On Weak r-Monotonicity*

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ABSTRACT

In this paper the relationship between weak r-monotonicity and $\{1\}$-monotonicity is discussed. In particular, an affirmative answer to a question raised by Werner in 1977 is given.

1. INTRODUCTION AND PRELIMINARIES

From L. Collatz we have the following definition. A square real matrix A is said to be *monotone* if $Ax \geqslant 0 \Rightarrow x \geqslant 0$; this implication is equivalent to A being invertible with $A^{-1} \geqslant 0$ (see [4]). Several generalizations of this concept exist where A is singular and in general rectangular (see, e.g., [1–3, 6–12]).

The purpose of this paper is to study the relationship between the following two generalizations of matrix monotonicity.

DEFINITION 1.1 (Berman and Plemmons [3]). A real $m \times n$ matrix A is $\{1\}$-*monotone* if it has a nonnegative $\{1\}$-inverse X, i.e. the equation $AXA = A$ is solvable for some $X \geqslant 0$.

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DEFINITION 1.2 (Werner [9]). A real $m \times n$ matrix A of rank r is *weak-r-monotone* if A has a monotone submatrix of order r.

In Section 2 we shall be concerned with the following question: *Suppose that A has rank r; is A {1}-monotone if and only if A is weak-r-monotone?* This question has already been considered in [9]. It will be shown that the "if" part of this question is true, whereas the "only if" part is (in general) false. In Section 3 we shall look for an additional assumption under which the equivalence under study becomes true. Our main result in this respect will give an affirmative answer to a conjecture raised in [9] (see also [10]). Some further known results then follow as special cases.

Suppose that the matrix A is in the space $\mathbb{R}^{m,n}$ of all real $m \times n$ matrices and has rank r, and let B be an $r \times r$ nonsingular submatrix of A. Then there exist matrices $U \in \mathbb{R}^{m-r,r}$, $V \in \mathbb{R}^{r,n-r}$ and permutation matrices P and Q of orders m and n, respectively, such that

$$A = P\begin{pmatrix} I \\ U \end{pmatrix} B(I \quad V)Q, \tag{1.1}$$

where I denotes the identity matrix of order r. In what follows we shall use the following known results.

LEMMA 1.3 (see Werner [11, Satz 5.3 and Satz 5.4]). *Suppose $A \in \mathbb{R}^{m,n}$ is of rank r. Let B be a nonsingular submatrix of A of order r, and consider the representation (1.1) of A determined by B. Denote by $A\{1\}$ the set of all $\{1\}$-inverses of A. Then*

(i) $$A\{1\} = \left\{ Q'\left(\begin{array}{c|c} B^{-1} - YU - VX - VZU & Y \\ \hline X & Z \end{array} \right) P' \,\middle|\, X \in \mathbb{R}^{n-r,r},\ Y \in \mathbb{R}^{r,m-r},\ Z \in \mathbb{R}^{n-r,m-r} \right\},$$

and consequently,

(ii) $$\{A\}A\{1\} = \left\{ P\begin{pmatrix} I \\ U \end{pmatrix}(I - LU \quad L)P' \,\middle|\, L \in \mathbb{R}^{r,m-r} \right\},$$

(iii) $$A\{1\}\{A\} = \left\{ Q'\begin{pmatrix} I - VK \\ K \end{pmatrix}(I \quad V)Q \,\middle|\, K \in \mathbb{R}^{n-r,r} \right\},$$

where $\{A\}$ stands for the singleton set consisting of A.

As a direct consequence of Lemma 1.3 we mention

LEMMA 1.4 (see Werner [11, Satz 6.1]). *Let A and B be as in Lemma 1.3, and consider the representation (1.1) of A determined by B. Then A is $\{1\}$-monotone if and only if we can find nonnegative matrices X, Y, and Z such that*

$$B^{-1} \geqslant YU + VX + VZU. \tag{1.2}$$

NOTE 1.5. In (1.1), Lemma 1.3 and Lemma 1.4, we interpret U, Y, Z, L as absent and YU, VZU, LU as zero if $\operatorname{rank}(A) = m$, and similarly for V, X, Z, K, VX, VZU, and VK if $\operatorname{rank}(A) = n$. Hence in particular $\{A\}A\{1\} = \{I\}$ if $\operatorname{rank}(A) = m$. Likewise $A\{1\}\{A\} = \{I\}$ if $\operatorname{rank}(A) = n$.

