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Dynamics of horizontal-like maps in higher dimension

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Abstract

We study the regularity of the Green currents and of the equilibrium measure associated to a horizontal-like map in $\mathbb{C}^k$, under a natural assumption on the dynamical degrees. We estimate the speed of convergence towards the Green currents, the decay of correlations for the equilibrium measure and the Lyapounov exponents. We show in particular that the equilibrium measure is hyperbolic. We also show that the Green currents are the unique invariant vertical and horizontal positive closed currents. The results apply, in particular, to Hénon-like maps, to regular polynomial automorphisms of $\mathbb{C}^k$ and to their small perturbations.

MSC: primary 37F; secondary 32U40, 32H50

Keywords: Horizontal-like map; Green current; Equilibrium measure; Entropy; Mixing; Lyapounov exponent; Structural disc of currents

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1. Introduction

The abstract theory of non-uniformly hyperbolic systems is well developed, see e.g. Katok and Hasselblatt [28], Pesin [32], L.-S. Young [38]. It is however difficult to show that a concrete example is a non-uniformly hyperbolic system. The main questions are to construct a measure of maximal entropy, to study the decay of correlations and to show that the Lyapounov exponents do not vanish. Such problems have been studied in dimension 2 for real Hénon maps by Benedicks, Carleson, L.-S. Young, Viana, etc., see e.g. [3,4,39]. In this paper we consider these questions for holomorphic horizontal-like maps in $\mathbb{C}^k$ using tools from complex analysis: positive closed currents, estimates for solutions of the dd^c-equation and appropriate spaces of test forms. The complex analytic methods permit to avoid the delicate arguments used in the real setting.

In [13] the first and the third authors studied the dynamics of polynomial-like maps in several complex variables using adapted spaces of test functions. This approach permits to study convergence problems, in particular, the decay of correlations for the measure of maximal entropy. Recall that a polynomial-like map is a proper holomorphic map $f:U \to V$ between convex open sets $U \Subset V$ (or more generally pseudoconvex open sets) in $\mathbb{C}^k$. Such a map is somehow "expanding," but it has in general a non-empty critical set; so, it is not uniformly hyperbolic in the dynamical sense, see [28]. It is shown in [13] that the measure of maximal entropy is hyperbolic if the topological degree is strictly larger than the other dynamical degrees. This condition is natural and is stable under small perturbations on the map. Holomorphic endomorphisms of $\mathbb{P}^k$ can be lifted to polynomial-like maps in some open sets of $\mathbb{C}^{k+1}$. So, their dynamical study is a special case of polynomial-like maps. Small transcendental perturbations of such maps provide large families of examples.

Here, we consider the quantitative aspects of the dynamics of horizontal-like maps f in any dimension, inside a product of convex open sets $D = M \times N$ in $\mathbb{C}^p \times \mathbb{C}^{k-p}$. They are basically holomorphic maps which are somehow "expanding" in p directions (horizontal directions) and "contracting" in the other $k-p$ directions (vertical directions), see Section 3 for the precise definition. They partially look like a horseshoe. But, the expansion and contraction are of global nature, and in general, these maps are not uniformly hyperbolic. Small perturbations of horizontal-like maps are horizontal-like provided that we shrink slightly the domain of definition. When $p = k$ we obtain polynomial-like maps.

Hénon maps in $\mathbb{C}^2$ were studied by Bedford, Lyubich and Smillie [2] with the equilibrium measure introduced by the third author of the present article, see also [22]. The case of horizontal-like maps in dimension 2, i.e. $k = 2$ and $p = 1$, has been studied by Dujardin with emphasis on biholomorphic maps (Hénon-like maps) [20] and was developed by Dujardin, the first and the third authors to deal with random iteration of meromorphic horizontal-like maps [12]. It turns out that horizontal-like maps are the building blocks for polynomial maps of "saddle type." In particular, they were used to study rates of escape to infinity for polynomial mappings in $\mathbb{C}^2$. The randomness comes from the indeterminacy points at infinity, see also [37].

