CARLEMAN'S INEQUALITY FOR FINITE SERIES

BY

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1. *Introduction.* CARLEMAN [2] proved the following inequality for convergent infinite series $\sum\limits_{1}^{\infty} a_n$ with positive terms:

$$\sum_{\nu=1}^{\infty} (a_1 \dots a_\nu)^{1/\nu} < e \sum_{\nu=1}^{\infty} a_\nu. \tag{1.1}$$

The constant e is best possible, although there is no convergent series for which equality holds.

If, however, we restrict the series to a finite number of terms, we obtain an inequality with a smaller best possible bound λ_n:

$$\sum_{\nu=1}^{n} (a_1 \dots a_\nu)^{1/\nu} \leqslant \lambda_n \sum_{\nu=1}^{n} a_\nu \tag{1.2}$$

for all $a_1 > 0, \dots, a_n > 0$. It is the purpose of this paper to establish the asymptotic behaviour of λ_n if $n \to \infty$. We shall show that

$$\lambda_n = e - \frac{2\pi^2 e}{(\log n)^2} + O\left(\frac{1}{(\log n)^3}\right). \tag{1.3}$$

The form of (1.3) reminds of similar results for the finite sections in Hilbert's inequality, where

$$\lambda_n = \pi - \tfrac{1}{2} \frac{\pi^5}{(\log n)^2} + O\left(\frac{\log\log n}{(\log n)^3}\right) \tag{1.4}$$

(see [1]), and in Hardy's inequality, where

$$\lambda_n = 4 - \frac{16\,\pi^2}{(\log n)^2} + O\left(\frac{\log\log n}{(\log n)^3}\right) \tag{1.5}$$

(see [4]). The questions about Hilbert's and Hardy's inequalities are problems on the largest eigenvalue of a matrix, and the methods by which (1.4) and (1.5) were derived depended essentially on that point of view. Carleman's inequality, however, is essentially different in nature, and our method for proving (1.3) will have nothing in common with the linear algebra methods used for the proofs of (1.4) and (1.5).

We shall show (sec. 2), by an argument almost completely copied from CARLEMAN's paper [2], that λ_n is the largest solution of the equation

$h_n(\lambda) = \log(n\lambda)$, where the functions $h_\nu(\lambda)$ are defined recursively by $h_1(\lambda) = 0$, and

$$(1.6) \qquad h_{\nu+1}(\lambda) - h_\nu(\lambda) = \frac{1}{\nu+1}\{\log(1-(\lambda\nu)^{-1}\, e^{h_\nu(\lambda)})^{-\nu} - h(\lambda)\}.$$

The rest of this paper is devoted to a close study of the asymptotics of this recurrence. It is a problem with several quite difficult aspects, and we present it in great detail since the methods may be applicable in many similar cases.

We shall show in sec. 3 that our final result (1.3) follows directly from the asymptotic behaviour of the logarithm of what we shall call the "breakdown index", i.e. the number of h_ν's we can evaluate from (1.6) (with $h_1 = 0$) without having to evaluate the logarithm of a non-positive number.

It was Prof. H. S. Wilf who suggested to the author to attack the asymptotic behaviour of λ_n by an investigation of the recurrence relations arising from Carleman's method (sec. 2 below).

2. *Carleman's analysis.* Carleman [2] attacked the problem of λ_n by the Lagrange multiplier method: λ_n is the maximum of the expression on the left-hand side of (1.2) under the condition that $\sum_1^n a_\nu = 1$. This easily leads to the result that λ_n is the largest positive value of λ such that the following set of equations has a solution with $a_1 > 0, \ldots, a_n > 0$:

$$\begin{aligned}
\lambda(a_1 - a_2) &= a_1 \\
2\lambda(a_2 - a_3) &= (a_1 a_2)^{1/2} \\
3\lambda(a_3 - a_4) &= (a_1 a_2 a_3)^{1/3} \\
&\cdots\cdots\cdots \\
(n-1)\,\lambda(a_{n-1} - a_n) &= (a_1 \ldots a_{n-1})^{1/(n-1)} \\
n\lambda a_n &= (a_1 \ldots a_n)^{1/n}.
\end{aligned}$$

