# A Note on Generalizations of Strict Dlagonal Dominance for Real Matrices 

M. Neumann<br>Department of Mathematics<br>The University of Nottingham<br>Nottingham NG7 2RD, England

Submitted by R. S. Varga


#### Abstract

We investigate classes of real square matrices possessing some weakened form of strict diagonal dominance of a real matrix whose diagonal entries are all positive. The intersection of each one of these classes with the set of all real matrices, with nonpositive off-diagonal elements, coincides with the set of all nonsingular Mmatrices.


## 1. INTRODUCTION

In [8] Varga showed that many of the generalizations (to the concept) of strict diagonal dominance of an $n \times n$ complex matrix can, indeed, be based upon the theory of nonsingular $M$-matrices. More recently Plemmons [5] has surveyed the numerous characterizations, which can be found in the works of authors from different disciplines, for an $n \times n$ real matrix, with nonpositive off-diagonal entries to be a nonsingular $M$ matrix.

In Plemmon's survey these diverse characterizations are grouped into classes, such that each class contains several conditions which are equivalent for arbitrary real (and square) matrices. Moreover, a directed graph is presented there to show the relationships between the classes. (Here we shall refer to the classes of characterizations as classes of matrices.)

Unfortunately, the classification in [5] is incomplete, particularly with respect to those classes of matrices which possess some weakened form of strict diagonal dominance of a real matrix whose diagonal entries are all positive. In fact, the author of [5] makes no secret of this incompleteness,
pointing out several questions, the answers to which were unknown to him at the time of preparing the survey.

This note grew out of an attempt to answer some of the questions raised in [5]. However, as happens on some occasions, we in turn raise certain questions, the answers to which are unknown to us at this time. Our main results are given in Lemmas 3 and 4 and Propositions 1 and 2. We make use of these results to present a refinement of the directed graph in [5] mentioned earlier.

## 2. NOTATIONS AND PRELIMINARIES

We shall denote by $R^{n}\left(C^{n}\right)$ and by $R^{n, n}\left(C^{n, n}\right)$ the $n$-dimensional real (complex) space and the set of all $n \times n$ real (complex) matrices, respectively. The $i$ th entry of a vector $x \in R^{n}$ will be denoted by $x_{i}$ or $(x)_{i}$, and sometimes it will be convenient to denote the $i, j$ th entry of a matrix $S \in R^{n, n}$ by $(S)_{i j}$. For any two nonnegative integers $m$ and $l$ with $l \geqslant m,[m, l]$ is the set of integers $m, m+1, \ldots, l$.

For $S \in R^{n, n}$, the symbols $S^{T}, \sigma(S)$ and $\rho(S)$ denote, respectively, the transpose, the spectrum and the spectral radius of $S$. If $x \in R^{n}$ and each component of $x$ is nonnegative (positive), we shall write $x \geqslant 0(x>0)$, and if each element of a matrix $S \in R^{n, n}$ is nonnegative we shall write $S \geqslant 0$. A diagonal matrix will be called positive if all its diagonal entries are positive numbers. If $S \in C^{n, n}$, then $|S|$ will denote the real matrix whose entries are the moduli of the entries of $S$.

Next consider the subset of $R^{n, n}$ defined by

$$
\mathrm{Z}^{n, n}=\left\{\mathrm{S}=\left(s_{i j}\right) \in R^{n, n}: s_{i j} \leqslant 0, i \neq j\right\} .
$$

Any matrix $S \in Z^{n, n}$ can be expressed in the form

$$
\begin{equation*}
S=t I-U \tag{2.1}
\end{equation*}
$$

where $U \geqslant 0$ and $t \geqslant 0$. A matrix $S \in Z^{n, n}$ is called an $M$-matrix if it has a representation (2.1) with $t \geqslant \rho(U)$. If $S$ is a nonsingular $M$-matrix, then for any representation (2.1) of $S$ we have that $t>\rho(U)$ (e.g. [5]). It is well known that $S \in Z^{n, n}$ is a nonsingular $M$-matrix if and only if $S$ is nonsingular and $S^{-1} \geqslant 0$ (see [2]). This characterization is often used in the literature as the definition of a nonsingular $M$-matrix (see, for example, [ 9 , Chapter 3]). We shall denote by $P$ the class of all nonsingular $M$-matrices. For $S=\left(s_{i j}\right) \in R^{n, n}$,
the comparison matrix $\mathfrak{M l}(\mathrm{S})$ for $S$ is the real matrix given by

$$
(\mathfrak{M}(S))_{i j}=\left\{\begin{array}{rll}
\left|s_{i i}\right| & \text { if } & i=j \\
-\left|s_{i j}\right| & \text { if } & i \neq j
\end{array}\right.
$$