In this paper, we continue our study in the higher-dimensional case. In order to simplify the notation, *we only consider invertible maps*. However, a large part of our study can be extended to the general case. Some basic objects and the first properties for such maps (Green currents $T_\pm$, equilibrium measure μ, entropy, mixing, etc.) were constructed and established in [17]. The Green current T_+ is positive closed of bidegree (p,p), invariant under f^* and is *vertical*: its support does not intersect the vertical boundary $\partial M \times N$ of D. The Green current T_- is positive closed of bidegree $(k-p, k-p)$, invariant under f_* and is *horizontal*. The equilibrium measure μ is an invariant probability measure which is equal to the wedge-product $T_+ \wedge T_-$ of the Green currents. The definition of wedge-product relies on an intersection theory for positive closed currents.

The main technical problem is the use of currents of bidegree (p, p), $p \geqslant 1$. For that purpose, a geometry on the space of positive closed (p, p)-currents was introduced using as basic objects: *structural discs of currents*. Roughly speaking, in order to travel from a positive closed current R_1 of bidimension $(k-p, k-p)$ to another one R_2, we construct a family of currents parametrized by a holomorphic disc $\Delta \subset \mathbb{C}$. These currents appear as the slices of a positive closed current $\mathscr{R}$ of bidimension $(k-p+1, k-p+1)$ in $\Delta \times D$; the currents R_1 and R_2 are seen as two points of the disc, i.e. two currents obtained by slicing $\mathscr{R}$ with $\{\theta_1\} \times D$ and $\{\theta_2\} \times D$ for some θ_1, θ_2 in Δ. We use properties of subharmonic functions on those structural discs in order to define the wedge-product of currents of higher bidegree and in order to prove the convergence results in the construction of $T_\pm$ and μ. More formally as in [19] we use super-functions, i.e. functions defined on horizontal currents which are p.s.h. on structural discs of currents.

In the present article, we study the quantitative properties of these basic dynamical objects. For a horizontal-like map f, one associates a main dynamical degree $d \geqslant 2$ which is an integer. The topological entropy of f and the entropy of μ are equal to $\log d$. We will define the other dynamical degrees $d_s^\pm$ in Section 3. One of our main results is the following.

Theorem 1.1. *Let f be an invertible horizontal-like map on a convex domain $D = M \times N$ in $\mathbb{C}^p \times \mathbb{C}^{k-p}$. Assume that the main dynamical degree d of f is strictly larger than the other dynamical degrees. Then the Green currents T_+ and T_- of f are the unique, up to a multiplicative constant, invariant vertical and horizontal positive closed currents of bidegrees (p, p) and $(k-p, k-p)$ respectively. The equilibrium measure μ of f is exponentially mixing and is hyperbolic. More precisely, μ admits $k-p$ strictly negative and p strictly positive Lyapounov exponents.*

We study the speed of convergence towards the Green currents $T_\pm$ and the equilibrium measure μ, and also the regularity of these objects. The regularity is studied by considering on which space of forms or functions the currents or measures act continuously. We show in particular that μ is PB, that is, plurisubharmonic functions (p.s.h. for short) are μ-integrable. The main tools here are estimates and localization of the support for good solutions of the dd^c-equation. We obtain these estimates through integral formulas (a classical result by Andreotti–Grauert is crucial here). They permit to apply the dd^c-method and the duality method as in [13,15–17]. The speed of convergence towards the Green currents is a basic ingredient in the proof of the decay of correlations for μ.

For Hénon like-maps ($k = 2$, $p = 1$), the hypothesis on the dynamical degrees is always satisfied. Theorem 1.1, except for the decay of correlations (exponential mixing), was proved in [20]. The decay of correlations for Hölder observables and for Hénon maps was investigated by the first author in [11]. The hyperbolicity of the equilibrium measure is considered in a very general context for meromorphic maps on compact Kähler manifolds by de Thélin [9]. We follow his method.

We end this introduction by giving another large family of examples. Consider a polynomial automorphism f of $\mathbb{C}^k$. We still denote by f its meromorphic extension to $\mathbb{P}^k$. When the indeterminacy sets I_+ and I_- of f and f^{-1} in the hyperplane at infinity L_∞ are non-empty and have no intersection, we say that f is *regular*. Then there is an integer p such that $\dim I_+ = k-p-1$ and $\dim I_- = p-1$. We refer to [35] for the basic dynamical objects and properties of such maps, see also Section 6 below. Let $z = (z_1, \dots, z_k)$ denote the coordinates in $\mathbb{C}^k$ and denote

$[z_0 : \cdots : z_k]$ the homogeneous coordinates of $\mathbb{P}^k$. The hyperplane at infinity $L_\infty := \mathbb{P}^k \setminus \mathbb{C}^k$ is given by the equation $z_0 = 0$.