Introducing an extra variable a_{n+1}, we can replace the last equation by

$$n\lambda(a_n - a_{n+1}) = (a_1 \ldots a_n)^{1/n}, \qquad a_{n+1} = 0.$$

This means that λ_n satisfies the condition that if we start from an arbitrary $a_1 > 0$, and if we evaluate $a_2, a_3, \ldots$ recursively from

$$(2.1) \qquad \nu\lambda(a_\nu - a_{\nu+1}) = (a_1 \ldots a_\nu)^{1/\nu},$$

then $a_1 > 0, a_2 > 0, \ldots, a_n > 0, a_{n+1} = 0$. Moreover, λ_n is the largest number with this property.

We simplify (2.1) by a substitution

$$(2.2) \qquad h_\nu = \nu^{-1} \log(a_1 \ldots a_\nu) - \log a_\nu \qquad (\nu = 1, \ldots, n).$$

Now h_{n+1} is not defined, but the condition $a_{n+1} = 0$ can be replaced by $n\lambda a_n = (a_1 \ldots a_n)^{1/n}$, that is, by $h_n = \log(n\lambda)$. The advantage of (2.2) is that we obtain a recursion expressing $h_{\nu+1}$ in terms of h_ν without using $h_1, \ldots, h_{\nu-1}$. It is obvious that $h_1=0$. The h_ν's depend on λ, of course, so we represent them by $h_\nu(\lambda)$. It is easily verified that (2.2) transforms into (1.6).

Carleman proved that $\lambda_n<e$, by the following argument. If for $\lambda \geqslant e$ we evaluate $h_2, h_3, \ldots$ consecutively from (1.6), we can show that

$$h_\nu(\lambda) < 1-\nu^{-1} \qquad (\lambda \geqslant e,\ \nu = 2, 3, \ldots), \tag{2.3}$$

and this implies that for no value of n we have $h_n(\lambda)=\log (n\lambda)$. In other words, our λ_n cannot be $\geqslant e$.

Formula (2.3) can be proved by induction. If $h_\nu(\lambda) \leqslant 1-\nu^{-1}$ for some ν, then we have

$$e^{1-h_\nu(\lambda)} > 1+\nu^{-1},$$

and, by (1.6), this is equivalent to $h_{\nu+1}(\lambda)<1-(\nu+1)^{-1}$.

From the fact that $\lambda_n<e$ $(n=1, 2, \ldots)$ we can derive (1.1), with $\leqslant$ instead of $<$, however. The fact that there is no convergent series producing equality in (1.1), requires some careful reasoning: under the assumption that the equality sign holds we have to show that (2.1) is true for all ν (with $\lambda=e$); this can be done by proving that the sum of the series is a differentiable function of $a_1, \ldots, a_{\nu+1}$, if $a_{\nu+2}, a_{\nu+3}, \ldots$ are kept constant. Once this has been done, one can show that the a_ν derived from (2.1) in this way, produce a divergent series (if $\lambda=e$). (Carleman proved $a_\nu/a_1>\nu^{-1}$; and in sec. 7 we shall determine the asymptotic behaviour of these a_ν).

A much simpler proof of Carleman's inequality (in the strong form (1.1)) was given later by Pólya (see [3]), but that method does not throw much light on our problem about the finite series.

3. *The breakdown index.* We take any value of $\lambda>0$, and we evaluate $h_2, h_3, \ldots$ consecutively from (1.6), starting with $h_1=0$. We are interested in real values of h_ν only, and therefore we say that the procedure breaks down at the first ν where $1-(\lambda\nu)^{-1} \exp (h_\nu(\lambda)) \leqslant 0$, or, what is the same thing, $h_\nu(\lambda) \geqslant \log (\lambda\nu)$. It may happen that it never breaks down: if $\lambda \geqslant e$ we have, by (2.3)

$$h_\nu(\lambda) \leqslant 1-\nu^{-1} < 1 \leqslant \log (\lambda\nu) \qquad (\nu=1, 2, \ldots). \tag{3.1}$$

We define the *breakdown index* N_λ as the smallest ν for which $h_\nu(\lambda) \geqslant \log (\lambda\nu)$ if there is such a ν, and we put $N_\lambda=\infty$ if $h_\nu(\lambda)<\log (\lambda\nu)$ for all ν. So for all $\lambda>0$ we can say that $h_\nu(\lambda)$ is defined for all $\nu<N_\lambda+1$.

