PERGAMON

Applied Mathematics Letters 12 (1999) 43–46

Applied Mathematics Letters

# Nonlocal Generalized Models for a Confined Plasma in a Tokamak

A. FERONE
Dipartimento di Matematica
Seconda Università di Napoli, Piazza Duomo
Caserta, Italy

M. A. JALAL
Université De Poitiers Mathématiques
40, Avenue du Recteur Pineau
86022 Poitiers, France

J. M. RAKOTOSON
Université De Poitiers Mathématiques
40, Avenue du Recteur Pineau
86022 Poitiers, France

R. VOLPICELLI
Dipartimento di Matematica ed Applicazioni "R. Caccioppoli"
Università di Napoli "Federico II", Complesso Monte S. Angelo
Via Cintia, 80126 Napoli, Italy
rako@matpts.univ-poitiers.fr

*(Received December 1997; revised and accepted January 1998)*

Communicated by R. Temam

**Abstract**—The following model appears in plasma physics for a Tokamak configuration: $-\Delta u + g(u) = 0$, $u \in \mathcal{V} = H_0^1(\Omega) \oplus \mathbb{R}$, $\int_{\partial\Omega} \frac{\partial u}{\partial n} = I > 0$, where $I$ is a given positive constant, which is equivalent to find a fixed point $u = F(u - g(u)) + \varphi_0$ where $F$ is a compact operator on $L^2(\Omega)$. According to Grad and Shafranov the nonlinearity $g$ can depend on $u_*$ which is the generalized inverse of the distribution function $m(t) = \text{meas}\{x : u(x) > t\} = |\{u > t\}|$ (see [1]). But in these cases the map $u \longrightarrow g(u)$ cannot be continuous on all the space $\mathcal{V}$ but only on a nonlinear nonclosed set $\mathcal{V}_0$. This implies that the standard direct method for fixed point cannot be applied to solve the preceding problem. Nevertheless, using the Galerkin method and a topological argument, we prove that there exists a solution $u$ fixed point of $u = F(u - g(u)) + \varphi_0$ under suitable assumptions on $g$.

The model we treat covers a large new class of nonlinearities including relative rearrangment and monotone rearrangment. The resolution of the concrete model needs an extension of the strong continuity result of the relative rearrangement map made in [2] (see Theorem 1.1 below for the definition and result).© 1998 Elsevier Science Ltd. All rights reserved.

**Keywords**—Plasma physics, Topological degree, Weighted relative rearrangement, Weighted monotone rearrangement.

## 1. THE GRAD-MERCIER-SHAFRANOV MODELING

We recall (see [3,4]) that a Tokamak is an axisymmetric torus in which a plasma is confined by a magnetic field due to currents in external coils. The magnetic induction $B$ and the current

0893-9659/98/\$—see front matter © 1998 Elsevier Science Ltd. All rights reserved.
PII: S0893-9659(98)00124-4

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

density $j$ follow the Maxwell laws, which read

$$\operatorname{div} B = 0, \tag{1.1}$$

$$\operatorname{rot} B = j. \tag{1.2}$$

If $(r, \theta, \zeta)$ denote the cylindrical coordinates, taking into account the axisymmetric configuration, relation (1.1) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_\zeta}{\partial \zeta} = 0, \tag{1.3}$$

where $B = (B_r, B_\theta, B_\zeta)$. This implies the existence of a function $u$, called the flux function, depending only on $(r, \zeta)$ such that:

$$B_\zeta = \frac{1}{r}\frac{\partial u}{\partial r}, \qquad B_r = -\frac{1}{r}\frac{\partial u}{\partial \zeta}.$$

From (1.2), one can derive the equation

$$-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right) - \frac{1}{r}\frac{\partial^2 u}{\partial \zeta^2} = j_\zeta,$$

where $j_\zeta$ is the toroidal component of the current density (i.e., the $\zeta$-component of $j$).