Corollary 1.2. *Let f be a regular polynomial automorphism of $\mathbb{C}^k$. Assume that the indeterminacy sets of f and f^{-1} are linear and defined by*

$$I_+ = \{z_0 = z_1 = \cdots = z_p = 0\} \quad \textit{and} \quad I_- = \{z_0 = z_{p+1} = \cdots = z_k = 0\}.$$

Let B_s^R denote the ball of center 0 and of radius R in $\mathbb{C}^s$. Then, if R is large enough, any holomorphic map on $B_p^R \times B_{k-p}^R$, close enough to f, is horizontal-like. Moreover, its equilibrium measure is exponentially mixing and hyperbolic.

Note that the above perturbation of f may be transcendental and that Corollary 1.2 produces large families of examples.

Here is a brief outline of the paper. In Section 2, the main tools, in particular, several classes of currents and the solution of the dd^c-equation, are introduced. In Section 3, we recall the dynamical objects associated to a horizontal-like map. Theorem 1.1 is proved in Sections 4 and 5. Corollary 1.2 is deduced from Theorem 1.1 and from Proposition 6.1 in the last section. Also in the last section open questions are stated.

1.1. Notation and convention

Throughout the paper, $D := M \times N$ is a bounded convex domain in $\mathbb{C}^p \times \mathbb{C}^{k-p}$. The estimates we obtain are valid in the interior of D and might be bad near the boundary, but this is harmless for the type of maps we consider. So, we sometimes reduce D slightly in order to have maps and currents defined in a neighbourhood of $\overline{D}$; this simplifies the exposition. We will also choose strictly convex domains with smooth boundary $M'' \Subset M' \Subset M$ and $N'' \Subset N' \Subset N$ and consider the domains $D' := M' \times N'$ and $D'' := M'' \times N''$. When we consider vertical currents R or horizontal currents S, Φ, our choice is so that R is supported on $M'' \times N$ and S, Φ are supported on $M \times N''$. When we consider a horizontal-like map f on D, we assume that $f^{-1}(D) \subset M'' \times N$ and $f(D) \subset M \times N''$. So, f restricted to D' or D'' is horizontal-like. The convex domains $\widetilde{M}$, $\widehat{M}$, $\widetilde{N}$, $\widehat{N}$ are chosen so that $M \Subset \widetilde{M} \Subset \widehat{M}$ and $N \Subset \widetilde{N} \Subset \widehat{N}$. Note also that *when we consider the convergence of a family of vertical or horizontal currents, we assume that they have support in the same vertical or horizontal set.*

2. Currents and dd^c-equation

In this section, we will introduce the tools used in this work. We will give some geometrical and analytical properties of several classes of currents. In particular, we will define structural discs of currents and solve the dd^c-equation with estimates and with controlled support. Recall that $d^c := \frac{i}{2\pi}(\overline{\partial} - \partial)$.

2.1. Vertical, horizontal currents and their intersection

We call *vertical* (respectively *horizontal*) *boundary* of D the sets $\partial_v D := \partial M \times N$ (respectively $\partial_h D := M \times \partial N$). A subset E of D is *vertical* (respectively *horizontal*) if $\overline{E}$ does not intersect $\overline{\partial_v D}$ (respectively $\overline{\partial_h D}$). Let π_1 and π_2 denote the canonical projections of D onto M

and N. Then E is vertical or horizontal if and only if $\pi_1(E) \Subset M$ or $\pi_2(E) \Subset N$. A current on D is *vertical* or *horizontal* if its support is vertical or horizontal. Let $\mathscr{C}_v(D)$ denote the cone of positive closed vertical currents of bidegree (p,p) on D. Consider a current R in $\mathscr{C}_v(D)$. Since π_2 is proper on $\operatorname{supp}(R)$, $(\pi_2)_*(R)$ is a positive closed current of bidegree $(0,0)$ on N. Hence, $(\pi_2)_*(R)$ is given by a constant function on N that we denote by $\|R\|_v$. Convergence in $\mathscr{C}_v(D)$ is the weak convergence of currents with support in a fixed vertical set.