If $\lambda \geqslant e$ we have $N_\lambda=\infty$ (cf. (2.3)); if $0<\lambda<e$, however, it will turn out that N_λ is finite. And we shall show (see sec. 6) that

$$\log N_\lambda = 2^{\frac12}\pi\left(\log \frac{e}{\lambda}\right)^{-\frac12} + O(1) \tag{3.2}$$

if $\lambda<e$, $\lambda \to e$.

It is convenient to have some monotonicity properties available, and indeed, as long as $0<\lambda\leqslant e$ these are easily obtained. We have $h_1(\lambda)=0$ $(0<\lambda\leqslant e)$, λ_1 is the largest λ for which $h_1(\lambda)=\log\lambda$, whence $\lambda_1=1$. Now $h_2(\lambda)$ is defined for $\lambda\geqslant\lambda_1$, and $h_2(\lambda)=\frac{1}{2}\log(1-\lambda^{-1})^{-1}$. This is decreasing for $\lambda\geqslant\lambda_1$, whereas $\log 2\lambda$ is increasing. Moreover $h_2(\lambda)\geqslant h_1(\lambda)$. As $h_2(\lambda_1)=\infty$, $h_2(e)<\log 2e$ (see (2.3)), there is exactly one value of λ for which $h_2(\lambda)==\log 2\lambda$, and that is our λ_2. This procedure can be continued. At each step we argue that h_ν is decreasing for $\lambda>\lambda_{\nu-1}$, that $h_\nu(\lambda_{\nu-1})=\infty$, and that $h_\nu(e)<\log(\nu e)$, and we infer that λ_ν is uniquely defined by $h_\nu(\lambda)==\log(\nu\lambda)$; moreover $h_{\nu+1}$ is again decreasing $(\lambda_\nu\geqslant\lambda)$, since

$$h_{\nu+1}(\lambda)=\nu(\nu+1)^{-1}\{h_\nu(\lambda)+\log(1-(\lambda\nu)^{-1}\exp h_\nu(\lambda))^{-1}\},$$

and both terms on the right are decreasing.

So by induction we obtain that

$$1=\lambda_1<\lambda_2<\lambda_3<\ldots<e$$

and that $h_{\nu+1}(\lambda)$ is defined and decreasing for $\lambda>\lambda_\nu$. Moreover $h_\nu(\lambda)>\log(\nu\lambda)$ if $0\leqslant\lambda<\lambda_\nu$, $h_\nu(\lambda_\nu)=\log(\nu\lambda_\nu)$, $h_\nu(\lambda)<\log(\nu\lambda_\nu)$ if $\lambda>\lambda_\nu$.

It follows that the breakdown index N_λ equals 1 if $\lambda\leqslant\lambda_1$, 2 if $\lambda_1<\lambda\leqslant\lambda_2$, etc.

Knowing this we can immediately translate (3.2) into an asymptotic formula for λ_n. If $\lambda\to e(\lambda<e)$, then $\log N_\lambda$ tends to infinity, and therefore the $\lambda_1,\lambda_2,\ldots$ cannot stay below a constant $<e$. So $\lambda_n\to e$. If $\lambda=\lambda_n$, then $N_\lambda=n$. It follows by (3.2) that

$$\log(e/\lambda_n)=2\pi^2(\log n+O(1))^{-2},$$

and this leads to (1.3).