2. $\{1\}$-MONOTONICITY IS NOT EQUIVALENT TO WEAK r-MONOTONICITY

We begin with

THEOREM 2.1. *Suppose that $A \in \mathbb{R}^{m,n}$ has rank r. If A is weak-r-monotone, then A is $\{1\}$-monotone.*

Proof. Let A be weak-r-monotone. Then, by Definition 1.2, there exists a nonsingular $r \times r$ submatrix B of A which is monotone, i.e. $B^{-1} \geqslant 0$. Consider the representation (1.1) of A determined by this matrix B. Set

$$G := Q' \left(\begin{array}{c|c} B^{-1} & 0 \\ \hline 0 & 0 \end{array} \right) P'.$$

Then $G \geqslant 0$ and $AGA = A$ [see also Lemma 1.3(i)], showing that A is $\{1\}$-monotone. ∎

That the converse of Theorem 2.1 is not true is illustrated by

EXAMPLE 2.2. Consider the matrix

$$A = 4^{-1/2} \begin{pmatrix} 1 & 3 & -1 & 1 \\ 2 & -2 & 2 & -2 \\ -1 & 1 & 1 & 3 \end{pmatrix}$$

of rank 3, and set

$$G := 4^{-1/2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $AGA = A$ and $G \geqslant 0$, A is $\{1\}$-monotone. However, A is not weak-r-monotone, because all nonsingular 3×3 submatrices of A fail to be monotone.

3. A CONDITION ON A UNDER WHICH A IS WEAK-r-MONOTONE IF IT IS $\{1\}$-MONOTONE

In this section the following concepts will play a key role.

DEFINITION 3.1. A matrix $A \in \mathbb{R}^{m,n}$ is a *P_+-matrix* if there exist $\{1\}$-inverses G and H of A such that

$$GA \geqslant 0, \qquad AH \geqslant 0. \tag{3.1a–b}$$

DEFINITION 3.2. A matrix $E \in \mathbb{R}^{r,t}$ of *rank* r is a *W-matrix* if E has a nonsingular submatrix F of order r such that

$$F^{-1}E \geqslant 0. \tag{3.2}$$

For P_+-matrices the author conjectured [10] (see also [9]) the following

THEOREM 3.3. *Let $A \in \mathbb{R}^{m,n}$ of rank r be a P_+-matrix. Then A is $\{1\}$-monotone if and only if A is weak-r-monotone.*

To establish this result we require several lemmas.

LEMMA 3.4. *Let $A \in \mathbb{R}^{m,n}$ be of rank r, let B be a nonsingular submatrix of A of order r, and consider the representation* (1.1) *of A determined by B. Then A is a P_+-matrix if and only if $(I\ V)$ and $(I\ U')'$ are both P_+-matrices.*

Proof. Observe first that by a theorem due to R. C. Bose (cf. Lemma 5.1 in [11]), G is a $\{1\}$-inverse of A if and only if QGP is a $\{1\}$-inverse of $\tilde{A} := P'AQ'$. Since P and Q are permutation matrices, it follows that A is a P_+-matrix if and only if

$$\tilde{A} = \begin{pmatrix} I \\ U \end{pmatrix} B (I \quad V) \tag{3.3}$$

is a P_+-matrix. By applying Lemma 1.3 to the representation (3.3) of $\tilde{A}$ determined by B as well as to $D := (I \ U')'$ and $E := (I \ V)$, we further obtain

$$\{\tilde{A}\}\tilde{A}\{1\} = \{D\}D\{1\}, \qquad \tilde{A}\{1\}\{\tilde{A}\} = E\{1\}\{E\}. \tag{3.4a–b}$$

Since $D\{1\}\{D\} = \{I\}$ and $\{E\}E\{1\} = \{I\}$ [see Note 1.5], our claim then follows by virtue of (3.4$a-b$). ■

LEMMA 3.5. *A matrix A is a P_+-matrix if and only if A' is a P_+-matrix.*

Proof. Observe that $G \in A\{1\}$ if and only if $G' \in A'\{1\}$. ■

LEMMA 3.6. *For $T_0 \in \mathbb{R}^{r,u}$, let $A_0 := (I \ T_0)$. Then A_0 is a P_+-matrix if and only if A_0 is a W-matrix.*