Next let $S=\left(s_{i j}\right) \in C^{n, n}$. Then $S$ is strictly diagonally dominant if

$$
\left|s_{i i}\right|>\sum_{\substack{i=1 \\ i \neq i}}^{n}\left|s_{i j}\right|, \quad i \in[1, n]
$$

In [1], Beauwens defines $S$ to be lower semistrictly diagonally dominant if

$$
\left|s_{i i}\right| \geqslant \sum_{\substack{i=1 \\ i \neq i}}^{n}\left|s_{i j}\right|, \quad i \in[1, n]
$$

and if

$$
\left|s_{i i}\right|>\sum_{i=1}^{i-1}\left|s_{i j}\right|, \quad i \in[1, n] .
$$

Beauwens defines $S$ to be semistrictly diagonally dominant if for some permutation $Q$, the matrix $Q S Q^{T}$ is lower semistrictly diagonally dominant. It follows that every strictly diagonally dominant matrix is semistrictly diagonally dominant. That the converse of this statement is not true is illustrated by taking

$$
S=\left(\begin{array}{ll}
1 & 1 \\
1 & 2
\end{array}\right)
$$

We shall adopt here the same letters of the alphabet used in [5] to denote classes of matrices possessing certain given properties. For convenience we display here those classes of matrices which are most relevant to this note.
$L$ : The class of matrices $S=\left(s_{i j}\right) \in R^{n, n}$ with the following properties: for each $S \in L$ there exists a vector $x>0$ such that

$$
\begin{equation*}
S x \ngtr 0 \tag{2.2}
\end{equation*}
$$

and such that if for some index $i_{0} \in[1, n]$,

$$
\begin{equation*}
(S x)_{i_{0}}=0 \tag{2.3}
\end{equation*}
$$

there exist indices

$$
\begin{equation*}
i_{1}, i_{2}, \ldots, i_{T} \in[1, n] \tag{2.4}
\end{equation*}
$$

such that

$$
\begin{equation*}
s_{i_{k} i_{k+1}} \neq 0, \quad k \in[0, r] \tag{2.5}
\end{equation*}
$$

and such that

$$
\begin{equation*}
(S x)_{i_{r}}>0 . \tag{2.6}
\end{equation*}
$$

M: The class of matrices $S=\left(s_{i j}\right) \in R^{n, n}$ with the following properties: for each $S \in M$ there exists a vector $x>0$ such that $S x \geqslant 0$ and such that

$$
\begin{equation*}
\sum_{i=1}^{i} s_{i j} x_{i}>0, \quad i \in[1, n] . \tag{2.7}
\end{equation*}
$$

$M^{\prime}$ : The class of matrices $S \in R^{n, n}$ with the property that for each matrix $S \in M^{\prime}$ there exists a permutation matrix $Q$ such that $Q S Q^{T} \in M$.
$N$ : The class of matrices $S \in R^{n, n}$ with positive diagonal entries such that for each matrix $S \in N$ there exists a positive diagonal matrix $D$ such that the matrix $S D$ is strictly diagonally dominant.

If $C_{1}$ and $C_{2}$ are any two classes of matrices in $R^{n, n}$, by $C_{1} \Rightarrow C_{2}$ we shall mean that a necessary condition for a matrix to be in $C_{1}$ is that it be in $C_{2}$.

## 3. THE MAIN RESULTS

In [4] Moylan showed that a real matrix $S$ with all its diagonal entries positive is in $N$ if and only if $\mathfrak{M}(S) \in P$. Our first result shows that in addition to the list of characterizations in [5], $N$ can be further characterized by properties which relate to semistrict diagonal dominance.

Lemma 1. Let S be a real matrix with all its diagonal entries positive. Then the following statements are equivalent:
(i) For some positive diagonal matrix $D$, the matrix SD is semistrictly diagonally dominant.
(ii) $S \in N$.