It may be remarked that for fixed $\lambda\leqslant e$ the $h_\nu(\lambda)$ are $\geqslant 0$ and increase if ν increases from 1 to N_λ. This follows from (1.6) by remarking that

$$\log(1-(\lambda\nu)^{-1}e^h)^{-\nu}>h \tag{3.3}$$

for all h satisfying $h>0$, $e^h<\lambda\nu$ (provided that $\lambda\leqslant e$). We can derive (3.3) as follows: since $(\lambda\nu)^{-1}e^h<1$ the left-hand side is at least $\nu(\lambda\nu)^{-1}e^h$, and as $e^h\geqslant eh$ for all $h\geqslant 0$, (3.3) follows.

The breakdown condition $h_\nu\geqslant\log(\lambda\nu)$ is slightly awkward. We are able to replace it by a simpler one, for example $h_\nu\geqslant 2$, by virtue of the following argument. Let $0<\lambda<e$, and assume that N is such that $N<N_\lambda$ and $h_N>2$. Then we have

$$\log N_\lambda-\log N<16. \tag{3.4}$$

For, if $N\leqslant\nu\leqslant N_\lambda$, we have, by (1.6)

$$h_{\nu+1}-h_\nu>\frac{1}{\nu+1}\{\lambda^{-1}e^{h_\nu}-h_\nu\}>\frac{1}{6(\nu+1)}h_\nu^2.$$

since $(e^{h-1}-h)h^{-2}$ increases for $h\geqslant 2$. We simplify this by considering $k_N, k_{N+1}, \ldots$ defined by $k_N=h_N$, and

$$k_{\nu+1}-k_\nu = k_\nu^2/(6(\nu+1)). \tag{3.5}$$

Obviously $k_\nu\leqslant h_\nu$ $(N\leqslant\nu\leqslant N_\lambda)$, and since $\log(\lambda\nu)\leqslant\nu$, we have $k_\nu\leqslant\nu$ for $N\leqslant\nu<N_\lambda$. Therefore (3.5) guarantees that $k_{\nu+1}<\frac{4}{3}k_\nu$ $(N\leqslant\nu<N_\lambda)$, whence (3.5) implies that

$$k_\nu^{-1}-k_{\nu+1}^{-1} > (8(\nu+1))^{-1} \qquad (N\leqslant\nu<N_\lambda).$$

Taking summation with respect to ν we infer that

$$\tfrac{1}{2} > \sum_{N\leqslant\nu<N_\lambda} (8(\nu+1))^{-1},$$

and this proves (3.4).

The fact that $\log N_\lambda-\log N=O(1)$ means that there is no harm in replacing the left-hand side of (3.2) by $\log N$. So if we find a value of ν for which $h_\nu\geqslant 2$, we need not investigate the rest of the sequence until breakdown.

4. *Heuristic treatment.* Our problem is, roughly, to determine how many steps we have to take in our recurrence (1.6) in order to push h_ν beyond the value 2, assuming that λ is fixed, $\lambda<e$ and λ close to e. The right-hand side of (1.6) equals

$$\frac{1}{\nu+1}\{\lambda^{-1}e^{h_\nu(\lambda)}-h_\nu+e^{2h_\nu(\lambda)}(2\lambda^2\nu)^{-1}+e^{3h_\nu(\lambda)}(3\lambda^3\nu^2)^{-1}+\ldots\}. \tag{4.1}$$

Now assume we are able to neglect the terms containing $\nu^{-1}, \nu^{-2}, \ldots$; then we have a recurrence which can be written as

$$\Delta h = (\nu+1)^{-1}(\lambda^{-1}e^h-h). \tag{4.2}$$

Next we consider ν as a continuous variable, and we replace (4.2) by the corresponding differential equation; that is, we replace Δh by $dh/d\nu$. Then we get

$$\frac{d\log(\nu+1)}{dh} = (\lambda^{-1}e^h-h)^{-1}.$$

This suggests that if N is the number of steps necessary to increase h from 0 to about 2, then $\log N$ is roughly equal to

$$\int_0^2 \frac{dh}{\lambda^{-1}e^h-h}. \tag{4.3}$$

The integrand has its maximum at $h=\log\lambda$, and this is close to 1. In the neighbourhood of that maximum it can be approximated by

$\frac{1}{2}(h-\log\lambda)^2+(1-\log\lambda)$, and therefore the value of (4.3) can be compared with

$$\int_{-\infty}^{\infty}\{\tfrac{1}{2}(h-\log\lambda)^2+(1-\log\lambda)\}^{-1}\,dh = 2^{\frac{1}{2}}\pi(\log(e/\lambda))^{-\frac{1}{2}},$$

and this leads to something like (3.2).