Proof of the "only if" part. Applying Lemma 1.3 to the representation $A_0 = (I \ T_0)$ results in $\{A_0\}A_0\{1\} = \{I\}$ [observe Note 1.5] and

$$A_0\{1\}\{A_0\} = \left\{ \begin{pmatrix} I - T_0 K \\ K \end{pmatrix} (I \quad T_0) \,\middle|\, K \in \mathbb{R}^{u,r} \right\},$$

so that A_0 is a P_+-matrix if and only if the following system of inequalities is consistent:

$$I \geqslant T_0 K, \qquad T_0 \geqslant T_0 K T_0, \qquad K \geqslant 0, \qquad K T_0 \geqslant 0. \tag{3.5a–d}$$

Assume that A_0 is a P_+-matrix, and let K_0 be a particular solution to (3.5a–d). For $Z := K_0 T_0 K_0$ we have $Z \leqslant K_0$ by virtue of (3.5 a,c). Inspection shows that Z solves (3.5a–d) if K_0 does. Define a sequence $\{K_n\}$ for $n \geqslant 1$ by setting $K_n := K_{n-1} T_0 K_{n-1}$. This sequence is decreasing and bounded below. Hence it converges, and by construction it follows that $Y := \lim K_n$

solves not only (3.5a–d) but also

$$K = KT_0K. \tag{3.5e}$$

This matrix Y enables us to construct a proper submatrix D_0 of A_0 such that $G_0A_0 \geqslant 0$ for some $\{1\}$-inverse G_0 of D_0. We need to consider two exhaustive cases.

Case I: Assume that $Y = 0$. Then, by (3.5b), $T_0 \geqslant 0$. Hence $A_0 \geqslant 0$, and $D_0 := I$ can serve as a proper nonsingular $r \times r$ submatrix of A_0 for which $D_0^{-1}A_0 \geqslant 0$; thus showing that A_0 is a W-matrix.

Case II: Assume that $Y \not\equiv 0$. In this case set $D_0 := (J_0 \; T_0)$, where J_0 is defined as a submatrix of I and contains its ith column e_i if and only if the ith column of Y is equal to the zero column. In D_0 as well as in what follows we interpret each block and each summand depending on J_0 as absent and zero, respectively, when Y contains no zero column. By definition of J_0, we have

$$YJ_0 = 0. \tag{3.6}$$

From $I - T_0Y \geqslant 0$, $Y \geqslant 0$, and $Y(I - T_0Y) = 0$ [observe (3.5a,c,e)] we conclude that

$$Ye_i \neq 0 \quad \Rightarrow \quad e_i'(I - T_0Y) = 0. \tag{3.7}$$

Hence

$$(I - J_0J_0')(I - T_0Y) = 0. \tag{3.8}$$

The equation $Y = YT_0Y$ [note (3.5e)] also yields

$$(T_0Y)^2 = T_0Y, \qquad (YT_0)^2 = YT_0. \tag{3.9a–b}$$

By means of (3.8), (3.9a), and (3.6) it may now be checked that

$$(J_0J_0' + T_0Y)(2I - J_0J_0' - T_0Y) = I,$$

thus showing that

$$(J_0J_0' + T_0Y)^{-1} = 2I - J_0J_0' - T_0Y. \tag{3.10}$$

The nonsingularity of $J_0 J_0' + T_0 Y = (J_0 \ T_0)(J_0 \ Y')'$ forces the matrix $D_0 = (J_0 \ T_0)$ to have full row rank, i.e. $\operatorname{rank}(D_0) = r$. For the matrix

$$G_0 := \begin{pmatrix} J_0' \\ Y \end{pmatrix} (J_0 J_0' + T_0 Y)^{-1} \tag{3.11}$$

we have $D_0 G_0 = I$ and therefore $D_0 G_0 D_0 = D_0$, i.e., G_0 is a $\{1\}$-inverse of D_0. That

$$G_0 = \begin{pmatrix} J_0'(I - T_0 Y) \\ Y \end{pmatrix} \geqslant 0 \tag{3.12a}$$

and

$$G_0 D_0 = \left(\begin{array}{c|c} J_0' J_0 & J_0'(I - T_0 Y) T_0 \\ \hline 0 & Y T_0 \end{array}\right) \geqslant 0 \tag{3.12b}$$

hold true can be seen by using the definition of J_0, Equations (3.6), (3.9b), (3.10), and the fact that Y is a solution to (3.5a–e). From (3.12a–b) and the definition of D_0 it is clear that