Proof. Clearly (ii) implies (i), so we show that (i) implies (ii). Let $Q$ be a permutation matrix such that the matrix $Q S D Q^{T}$ is lower semistrictly diagonally dominant. By the Corollary in [1, p. 111] it follows that $Q \mathfrak{M}(S D) Q^{T}=$ $\mathfrak{M}\left(Q S D Q^{T}\right) \in P$, and hence by the properties of $P, \mathfrak{M}(S D) \in P$. Finally, since $D^{-1}$ is a positive diagonal matrix, we have that

$$
\mathfrak{M}(\mathrm{S})=\mathfrak{M}(\mathrm{SD}) D^{-1} \in P
$$

by [3, Theorem 4.9], and hence $S \in N$ by [4].
Lemma 1 has several simple corollaries. We only give here two of the possible ones.

Corollary 1. Let $S \in C^{n, n}$. Then the following statements are equivalent:
(i) There exists a nonsingular diagonal matrix $D \in C^{n, n}$ such that the matrix SD is semistrictly diagonally dominant.
(ii) For some nonsingular diagonal matrix $D \in C^{n, n}$, the matrix $S D$ is strictly diagonally dominant.

Proof. That (ii) implies (i) is obvious. (i) implies (ii): Since $S D$ is semistrictly diagonally dominant, it follows at once that the diagonal entries of the matrix $S$ are all nonzero. Thus, since $|S D|=|S||D|$, the matrix $|S|$ satisfies the conditions of statement (i) in Lemma 1. But then there exists a positive diagonal matrix, say $D^{\prime}$, such that the matrix $|\mathrm{S}| D^{\prime}$ is strictly diagonally dominant and hence $S D^{\prime}$ is strictly diagonally dominant.

Corollary 1 leads us to the conclusion that the concepts of strict diagonal dominance, lower semistrict diagonal dominance and semistrict diagonal dominance for a complex matrix are equivalent to each other up to a multiple by a nonsingular diagonal matrix $D \in C^{n, n}$. We wish to remark that Lemma 1 and Corollary 1 could also be derived from Beauwens's results in [1] or from Varga's results in [8].

The second corollary is more of an outcome of Corollary 1.
Corollary 2. Suppose that $S \in C^{n, n}$ is semistrictly diagonally dominant. Then the point Jacobi and the point Causs-Seidel iteration matrices associated with S are both convergent.

Proof. Using similarity arguments, the proof easily follows from [9, Theorem 3.4] and Corollary 1.

One of the principal goals of this note is to show that $M \Rightarrow L$. This will be accomplished in Lemma 3 below. Beforehand, we offer an observation which may be helpful in determining whether a matrix $S \in R^{n, n}$ is in $L$ and which generalizes a result of Vandergraft in [6].

Lemma 2. Let $S=\left(s_{i j}\right) \in R^{n, n}$, and suppose that there exists a nonzero vector $x \in R^{n}$ such that

$$
\Omega \equiv\left\{k \in[1, n]:(S x)_{k} \neq 0\right\} \neq \varnothing .
$$

Then for each index $i \in[1, n] \backslash \Omega$ there exist indices $i_{1}, \ldots, i_{r} \in[1, n]$, with $i_{r} \in \Omega$, such that the product

$$
s_{i_{i}}, s_{i_{i}, i_{2}} \cdots s_{i_{i-1} i_{i}} \neq 0
$$

if and only if the vector $S x$ does not belong to any proper subspace of $R^{n}$, spanned by unit coordinate vectors, which is invariant under $S$.

The proof of Lemma 2 is not dissimilar to the proof of [6, Theorem 4.1] and is therefore omitted.

Lemma 3. Suppose that $S=\left(s_{i j}\right) \in R^{n, n}$ is in $M$. Then $S \in L$. More specifically, if $x>0$ is a vector such that $S x \geqslant 0$ and such that (2.7) holds, then for each index $i_{0} \in[1, n-1]$ for which (2.3) holds, there exists a strictly increasing sequence of integers (2.4) such that (2.5) and (2.6) are satisfied.