Recall from Theorem 2.1 in [17] that the slice measure $\langle R, \pi_2, w\rangle$ is defined for every $w \in N$, and that its mass is equal to $\|R\|_v$ which is independent of w. We say that $\|R\|_v$ is the *slice mass* of R. For every smooth probability measure Ω with compact support in N, we have $\|R\|_v := \langle R, (\pi_2)^*(\Omega)\rangle$. When $\|R\|_v = 1$ we say that R is *normalized*. Let $\mathscr{C}_v^1(D)$ denote the set of such currents. This convex set is relatively compact in the cone of positive closed currents on D. In particular, the mass of normalized currents R on a compact set of D is bounded uniformly on R. In order to avoid convergence problems on the boundary, we will also use the convex set $\mathscr{C}_v^1(M \times \overline{N})$ of positive closed currents which are vertical in $M \times \widetilde{N}$ with slice mass 1 for some neighbourhood $\widetilde{N}$ of $\overline{N}$.

The slice mass $\|\cdot\|_h$, the sets $\mathscr{C}_h(D)$, $\mathscr{C}_h^1(D)$ and the convergence for horizontal currents of bidegree $(k-p, k-p)$ are defined similarly. If R is a current in $\mathscr{C}_v(D)$ and S is a current in $\mathscr{C}_h(D)$ we can define the intersection $R \wedge S$. This is a positive measure of mass $\|R\|_v\|S\|_h$ with support in $\operatorname{supp}(R) \cap \operatorname{supp}(S)$, see [17]. It depends linearly on R and on S and is continuous with respect to the *plurifine topology* in the following sense. Let (R_θ) and $(S_{\theta'})$ be structural discs in $\mathscr{C}_v^1(D)$ and $\mathscr{C}_h^1(D)$, see the definition below. Assume that $\operatorname{supp}(R_\theta) \cap \operatorname{supp}(S_{\theta'})$ is contained in an open set $\Omega \Subset D$. If φ is a p.s.h. function on a neighbourhood of $\overline{\Omega}$, then $\langle R_\theta \wedge S_{\theta'}, \varphi\rangle$ is a p.s.h. function of (θ, θ') or equal to $-\infty$, see Proposition 3.4 and Remark 3.8 in [17]. Basically, for a suitable choice, with $R_1 = R$, $S_1 = S$ and R_θ, $S_{\theta'}$ smooth when $\theta \neq 1$, $\theta' \neq 1$, we obtain $R \wedge S$ as the limit of $R_\theta \wedge S_{\theta'}$, $R_\theta \wedge S$, $R \wedge S_{\theta'}$ for $\theta \to 1$ and $\theta' \to 1$. It is also shown in [17] that for a p.s.h. function φ on D

$$\langle R \wedge S, \varphi\rangle = \limsup \langle R' \wedge S', \varphi\rangle = \limsup \langle R' \wedge S, \varphi\rangle = \limsup \langle R \wedge S', \varphi\rangle,$$

with R', S' smooth in $\mathscr{C}_v(D)$, $\mathscr{C}_h(D)$ converging respectively to R and S.

2.2. *Structural discs of currents*

Let X be a complex manifold. Consider a positive closed (p,p)-current $\mathscr{R}$ in $X \times D$. We assume that the support of $\mathscr{R}$ is contained in $X \times M' \times N$ for some open set $M' \Subset M$. Let $\pi : X \times D \to X$ denote the canonical projection. It is shown in [17] that the slice $\langle \mathscr{R}, \pi, x\rangle$ exists for every $x \in X$. They can be considered as the intersection of $\mathscr{R}$ with the current of integration on $\pi^{-1}(x)$. This is a positive closed (p,p)-current on $\{x\} \times D$ that we identify with a current R_x on D which is vertical. When $\mathscr{R}$ is a smooth form, the slice R_x is simply the restriction of $\mathscr{R}$ to $\pi^{-1}(x)$. The slice mass of R_x does not depend on x. So, multiplying $\mathscr{R}$ with a constant, we can assume that this mass is 1. We obtain a map $\tau : X \to \mathscr{C}_v^1(D)$ with $\tau(x) := R_x$. In general, R_x does not depend continuously on x with respect to the usual topology on X. The dependence is continuous with respect to the *plurifine topology*, i.e. the coarsest topology for which p.s.h. functions on X are continuous. We call *structural variety* of $\mathscr{C}_v^1(D)$ the map τ or the family (R_x). This notion can be easily extended to $\mathscr{C}_v^1(M \times \overline{N})$.