There are various doubtful steps in this argument, but the only one that presents a serious difficulty is the first one. The terms with ν^{-1}, ν^{-2}, ... in (4.1) can be expected to give only a small contribution if ν is large, but the question (to be settled in sec. 5) is whether this contribution is small compared to $\lambda^{-1}\,e^{h_\nu(\lambda)}-h_\nu(\lambda)$. The latter expression can be small if both $h_\nu-1$ and $\lambda-e$ are small, and it is especially in that region that the integrand of (4.3) produces its maximal effect.

5. *Preliminary results*

Lemma 1. If $p>0$ is given, then we can find an integer $\mu>p$ and a number $\beta(1<\beta<e)$ such that

$$\tfrac{1}{2} < h_\mu(\lambda) < \log\lambda-\mu^{-1} \qquad (\beta<\lambda\leqslant e). \tag{5.1}$$

Proof. We have $0\leqslant h_\nu(e)<1-\nu^{-1}$ $(\nu=2, 3, \ldots)$. And not all $h_\nu(e)$ are $\leqslant\frac{1}{2}$. For, (1.6) implies that

$$h_{\nu+1}(e)-h_\nu(e) > (\nu+1)^{-1}\,(e^{-\frac{1}{2}}-\tfrac{1}{2})$$

as long as $h_\nu(e)<\frac{1}{2}$, and as $\sum_1^\infty(\nu+1)^{-1}=\infty$, this cannot apply to all positive integers ν. So there is an integer $\mu>p$ for which

$$\tfrac{1}{2} < h_\mu(e) < 1-\mu^{-1}.$$

Having fixed μ this way, we remark that $h_\mu(\lambda)$ is continuous at $\lambda=e$, and the lemma follows.

Lemma 2. There exist numbers $\beta(1<\beta<e)$ and $c(c>0)$ such that for all λ satisfying $\beta<\lambda\leqslant e$, and for all ν satisfying $1\leqslant\nu\leqslant N_\lambda$ (N_λ is the breakdown index) we have

$$\lambda^{-1}\,e^{h_\nu(\lambda)}-h_\nu(\lambda) > c\,\nu^{-\frac{1}{2}}. \tag{5.2}$$

Proof. We apply lemma 1 with $p=2$, whence we obtain values of $\mu(\mu\geqslant 3)$ and β. For the time being we keep λ fixed $(\beta<\lambda<e)$ and we write h_ν instead of $h_\nu(\lambda)$.

As we remarked in sec. 3, the sequence h_μ, $h_{\mu+1}$, ... is increasing, possibly until breakdown. We shall now first consider those integers $\nu\geqslant\mu$ for which $h_\nu<\log\lambda$.

For those ν we can prove

$$h_{\nu+1}-h_\nu < (\nu+1)^{-1}\,\{\tfrac{1}{2}(\log\lambda-h)^2+\log(e/\lambda)+\tfrac{3}{4}\,\nu^{-1}\}. \tag{5.3}$$

This follows by expanding the logarithm in (1.6) (cf. (4.1)) and remarking that $e^{-u}<1-u+\frac{1}{2}u^2$, where $u=\log\lambda-h_\nu$, and that

$$\tfrac{1}{2}\,e^{-u}\,\nu^{-1}+\tfrac{1}{3}\,e^{-2u}\,\nu^{-2}+\tfrac{1}{4}\,e^{-3u}\,\nu^{-3}+\ldots<\tfrac{3}{4},$$

because of $e^{-u}<1,\ \nu\geqslant\mu\geqslant 3$.