Proof. If $S x>0$, then $S \in L$ and there is nothing to prove. Suppose then that (2.3) holds for some index $i_{0} \in[1, n-1]$. If $s_{i_{o} n} \neq 0$, then since $(S x)_{n}>0$ by (2.7), Equations (2.5) and (2.6) hold with $r=1$, and so $S \in L$. Assume therefore that $s_{i_{0} n}=0$. By (2.7)

$$
\sum_{i=1}^{i_{0}} s_{i_{0} j} x_{i}>0
$$

and so by (2.3) there exists an index $i_{1} \in\left[i_{0}+1, n-1\right]$ such that

$$
\begin{equation*}
s_{i_{0} i_{1}} \neq \mathbf{0} \tag{3.1}
\end{equation*}
$$

If ( $S x)_{i_{1}}>0$, then again (2.5) and (2.6) hold with $r=1$, and hence $S \in L$. Suppose then that

$$
(S x)_{i_{1}}=0
$$

and apply to the index $i_{1}$ similar considerations to those applied to the index $i_{0}$, and so on. Since the sequence of indices $\left\{i_{k}\right\}$ thus generated is strictly increasing, the process must terminate after a finite number of steps, say $r$, with $i_{r}=n$ if at no previous stage $s_{i_{k-1}} \neq 0$ or $s_{i_{k-1} n}=0$ but $(S x)_{i_{k}}>0$, $k \in[1, r-1]$. Thus $S \in L$ and the proof is complete.

The next result will allow us to determine the relationship between the classes $N, M, M^{\prime}$ and $L$.

Lemma 4. Let $S=\left(s_{i j}\right) \in R^{n, n}$. Then $S \in L$ if and only if for every permutation matrix $Q$, the matrix

$$
Q S Q^{T} \in L
$$

Proof. The proof of the "if" part is obvious.
"Only if": Let $Q$ be an $n$th order permutation matrix, and let $\mu$ be the permutation function associated with $Q$. Let $x>0$ be a vector such that (2.2) through (2.6) hold (with respect to $x$ ). Set $x_{Q} \equiv Q x$ and $S_{Q} \equiv Q S Q^{T}$. Then $x_{Q}>0$, and since $S_{Q} x_{Q}=Q S x$, we have from (2.2) that $S_{Q} x_{Q} \geqslant 0$ and that the set

$$
\left\{j \in[1, n]:\left(S_{Q} x_{Q}\right)_{j}>0\right\} \neq \varnothing
$$

Suppose that for some index $j_{0} \in[1, n]$,

$$
\left(S_{Q} x_{Q}\right)_{i_{0}}=0 .
$$

Then since

$$
(S x)_{\mu^{-1}\left(j_{0}\right)}=\left(S Q^{T} x_{Q}\right)_{\mu^{-1}\left(j_{0}\right)}=\left(S_{Q} x_{Q}\right)_{i_{0}}
$$

it follows from (2.3) through (2.6) that there exists indices $i_{1}, \ldots, i_{r} \in[1, n]$ such that

$$
s_{\mu^{-1}\left(j_{0}\right) i_{1}}, s_{i_{1} i_{2}}, \ldots, s_{i_{i-1} i_{i}} \neq 0
$$

and such that

$$
(S x)_{i_{r}}>0 .
$$

But then the elements

$$
\left(S_{Q}\right)_{i_{0} \mu\left(i_{1}\right)},\left(S_{Q}\right)_{\mu\left(i_{1}\right) \mu\left(i_{2}\right)} \ldots,\left(\mathrm{S}_{Q}\right)_{\mu\left(i_{r-1}\right) \mu\left(i_{r}\right)}
$$

of the matrix $S_{Q}$ are all nonzero, and

$$
\left(\mathrm{S}_{Q} x_{Q}\right)_{\mu\left(i_{r}\right)}=\left(\mathrm{S} Q^{T} x_{Q}\right)_{i_{r}}=(\mathrm{S} x)_{i_{i}} .
$$

This concludes the proof.
We thus have the following result:

Proposition 1. In $R^{n, n}, N \Rightarrow M \Rightarrow M^{\prime} \Rightarrow L$.
Proof. That $N \Rightarrow M \Rightarrow M^{\prime}$ follows immediately from the definitions of these classes, and that $M^{\prime} \Rightarrow L$ follows from Lemmas 3 and 4.