Consider a vertical positive closed (p,p)-current R in $\mathscr{C}_v^1(M \times \overline{N})$. So, R is a vertical current of slice mass 1 on $M' \times \widetilde{N}$ for some convex open sets $M' \Subset M$ and $\widetilde{N} \Supset N$. Let Δ denote a small neighbourhood of the interval $[0,1]$ in $\mathbb{C}$. We constructed in [17] a *particular structural disc* $(R_\theta)_{\theta\in\Delta}$ in $\mathscr{C}_v^1(M\times\overline{N})$ parametrized by Δ such that $R_1=R$ and R_0 is independent of R. The current R_θ is obtained as a regularization of R. More precisely, we consider some holomorphic family of linear endomorphisms $h_{a,b,\theta}:\mathbb{C}^k\to\mathbb{C}^k$ parametrized by $(a,b,\theta)\in\mathbb{C}^p\times\mathbb{C}^{k-p}\times\Delta$ with $h_{a,b,1}=\mathrm{id}$. The current R_θ is obtained using a smooth probability measure ν with compact support in $\mathbb{C}^p\times\mathbb{C}^{k-p}$:

$$R_\theta := \int (h_{a,b,\theta})_*(R)\, d\nu(a,b).$$

The convexity of $M\times N$ and the fact that R is defined on $M'\times\widetilde{N}$ permit to define the smoothing and to obtain vertical currents R_θ in $\mathscr{C}_v^1(M\times\overline{N})$. The size of Δ depends only on M, M', N and $\widetilde{N}$. The considered structural discs satisfy the following important properties. *The currents R_θ depend continuously on θ, linearly on R and are smooth for $\theta\neq 1$. The continuity is with respect to the weak topology on R_θ and the usual topology on θ. Moreover, R_θ depend continuously on θ and on R with respect to the usual topology on $\theta\in\Delta\setminus\{1\}$, the $\mathscr{C}^\infty$ topology on R_θ and the weak topology on R. When R is smooth, the last property also holds for $\theta\in\Delta$.*

2.3. PSH currents and p.s.h. functions

A real $(k-p,k-p)$-horizontal current Φ on D is called *PSH* if $dd^c\Phi\geqslant 0$.[1] Let $\mathrm{PSH}_h(D)$ denote the set of horizontal PSH currents. It is endowed with the following topology. A sequence (Φ_n) converges to Φ in $\mathrm{PSH}_h(D)$ if $\Phi_n\to\Phi$ weakly and if Φ_n and Φ have their supports in a fixed horizontal set of D.

Recall that an upper semi-continuous function $\phi: D\to\mathbb{R}\cup\{-\infty\}$ is p.s.h. if it is not identically $-\infty$ and if its restriction to any holomorphic disc in D is subharmonic or equal to $-\infty$. Let $\mathrm{PSH}(D)$ denote the cone of such functions. It is relatively compact in $L^p_{\mathrm{loc}}(D)$ for $1\leqslant p<+\infty$. Note that an L^1_{loc} function $\phi: D\to\mathbb{R}\cup\{-\infty\}$ is p.s.h. if it is strongly upper semi-continuous and if $dd^c\phi$ is a positive closed current. The strong upper semi-continuity means $\phi(a)=\limsup_{z\to a}\phi(z)$ for $a\in D$ and $z\in A$ where A is any measurable subset of full measure in D. Denote by $\mathrm{PSH}(\overline{D})$ the cone of p.s.h. functions defined in a neighbourhood of $\overline{D}$.

2.4. Extension of spaces of test forms and super-functions

Let R be a current in $\mathscr{C}_v^1(D)$. It acts on horizontal smooth forms of bidegree $(k-p,k-p)$. We will extend this space of test forms. Let $\mathscr{H}_h(D)$ denote the space of real horizontal currents Φ of bidegree $(k-p,k-p)$ with $dd^c\Phi=0$. We consider the following topology on $\mathscr{H}_h(D)$: a sequence (Φ_n) converges to Φ in $\mathscr{H}_h(D)$ if $\Phi_n\to\Phi$ weakly and Φ_n have support in a fixed horizontal set.