If $p$ denotes the plasma kinetic pressure, then

$$\operatorname{grad} p = j \wedge B. \tag{1.4}$$

Thus, the toroidal current $j_\zeta$ satisfies the Grad-Mercier-Shafranov relation

$$j_\zeta(r, \zeta) = r\frac{\partial p}{\partial u} + \frac{1}{2r}\frac{\partial}{\partial u}(rB_\zeta)^2$$

$p$ and $rB_\zeta$ depend on the shape of flux lines. In Temam [4] or [5], the case where $j_\zeta = -\lambda u_-$ was considered. Here we will consider $j_\zeta = -g(u)$. The mathematical problem satisfied by $u$ is then given in the cross section of the torus denoted by $\Omega$ and the boundary conditions are the same as in [5], that is

$$\left.\begin{aligned} u &= \gamma \\ \int_{\partial\Omega} \frac{1}{r}\frac{\partial u}{\partial n} &= I, \end{aligned}\right\} \quad \text{on } \partial\Omega,$$

where $\gamma$ is an unknown constant and $I$ is a given positive constant. The proof of the main example needs an extension of the strong continuity result of the relative rearrangement map introduced in [2], as in Theorem 1.1.

THEOREM 1.1. *Let $\Omega$ be a bounded, open, and connected set in $\mathbb{R}^N$ with Lipschitz boundary. Let $u \in W^{1,r}(\Omega)$, $1 < r \le +\infty$ and let $\{u_j\}_{j\ge 0}$ be a sequence of functions in $W^{1,r}(\Omega)$ such that $|\{x \in \Omega : Du_j(x) = 0\}| = |\{x \in \Omega : Du(x) = 0\}| = 0$. If $u_j \to u$ in $W^{1,r}(\Omega)$ and $b \in L^p(\Omega)$, $1 \le p < +\infty$, then*

$$b_{*u_j}(|u_j > u_j(x)|) \to b_{*u}(|u > u(x)|) \text{ strongly in } L^p(\Omega)$$

*and*

$$b_{*u_j} \to b_{*u} \text{ strongly in } L^p(\Omega_*).$$

*Here, $b_{*u}$ is the weak derivative of the function $w$ given by*

$$w(\sigma) = \int_{u > u_*(\sigma)} b(x)\,dx, \qquad \forall \sigma \in [0, \operatorname{meas}(\Omega)]$$

*and the measure of set $E$ is denoted by $|E|$.*

## 2. EXISTENCE RESULTS IN A GENERAL FRAMEWORK

Let $\Omega$ be a bounded, open, and connected set in $\mathbb{R}^N$, $N \geq 2$, with a smooth boundary $\partial\Omega$.

We denote by $(u, v)$ the scalar product in $L^2(\Omega)$, and by $||u||_r$ and $||u||_{H^r}$ the norms in $L^r(\Omega)$ and in $H^r(\Omega)$, respectively, and $((u, v))$ the usual scalar product in $H^1(\Omega)$. We will also consider the space

$$\mathcal{V} = H_0^1(\Omega) \oplus \mathbb{R} = \{v \in H^1(\Omega),\ v = \text{constant on } \partial\Omega\},$$

the subsets

$$\mathcal{V}_0 = \{v \in \mathcal{V} : |\{x \in \Omega : Dv(x) = 0\}| = 0\},$$
$$\mathcal{V}_0 \cup \mathbb{R} = \mathcal{V}_0 \cup \{v \in \mathcal{V} : v = \text{cst}\}.$$

$\Delta$ is the Laplace operator, $\frac{\partial}{\partial n}$ the outward normal derivative on $\partial\Omega$ and $I$ is a given positive constant. Let us consider the following problem:

$$(\mathcal{P}_\delta) \qquad \begin{aligned} &-\Delta u + g(u) = 0, \qquad \text{in } \Omega,\\ &u \in \mathcal{V}_0 \cap W^{2,2}(\Omega),\\ &\int_{\partial\Omega} \frac{\partial u}{\partial n}\, d\sigma = I. \end{aligned}$$

The operator $g$ is assumed to satisfy the following conditions.

