on S. In other words, let τ be the homemorphic $\tau : S \to I$ and σ be the shift map of S, we have*

$$\tilde{\phi}\tau = \tau\sigma.$$

For a proof of this theorem, see [9].

This theorem differs a little from the standard Smale–Birkhoff homoclinic theorem in that the shift space here has infinitely many symbols. Here infinitely many symbols are needed to show the existence of the oscillatory solutions. Note that for any $q \in I$, if $k(q)$ is large, then the orbit must pass very close to the origin. The next theorem follows immediately:

THEOREM 4. *Let I be the set given by Theorem 3 and let $q \in I$ be a point such that its corresponding symbol sequence in S is unbounded, then the corresponding orbit is oscillatory. Hence there are uncountable many points of I which correspond to the oscillatory solutions.*

Note that a sequence $s = (..., s_3, s_2, s_1, s_0, s_1, s_2, s_3, ...)$ is said to unbounded if sup $\{s_i, i \in Z\} = \infty$.

It is easy to construct the capture orbit. One only need consider the small rectangles R' and R'' as shown in Fig. 2.

5. Non-Integrability of the Restricted Three-Body Problem

The notion of integrability is of great interest historically. While others were trying to find other integrals for the restricted three-body problem, Poincaré (1989) [10] showed, remarkably, that the problem is "non-integrable" in the sense that there is no other real analytic integral which is also analytic in the masses of the two primaries, besides the well-known Jacobi integral. However, for any particular values of μ, it was not known whether it is integrable. Because the presence of transversal homoclinic points prevents the existence of any real analytic integral, we have the following theorem:

Theorem 5. *Besides the well-known Jacobi integral, there is no additional real analytic integral for the circular restricted three-body problem for any values of the mass ratio* $\mu \in [0, 1]$ *of the primaries excluding possibly some finite number of values.*

This theorem extends the classical theorem of Poincaré's to assert that for all but finite number of values of the ratio of the masses of P_1, and P_2, the restricted three-body problem is "non-integrable." The author believes that this theorem is true for all values of μ. In fact, the author thinks that for any fixed μ, there exist transversal homoclinic points to the period orbit γ for almost all Jacobi constants.

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