Since $\lambda<e,\ \frac{1}{2}<h_\mu\leqslant h_\nu<\log\lambda$, we have $0<\log\lambda-h_\nu<\frac{1}{2}$, and therefore we can replace (5.3) by the *linear* recurrence relation

$$h_{\nu+1}-h_\nu<(\nu+1)^{-1}\{\tfrac{1}{4}(\log\lambda-h_\nu)+\log(e/\lambda)+\tfrac{3}{4}\,\nu^{-1}\}. \tag{5.4}$$

Putting

$$\log\lambda-h_\nu+4\log(e/\lambda)-\nu^{-1}=t_\nu,$$

(5.4) transforms into

$$t_{\nu+1}>t_\nu\,\{1-\tfrac{1}{4}(\nu+1)^{-1}\}. \tag{5.5}$$

By (5.1) we have $t_\mu>0$, and it follows that $t_\nu>0$ for all ν under consideration.

We have $1-\frac{1}{4}x>(1-x)^{\frac{1}{4}}\ (0<x<1)$, whence (5.5) shows that

$$t_{\nu+1}>t_\nu\,\nu^{\frac{1}{4}}(\nu+1)^{-\frac{1}{4}}.$$

It easily follows that

$$t_{\nu+1}\geqslant(\nu+1)^{-\frac{1}{4}}\,\mu^{\frac{1}{4}}\,t_\mu \tag{5.6}$$

for all ν under consideration, i.e. for all ν for which $h_\nu<\log\lambda$. This is certainly satisfied if $t_\nu>4\log(e/\lambda)$, and (5.6) guarantees that this is true as long as $\nu^{-\frac{1}{4}}\mu^{\frac{1}{4}}t_\mu>4\log(e/\lambda)$. Therefore

$$t_\nu\geqslant\nu^{-\frac{1}{4}}\,\mu^{\frac{1}{4}}\,t_\mu\qquad(\mu\leqslant\nu<\mu\,t_\mu{}^4\,(4\log(e/\lambda))^{-4}), \tag{5.7}$$

and we are sure that no breakdown occurs in this range.

We can now attack (5.2). If $0<h<\log\lambda$, we have

$$e^{h-\log\lambda}-h>\log(e/\lambda)+\tfrac{1}{3}(\log\lambda-h)^2$$

(for $e^{-u}>1-u+\frac{1}{3}u^2$ if $0<u<1$), and as $0\leqslant\log(e/\lambda)<1$ we obtain

$$e^{h-\log\lambda}-h>(\log(e/\lambda))^2+\tfrac{1}{3}(\log\lambda-h)^2>\tfrac{1}{32}(4\log(e/\lambda)+\log\lambda-h)^2,$$

since $u^2+\frac{1}{3}v^2>u^2+(\frac{1}{4}v)^2>\frac{1}{2}(u+\frac{1}{4}v)^2$. Applying this with $h=h_\nu$, and remarking that (5.7) leads to

$$4\log(e/\lambda)+\log\lambda-h_\nu>\nu^{-\frac{1}{4}}\,\mu^{\frac{1}{4}}\,t_\mu,$$

we obtain that the left-hand side of (5.2) is at least

$$(\tfrac{1}{32}\,\mu^{\frac{1}{2}}\,t_\mu{}^2)\nu^{-\frac{1}{2}}.$$

This holds in the region indicated in (5.7). The remaining regions do not cause much trouble. First, for the values $1\leqslant\nu<\mu$ we have $h_\nu(\lambda)<$

$<\log \lambda - \mu^{-1}$ (see (5.1) and use the fact that h_ν increases if ν increases (sec. 3)). It follows that

$$\lambda^{-1} e^{h_\nu} - h_\nu > \tfrac{1}{3}(\log \lambda - h_\nu)^2 > \tfrac{1}{3}\mu^{-2} \geqslant (\tfrac{1}{3}\mu^{-2})\nu^{-\frac{1}{2}}$$

if $1 \leqslant \nu < \mu$. In the second place, we have to consider the following range (which is empty if $\lambda = e$)

$$\mu\, t_\mu{}^4 \,(4 \log (e/\lambda))^{-4} \leqslant \nu < N_\lambda.$$

Here we use that $\lambda^{-1} e^h - h \geqslant \log (e/\lambda)$ for all h, whence (5.1) is realised with

$$c = \mu^{\frac{1}{2}}\, t_\mu{}^2\, 4^{-2} \,(\log (e/\beta))^{-1}.$$

In all three cases the constant is independent of λ and ν, so this completes the proof of the lemma.