(H1) $g : \mathcal{V} \to L^1(\Omega)$ is such that its restriction to $\mathcal{V}_0 \cup \mathbb{R}$ is a continuous operator, in the sense that, if $v \in \mathcal{V}_0 \cup \mathbb{R}$ and $\{v_k\}_{k\in\mathbb{N}}$ is a sequence in $\mathcal{V}_0 \cup \mathbb{R}$ such that $v_k \to v$ in $H^1(\Omega)$-strong, then $g(v_k) \to g(v)$ in $L^1(\Omega)$-strong.

(H2) There exist positive constants $\delta$, $\mu_0$ and $\delta'$ such that

$$I - \delta|\Omega| > 0, \tag{2.1}$$

$$\begin{aligned} &g(v)(x) \geq \delta' + \mu_0 v_-(x), \qquad &&\text{a.e. in } \Omega,\\ &g(v)(x) = \delta, &&\text{a.e. in } \{x \in \Omega : v(x) \geq 0\}, \end{aligned} \tag{2.2}$$

for all $v \in \mathcal{V}$.

(H3) For all $\eta > 0$, there exists a constant $C_\eta > 0$, such that

$$||g(v)||_2^{1/p} \leq \eta ||v_-||_{2p} + C_\eta, \tag{2.3}$$

for all $v \in \mathcal{V}_0$, where, here and in the following, $p = N/N-2$ if $N > 2$, $p \geq 1$ if $N = 2$.

The the following theorem holds.

THEOREM 2.1. *Let $\Omega$ be an open, bounded and connected set of $\mathbb{R}^N$, $N \geq 2$ with smooth boundary $\partial\Omega$ (for example $C^2$). Let us consider the problem $(\mathcal{P}_\delta)$ under the assumptions (H1)–(H3). Then there exists at least one solution $u \in \mathcal{V}_0 \cap W^{2,2}(\Omega)$, of the problem $(\mathcal{P}_\delta)$.*

## 3. NONSTANDARD GRAD-MERCIER-SHAFRANOV MODELS

In this section, we want to apply the previous results in order to give an existence result for a nonstandard Grad-Mercier-Shafranov problem of the following type (see [6,7] for nonlocal models and [5,8] for local models)

$$z \qquad \begin{aligned} &-\Delta u + G(\cdot, \beta(u)) = 0, \quad \text{in } \Omega,\\ &u \in \mathcal{V}_0 \cap W^{2,2}(\Omega),\\ &\int_{\partial\Omega} \frac{\partial u}{\partial n}\, d\sigma = I, \end{aligned} \tag{$\mathcal{GS}$}$$

where $\beta : H^1(\Omega) \to \mathbb{R}^3$ and $G : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ is a Caratheodory function, that is, for all $\xi \equiv (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, the map $x \to G(x, \xi)$ is measurable and, for almost every $x \in \Omega$, the map $\xi \to G(x, \xi)$ is continuous.

Let $\tau : L^2(\Omega) \to L^2(\Omega)$ be an operator vanishing on constant functions and such that there exist $\alpha_1, \alpha_2 > 0$ satisfying

$$\begin{aligned} &\tau(v)(x) = 0, \qquad \text{if } v(x) \geq 0, \\ &\|\tau(v)\|_2 \leq \alpha_1 \|v_-\|_2 + \alpha_2. \end{aligned} \tag{3.1}$$

For $v \in L^2(\Omega)$, let $k_0(v)(x) = \min(|v > v(x)|^{1-1/N}, (|\Omega| - |v \geq v(x)|)^{1-1/N})$ and $b \in L^\infty(\Omega)$. We consider the problem $(\mathcal{G}S)$ under the following assumptions.

$(GS_1)$ $\beta(v) = (v_-, \tau(v) b_{*v}(|v > v(\cdot)|), k_0(v_-) \frac{dv_*}{ds}(|v > v(\cdot)|))$.