$$G_0 A_0 \geqslant 0. \tag{3.13}$$

Now we proceed as follows. If D_0 is square and consequently nonsingular, then $G_0 = D_0^{-1}$, and it follows from (3.13) that A_0 is a W-matrix. It only remains to consider the case where D_0 is not square. By construction [observe (3.11) and (3.12a–b)] we have

$$G_0 D_0 \geqslant 0, \qquad D_0 G_0 = I, \quad \text{and} \quad G_0 \geqslant 0,$$

showing that the proper submatrix D_0 of A_0 is a $\{1\}$-monotone P_+-matrix. Since $\operatorname{rank}(D_0) = r$, there is a nonsingular $r \times r$ submatrix B_1 of D_0. So we can consider the representation

$$D_0 = B_1 (I \quad T_1) R_1$$

of D_0 determined by B_1, where R_1 is a suitable permutation matrix. From Lemma 3.4 it follows that

$$A_1 := (I \quad T_1)$$

is a P_+-matrix because D_0 is a P_+-matrix. If we apply the above steps not to A_0 but to A_1, we obtain a proper submatrix D_1 of A_1 such that $\operatorname{rank}(D_1) = r$ and

$$G_1 A_1 \geqslant 0 \tag{3.14}$$

for some $\{1\}$-inverse G_1 of D_1. Moreover

$$G_1 D_1 \geqslant 0, \qquad D_1 G_1 = I, \quad \text{and} \quad G_1 \geqslant 0,$$

so that D_1 is a $\{1\}$-monotone P_+-matrix. If D_1 is square, $G_1 = D_1^{-1}$, so that A_1 is a W-matrix by virture of (3.14). Otherwise, we proceed with D_1 as above with D_0. Doing this repeatedly results in a sequence of P_+-matrices $A_0, A_1, A_2, \dots$. Observe, however, that after a finite number of iterations we must arrive at a W-matrix, say A_k. This happens because by construction each matrix D_{i+1} of full row rank r is a proper $\{1\}$-monotone submatrix of

$$A_{i+1} := \begin{pmatrix} I & T_{i+1} \end{pmatrix}$$

in the representation

$$D_i = B_{i+1} A_{i+1} R_{i+1} = \begin{pmatrix} J_i & T_i \end{pmatrix} \tag{3.15}$$

of the $\{1\}$-monotone P_+-matrix D_i determined by the nonsingular $r \times r$ matrix B_{i+1}. Since A_k is a W-matrix, the proof of the "only if" part is completed by showing that A_i is a W-matrix if A_{i+1} is such a matrix. For this purpose, assume that A_{i+1} is a W-matrix, i.e. $F_{i+1}^{-1} A_{i+1} \geqslant 0$ for some nonsingular $r \times r$ submatrix F_{i+1} of $A_{i+1} = (I \;\, T_{i+1})$. Then $0 \leqslant F_{i+1}^{-1} A_{i+1} = (I \;\, \tilde{T}_{i+1}) P_{i+1}$ for some permutation matrix P_{i+1}. Evidently

$$\tilde{T}_{i+1} \geqslant 0, \qquad D_i = B_{i+1} F_{i+1} \begin{pmatrix} I & \tilde{T}_{i+1} \end{pmatrix} P_{i+1} R_{i+1} \tag{3.16a–b}$$

[note (3.15)], showing in particular that $B_{i+1} F_{i+1}$ is a nonsingular submatrix of D_i. Next consider the representation (3.16b) of D_i determined by $B_{i+1} F_{i+1}$. Since D_i is $\{1\}$-monotone, it follows from Lemma 1.4 that we can find a nonnegative matrix X such that $(B_{i+1} F_{i+1})^{-1} \geqslant \tilde{T}_{i+1} X$. Thus

$$(B_{i+1} F_{i+1})^{-1} \geqslant 0 \tag{3.17}$$

because $\tilde{T}_{i+1} \geqslant 0$ [see (3.16a)]. From (3.16a–b) we further obtain

$$(B_{i+1} F_{i+1})^{-1} D_i \geqslant 0. \tag{3.18}$$

Consequently [observe (3.17)–(3.18)],

$$(B_{i+1}F_{i+1})^{-1}A_i \geqslant 0, \tag{3.19}$$

because $D_i = (J_i \ T_i)$ is a submatrix of $A_i = (I \ T_i)$. Since $B_{i+1}F_{i+1}$ is a nonsingular submatrix of A_i, it follows from (3.19) that A_i is a W-matrix.