Remark. Note that if $S \in M^{\prime}$, then $Q S Q^{T} \in M^{\prime}$ for any permutation matrix $Q$. Hence both classes $M^{\prime}$ and $L$ are closed under same reordering of rows and columns of members in these classes. The class $M$ does not possess a similar property, as illustrated by taking

$$
S=\left(\begin{array}{ll}
1 & 0 \\
1 & 0
\end{array}\right) \quad \text { and } \quad Q=\left(\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right) .
$$

Next, we investigate the relationship of the classes $M$ and $M^{\prime}$ to some of the other properties of matrices studied in [5].

Proposition 2. Let $S \in R^{n, n}$.
(a) For S to belong to $M, \mathrm{~S}$ need not have any of the following properties:
( $\mathrm{i}_{\mathrm{a}}$ ) $\mathrm{S}+\alpha I$ is nonsingular for all $\alpha \geqslant 0$.
(iia) There exist a permutation matrix $Q$ and lower and upper triangular matrices $E_{1}$ and $E_{2}$ respectively, with all their diagonal entries positive, such that $Q S Q^{T}=E_{1} E_{2}$.
(iiia) Every splitting of $S$ into $S=R-T$, where $T \geqslant 0$ and $R$ is nonsingular with $R^{-1} \geqslant 0$ (i.e., the splitting $S=R-T$ is regular), has $\rho\left(R^{-1} T\right)<1$.
(b) For $S \in R^{n, n}$ to have any one of the following properties:
( $\mathrm{i}_{\mathrm{b}}$ ) S is nonsingular and $\mathrm{S}^{-1} \geqslant 0$,
(ii ${ }_{\mathrm{b}}$ ) there exists a positive diagonal matrix $D$ such that the matrix $S D+D \mathrm{~S}^{T}$ is positive definite,
$S$ need not be in $M$.
(c) For S to have any one of the following properties:
( $i_{c}$ ) (see ( $\left.i_{b}\right)$ ),
(iic) each eigenvalue of S has a positive real part,
(iii.) for each $k \in[1, n]$, the sum of all the principal minors of $S$ of order $k$ is positive,
$\left(\mathrm{iv}_{\mathrm{c}}\right) \mathrm{S}=E_{1} E_{2}$, where $E_{1}$ and $E_{2}$ are lower and upper triangular matrices respectively, with all their diagonal entries positive,
$S$ need not be in $M^{\prime}$.

Proof. (a): Consider the matrix

$$
S=\left(\begin{array}{rrr}
1 & -2 & 1 \\
-2 & 4 & 2 \\
0 & -2 & 3
\end{array}\right)
$$

and let $x=(1,1,1)^{T}$. Then $S x \geqslant 0$. Moreover the matrix $S$ and the vector $x$ satisfy (2.7), showing that $S \in M$. Since $S$ is singular, $S$ cannot satisfy ( $i_{a}$ ) or (iia). Consider the splitting of $S$ into

$$
S=\left[\begin{array}{rrr}
8 & -2 & 1 \\
-2 & 4 & -2 \\
1 & -2 & 9
\end{array}\right]-\left[\begin{array}{lll}
7 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 6
\end{array}\right] \equiv R-T .
$$

Since

$$
R^{-1}=\frac{1}{224}\left[\begin{array}{rrr}
32 & 16 & 0 \\
16 & 71 & 14 \\
0 & 14 & 28
\end{array}\right)
$$

the splitting $S=R-T$ is regular. Now $\sigma\left(R^{-1} T\right)=\left\{1, \frac{3}{4}, 0\right\}$, and hence $S$ does not satisfy (iiia).
(b): Let

$$
S=\left(\begin{array}{rr}
-1 & 1  \tag{3.2}\\
1 & 0
\end{array}\right)
$$

Then $S$ satisfies ( $\mathrm{i}_{\mathrm{b}}$ ) but $\mathrm{S} \notin M$. Next suppose that

$$
S=\left(\begin{array}{rrr}
1 & -\frac{1}{2} & -\frac{1}{3}  \tag{3.3}\\
-\frac{5}{6} & 1 & 1 \\
-1 & -1 & 1
\end{array}\right)
$$