Lemma 3. There exists a number $\beta (1 < \beta < e)$ such that for every λ satisfying $\beta < \lambda < e$ there exists an index $N < N_\lambda$ with $h_N > 2$.

Proof. We apply lemma 1 with $p = 2e^3$, and lemma 1 provides us with $\mu (\mu > 2e^3)$ and β such that (5.1) holds.

Next consider the numbers $h_\mu, h_{\mu+1}, h_{\mu+2}, \ldots$ as far as they are <3. If $\nu \geqslant \mu$, $h_\nu < 3$, we have

$$(\lambda\nu)^{-1} e^{h_\nu} < \tfrac{1}{2}, \tag{5.8}$$

and we easily evaluate from (1.6), using $\lambda^{-1} e^h - h \geqslant \log (e/\lambda)$, and using the development (4.1), that

$$(\nu+1)^{-1} \log (e/\lambda) < h_{\nu+1} - h_\nu < 1.$$

The lower estimate shows that not for all $\nu \geqslant \mu$ we have $h_\nu \leqslant 2$, since $\sum_\mu^\infty (h_{\nu+1} - h_\nu)$ would diverge. And if h_σ is the last one below 2, then $h_{\sigma+1}$ is still below 3, the difference being less then 1. So we can take $N = \sigma + 1$. It should be remarked that (5.8) guarantees that $\nu < N_\lambda$.

6. *Behaviour of* $\theta(h_\nu)$. As suggested by the discussion in sec. 4, we shall study $\theta(h_\nu)$, where θ is the function defined by

$$\theta(y) = \int_0^y \frac{dx}{e^x/\lambda - x}. \tag{6.1}$$

We first simplify the recurrence formula (1.6). Assuming

$$1 < \lambda \leqslant e,\ \nu > 2e^2,\ h_\nu < 2, \tag{6.2}$$

we have

$$h_{\nu+1} - h_\nu = (\nu+1)^{-1} \{\lambda^{-1} e^{h_\nu} - h_\nu + \varrho_\nu\},$$

where

$$|\varrho_\nu| = \nu \sum_{j=2}^{\infty} (e^{h_\nu} \lambda^{-1} \nu^{-1})^j j^{-1} < e^4 \nu^{-1}.$$

We deduce that

$$|h_{\nu+1}-h_\nu| \leqslant (\nu+1)^{-1}\,(e^2+e^4),$$

and from this rough estimate we easily obtain

$$|\lambda^{-1}\,e^x-x-(\lambda^{-1}\,e^{h_\nu}-h_\nu)| \leqslant c_1\,\nu^{-1} \qquad (h_\nu \leqslant x \leqslant h_{\nu+1}),$$

with c_1 not depending on λ or ν (still assuming (6.2) and $\beta<\lambda\leqslant e$).

We next apply the mean value theorem:

$$\theta(h_{\nu+1})-\theta(h_\nu) = (h_{\nu+1}-h_\nu)\,\theta'(x)$$

with some x between h_ν and $h_{\nu+1}$. Hence

$$\theta(h_{\nu+1})-\theta(h_\nu) = (\nu+1)^{-1}\,\,(H+\varrho_\nu)/(H+\sigma_\nu), \tag{6.3}$$

where $H=\lambda^{-1}\,e^{h_\nu}-h_\nu$, and $|\varrho_\nu|<e^4\,\nu^{-1}$, $|\sigma_\nu|<c_1\,\nu^{-1}$.