Proof of the "if" part. Let A_0 be a W-matrix, i.e., assume that there is a nonsingular $r \times r$ submatrix F_0 of A_0 such that $F_0^{-1}A_0 \geqslant 0$. Then $0 \leqslant F_0^{-1}A_0 = (I \ \tilde{T}_0)P_0$ for some permutation matrix P_0. Hence $A_0 = F_0(I \ \tilde{T}_0)P_0$ where $\tilde{T}_0 \geqslant 0$. Since $G = (I \ 0)'$ is a nonnegative $\{1\}$-inverse of the nonnegative matrix $(I \ \tilde{T}_0)$, it is clear that $(I \ \tilde{T}_0)$ is a P_+-matrix. Our claim now follows by virture of Lemma 3.4. ∎

We are now in a position to prove our main result.

Proof of Theorem 3.3. Sufficiency is clear by Lemma 2.1. To prove necessity, let $A \in \mathbb{R}^{m,n}$ of rank r be a $\{1\}$-monotone P_+-matrix. Since A has rank r, there is a nonsingular $r \times r$ submatrix B of A. Consider the representation

$$A = P\begin{pmatrix} I \\ U \end{pmatrix} B(I \quad V)Q \tag{1.1}$$

of A determined by B. From Lemma 3.4 in conjunction with Lemma 3.5 it follows that

$$D := (I \quad V) \quad \text{and} \quad E' := (I \quad U')$$

are P_+-matrices. By Lemma 3.6 we know that D and E' are W-matrices. Consequently, we can find nonsingular $r \times r$ submatrices M and N' of D and E', respectively, such that $M^{-1}D \geqslant 0$ and $N'^{-1}E' \geqslant 0$. Then

$$M^{-1}D = (I \quad \tilde{V})\tilde{Q} \quad \text{and} \quad N'^{-1}E' = (I \quad \tilde{U}')\tilde{P}'$$

for some nonnegative matrices $\tilde{U}$ and $\tilde{V}$ as well as for some permutation matrices $\tilde{P}$ and $\tilde{Q}$. Hence

$$D = M(I \quad \tilde{V})\tilde{Q}, \qquad E' = N'(I \quad \tilde{U}')\tilde{P}', \tag{3.20a–b}$$

where

$$\tilde{V} \geqslant 0, \qquad \tilde{U} \geqslant 0. \tag{3.20c–d}$$

Inserting (3.20a–b) in (1.1) results in

$$A = \bar{P}\begin{pmatrix} I \\ \tilde{U} \end{pmatrix} NBM(I \quad \tilde{V})\bar{Q}, \tag{3.21}$$

where $\bar{P} := P\tilde{P}$ and $\bar{Q} := \tilde{Q}Q$. We now apply Lemma 1.4 to the representation (3.21) of A determined by NBM. Since A is $\{1\}$-monotone, there exist nonnegative matrices X, Y, and Z such that

$$(NBM)^{-1} \geqslant Y\tilde{U} + \tilde{V}X + \tilde{V}Z\tilde{U}.$$

Consequently,

$$(NBM)^{-1} \geqslant 0,$$

by virtue of (3.20c–d). Hence A is weak-r-monotone. ■

We remark that Theorem 3.3 was used in [12] to prove several interesting results concerning the Drazin monotonicity of property-n matrices.

Theorem 3.3 admits a corollary for nonnegative matrices.

Corollary 3.7. *Let $A \in \mathbb{R}^{m,n}$ of rank r be nonnegative. Then A is $\{1\}$-monotone if and only if A is weak-r-monotone.*

Proof. Sufficiency is clear by Lemma 2.1. To establish necessity, let A be a nonnegative $\{1\}$-monotone matrix of rank r. Then there is a nonnegative $\{1\}$-inverse G of A. Consequently, $AG \geqslant 0$ and $GA \geqslant 0$, showing that A is a P_+-matrix. The corollary now follows from Theorem 3.3. ■

A different proof of this, using a result by Flor [5], may be found in [9, p. 81]. We conclude this paper with a further known corollary to Theorem 3.3.

Corollary 3.8 (cf. [10, Satz 3.5.8]). *For $A \in \mathbb{R}^{m,n}$ of rank r, let AA^+ and A^+A be both nonnegative matrices. Then A is $\{1\}$-monotone if and only if A is weak-r-monotone.*

Proof. Observe that the *Moore-Penrose inverse* A^+ of A (see, e.g., [3]) is a particular $\{1\}$-inverse of A. ∎

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