It is easy to verify the matrix $S+S^{T}$ is positive definite, so that $S$ satisfies (ii $\mathrm{i}_{\mathrm{b}}$ ). Assume now that there exists a vector $x=\left(x_{1}, x_{2}, x_{3}\right)^{T}>0$ such that

$$
\begin{equation*}
S x \geqslant 0 \tag{3.4}
\end{equation*}
$$

and such that (2.7) is satisfied for $i \in[1,3]$. Then from (3.3), (3.4) and (2.7), the component of $x$ have to satisfy the following inequalities:

$$
\begin{gather*}
x_{1} \geqslant \frac{1}{2} x_{2}+\frac{1}{3} x_{3}  \tag{3.5}\\
x_{2}>\frac{5}{6} x_{1} \tag{3.6}
\end{gather*}
$$

and

$$
\begin{equation*}
x_{3}>x_{1}+x_{2} . \tag{3.7}
\end{equation*}
$$

Inserting (3.7) in (3.5) shows that the components of $x$ have also to satisfy

$$
\begin{equation*}
x_{1}>\frac{1}{3} x_{1}+\frac{5}{6} x_{2} \tag{3.8}
\end{equation*}
$$

But then inscrting (3.6) in (3.8) we sce that

$$
x_{1}>\frac{37}{36} x_{1}
$$

which is impossible. Thus $S \notin M$.
(c): Let $S$ be as in (3.2). Then $S$ satisfies ( $\mathbf{i}_{\mathrm{b}}$ ), but $S \notin M^{\prime}$. Next let

$$
S=\left(\begin{array}{rr}
1 & 1 \\
-1 & -\frac{1}{2}
\end{array}\right)
$$

Then $S$ satisfies the properties in (iii) through (iv ${ }_{c}$ ), but clearly $S \notin M^{\prime}$. This completes the proof.


Fig. 1.
The results of Propositions 1 and 2 and of Theorem 1 in [5] yield the directed graph in Fig. 1 for the relationship between the classes of matrices introduced in [5].

We remark that the implication $K \Rightarrow G$ is due to Vandergraft [7].

Corollary 3. In $R^{n, n}$,

$$
P=N \cap Z^{n, n}=M \cap Z^{n, n}=M^{\prime} \cap Z^{n, n}=L \cap Z^{n, n}
$$

Proof. The proof follows from Proposition 1 and from Theorem 1 in [5].

## 4. OPEN QUESTIONS

(1) If all the principal minors of $S \in R^{n, n}$ are positive (i.e., $S \in A$ ), does $S \in M^{\prime}$ or not?
(2) If $S$ satisfies property ( $\mathrm{ii}_{\mathrm{b}}$ ) (i.e., $\mathrm{S} \in I$ ), does $S \in M^{\prime}$ or not?

We remark that C. R. Johnson and the author know that the answer to both questions is in the affirmative for $n \leqslant 3$. Indeed, permuting the first row and first column with the second row and column, respectively, in the matrix
$S$ given by (3.3) yields a matrix in $M$ [with respect, for example, to the vector $x=(1,2,3+\varepsilon)^{T}$ for $\varepsilon>0$ sufficiently small].

The author wishes to thank Professor R. J. Plemmons of the University of Tennessee at Knoxville for his encouragement and support in writing this note. Professor R. Beauwens and D. A. Burgess are thanked for their help and interest. Finally, the author wishes to thank Professor R. S. Varga for his comments concerning the original draft of this note.

## REFERENCES

1 R. Beauwens, Semistrict diagonal dominance, SIAM J. Numer. Anal. 13:109-112 (1976).

2 K. Fan, Topological proofs for certain theorems on matrices with nonnegative elements, Monatsh. Math. 62:219-237 (1958).
3 M. Fiedler and V. Ptak, On matrices with nonpositive off-diagonal elements and positive principal minors, Czech. Math. J. 12:382-400 (1962).
4 P. J. Moylan, Matrices with positive principal minors, Linear Algebra Appl. 17:53-58 (1977).
5 R. J. Plemmons, $M$-matrix characterization I: Nonsingular M-matrices, Linear Algebra Appl. 18:175-188 (1977).
6 J. S. Vandergraft, A note on irreducibility for linear operators on a partially ordered finite dimensional vector space, Linear Algebra Appl. 13:139-146 (1976).
7 J. S. Vandergraft, Applications of partial orderings to the study of positive definiteness, monotonicity, and convergence of iterative methods for linear systems, SIAM J. Numer. Anal. 9:97-104 (1972).
8 R. S. Varga, On recurring theorems on diagonal dominance, Linear Algebra Appl. 13:1-9 (1976).
9 R. S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, N.J., 1962.

Received 29 October 1977; revised 15 May 1978