$(GS_2)$ There exist positive constants $\delta, \delta', \mu_0, \kappa_1, \kappa_2$ such that

$$I - \delta|\Omega| > 0, \tag{3.2}$$

$$\begin{aligned} \delta' + \mu_0|\xi_1| \leq G(x, \xi) \leq \kappa_1(|\xi_1| + |\xi_2|) + \kappa_2, \qquad &\text{a.e. } x \in \Omega, \forall \xi \equiv (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3, \\ G(x, 0) = \delta, \qquad &\text{a.e. } x \in \Omega. \end{aligned} \tag{3.3}$$

Let us define the following operator:

$$g : v \in \mathcal{V} \to G(\cdot, \beta(v)(\cdot)) \in L^1(\Omega). \tag{3.4}$$

LEMMA 3.1. *Under the assumptions $(GS_1)$,$(GS_2)$, the operator $g$ defined in (3.4) satisfies the assumptions (H1)–(H3) given in Section 2.*

THEOREM 3.1. *Under the assumptions $(GS_1)$,$(GS_2)$ and (3.2), there exists at least one solution of the problem $(\mathcal{G}S)$.*

This model in a Tokamak is different from the model appearing in a Stellerator (see [9,10]). Nevertheless, many techniques used in these papers are useful in this case.

## REFERENCES

1. K.M. Chong and N.M. Rice, Equimeasurable rearrangements of functions, In *Queen's Papers in Pure and Applied Mathematics*, Volume 28, Queen's University, Ontario, (1971).
2. J.M. Rakotoson, Strong continuity of the relative rearrangement maps and application to a Galerkin approach for nonlocal problems, *Appl. Math. Lett.* **8** (6), 61–63, (1995).
3. H. Grad, Plasma transport in three dimensions, *Ann. New York Acad. Sci.* **357**, 223–235, (1980).
4. R. Temam, A nonlinear eigenvalue problem: The shape at equilibrium of a confined plasma, *Arc. Rat. Mech. Anal.* **60**, 51–73, (1975).
5. R. Temam, Remarks on a free boundary problem arising in plasma physics, *Comm. Part. Diff. Equat.* **2**, 563–585, (1977).
6. J. Mossino, Inégalités isopérimétriques et applications en physique, In *Collection Travaux en Cours*, Hermann, Paris, (1984).
7. J.M. Rakotoson, Un modéle non local en physique des plasmas: Résolution par une méthode de degré topologique, *Acta Applicandae Mathematicae* **4**, 1–14, (1985).
8. H. Berestycki and H. Brezis, On a free boundary problem arising in plasma physics, *Nonlinear Analysis T.M.A.* **4**, 415–436, (1980).
9. J.I. Diaz and J.M. Rakotoson, On a nonlocal stationary free boundary problem arising in the confinement of a plasma in a Stellerator Geometry, *Arch. Rat. Mech. Anal.* **134**, 53–95, (1996).
10. J.I. Diaz, J.F. Padial and J.M. Rakotoson, Mathematical treatment of the magnetic confinement in a current carrying stellarator, *Nonliner Analysis T.M.A.* (to appear).
11. A. Alvino, P.L. Lions and G. Trombetti, On optimization problems with prescribed rearrangements, *Nonlinear Analysis T.M.A.* **13**, 185–220, (1989).
12. A. Ferone and R. Volpicelli, Some relations between pseudo-rearrangement and relative rearrangement, (Preprint).
13. J. Mossino and R. Temam, Directional derivative of the increasing rearrangement mapping and applications to a queer differential equation in plasma physics, *Duke Math. J.* **48**, 475–495, (1981).
14. J.M. Rakotoson, Relative rearrangement for highly nonlinear equations, *Nonlinear Analysis T.M.A.* **24**, 493–507, (1995).
15. J.M. Rakotoson and R. Temam, A co-area formula with applications to monotone rearrangement and to regularity, *Arch. Rat. Mec. AnaL.* **109**, 213–238, (1990).