Introducing the extra assumption that $\beta<\lambda\leqslant e$ (see lemma 1), we now have, by (5.2),

$$|\varrho_\nu| < e^4\,c^{-1}\,\nu^{-\frac12}\,H, \qquad |\sigma_\nu| < c_1\,c^{-1}\,\nu^{-\frac12}\,H.$$

Using this in connection with (6.3), we observe that we can find an integer $\varkappa>2e^2$, $\varkappa$ not depending on λ, such that under the extra assumptions $h_\nu<2$, $\nu>\varkappa$ we have

$$|\theta(h_{\nu+1})-\theta(h_\nu)-\log\,((\nu+1)/\nu)| < c_3\,\nu^{-3/2}.$$

Now assuming $\lambda<e$, we take the sum over the values $\varkappa\leqslant\nu<N$, where N is the first index with $h_N>2$ (see lemma 2). It results that

$$|\theta(h_N)-\log N| < c_4+\log\varkappa+\theta(h_\varkappa) \tag{6.4}$$

where c_4 is an absolute constant.

By lemma 1, we can determine $\beta_1(\beta<\beta_1<e)$ such that $h_\varkappa(\lambda)<\log\lambda$ for all λ satisfying $\beta_1<\lambda<e$. Therefore, in the integral defining $\theta(h_\varkappa)$, the maximum of the integrand is attained at $x=h_\varkappa$, whence, by (5.1),

$$c_4+\log\varkappa+\theta(h_\varkappa) < c_4+\log\varkappa+(\log\lambda)\cdot(c\varkappa^{-\frac12})^{-\frac12} < c_5,$$

where c_5 is an absolute constant.

It is not difficult to find the asymptotic behaviour of $\theta(\infty)$, if $\lambda<e$, $\lambda\to e$, by routine methods (cf. sec. 4). They lead to

$$\theta(\infty) = \int_0^\infty \frac{dx}{e^x/\lambda-x} = 2^{\frac12}\,\pi(\log\,(e/\lambda))^{-\frac12}+O(1) \qquad (\lambda<e,\,\lambda\to e), \tag{6.5}$$

and it is also easy to see that

$$\theta(\infty)-\theta(2) = O(1).$$

As $\theta(2)\leqslant\theta(h_N)\leqslant\theta(\infty)$, (6.2) now leads to

$$\log N = 2^{\frac12}\,\pi(\log\,(e/\lambda))^{-\frac12}+O(1).$$

According to the discussion in sec. 3, this completes the proof of (3.2), and it was already shown there that (3.2) leads to our main result (1.3).

7. *The case* $\lambda = e$. If $\lambda = e$ the behaviour of the h_ν is different: they *all* stay below 1, and tend to 1, and the asymptotic problem is how $1 - h_\nu$ behaves if $\nu \to \infty$. The answer is immediately obtained from sec. 6, where $\lambda = e$ was not excluded. We obtain (cf. (6.4)), now for *all* ν,

$$|\theta(h_\nu) - \log \nu| < c_6,$$

where c_6 is an absolute constant.

Instead of (6.5), we now have to use the behaviour of $\theta(h)$ if $\lambda = e$, $h \to 1$, given by

$$\theta(h) = \frac{2}{1-h} + \frac{2}{3} \log \frac{1}{1-h} + O(1) \qquad (0 \leqslant h < 1),$$

and we easily derive

$$h_\nu = 1 - \frac{2}{\log \nu} - \frac{4}{3} \frac{\log \log \nu}{(\log \nu)^2} + O\left(\frac{1}{(\log \nu)^2}\right).$$

From (1.6) it now follows that $h_{\nu+1} - h_\nu = O(\nu^{-1} (\log \nu)^{-2})$, and so we obtain for the a_ν (see (2.2))

$$\log a_\nu - \log a_{\nu+1} = \nu^{-1} h_\nu + (h_\nu - h_{\nu-1}) = \nu^{-1}(h_\nu + O((\log \nu)^{-2})).$$

By standard summation methods we can finally show that there exists a constant A such that

$$a_\nu = A\, \nu^{-1} \{(\log \nu)^2 - \tfrac{4}{3} (\log \nu) (\log \log \nu) + O(\log \nu)\}.$$

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