Proposition 2.1. *The action of R can be extended in a unique way to a positive continuous linear form on $\mathscr{H}_h(D)$. Moreover, $(R,\Phi)\mapsto\langle R,\Phi\rangle$ with $\Phi\in\mathscr{H}_h(D)$ is bilinear and continuous in (R,Φ). In particular, $\langle R,\Phi\rangle$ is bounded on compact subsets of $\mathscr{C}_v^1(D)\times\mathscr{H}_h(D)$.*

[1] In other situations, we often assume that Φ is of order 0 or negative. This is necessary in particular when one defines the pull-back by a non-invertible map [18]. Note that a p.s.h. function is defined everywhere but not a PSH current.

Proof. Observe that if Φ is a current in $\mathscr{H}_h(D)$ we can use a slight dilation and a convolution in order to regularize Φ. So, there are smooth forms Φ_n converging to Φ in $\mathscr{H}_h(D)$. This implies the uniqueness, the linearity and the positivity of the extension. Recall that the positivity means $\langle R, \Phi\rangle \geqslant 0$ for $\Phi \geqslant 0$. We prove now the existence of the extension on $\mathscr{H}_h(D)$ and the continuity.

Shrinking D allows to assume that R is defined on $M' \times \widetilde{N}$ with $M' \Subset M$ and $\widetilde{N} \Supset N$. Consider the structural disc (R_θ) as above. Define $h(\theta) := \langle R_\theta, \Phi\rangle$. As in [17, Theorem 2.1], h is a harmonic function on $\Delta \setminus \{1\}$. If Φ is smooth, the function is defined and is harmonic on Δ. Define $h_n(\theta) := \langle R_\theta, \Phi_n\rangle$. The above description of properties of R_θ implies that h_n converge locally uniformly to h on $\Delta \setminus \{1\}$. Since h_n are harmonic on Δ and locally uniformly bounded, by maximum principle, the limit h can be extended to a harmonic function on Δ and h_n converge to h on Δ. Observe that the limit does not depend on the choice of Φ_n.

We have $\langle R, \Phi\rangle = h(1)$ when Φ is smooth. Define $\langle R, \Phi\rangle := h(1)$ the extension of R to all Φ in $\mathscr{H}_h(D)$. Recall that R_θ, for $\theta \neq 1$, depends continuously on R with respect to the $\mathscr{C}^\infty$ topology on R_θ. Hence, h depends continuously on (R, Φ). The continuity of $\langle R, \Phi\rangle$ follows. □

We will extend R to a linear form on $\mathrm{PSH}_h(D)$, but the extension can take the value $-\infty$. Recall that R is a current on $M' \times \widetilde{N}$.

Proposition 2.2. *The limit $\langle R, \Phi\rangle := \limsup\langle R, \Phi'\rangle$ with Φ' smooth converging to Φ in $\mathrm{PSH}_h(M' \times N)$, defines an extension of R to $\mathrm{PSH}_h(D)$. The extension depends linearly on R, Φ. It takes values in $\mathbb{R} \cup \{-\infty\}$ and does not depend on the choice of M' and $\widetilde{N}$. The function $\theta \mapsto \langle R_\theta, \Phi\rangle$ is subharmonic on Δ and we have $\langle R, \Phi\rangle = \limsup\langle R', \Phi\rangle$ with $R' \to R$ in $\mathscr{C}_v^1(M \times \overline{N})$.*

Proof. We can assume that Φ is supported on $\widetilde{M} \times N'$ and that R is vertical in $M'' \times \widetilde{N}$. So, we can assume that the considered currents Φ' are horizontal on D. Consider first the case where Φ is smooth. Let Φ_n be a sequence of smooth forms converging to Φ in $\mathrm{PSH}_h(D)$. Define $h(\theta) := \langle R_\theta, \Phi\rangle$ and $h_n(\theta) := \langle R_\theta, \Phi_n\rangle$. These functions are subharmonic and continuous on Δ, see [17, Theorem 2.1] (the subharmonicity is deduced from the positivity of $dd^c(\mathscr{R} \wedge \Phi_n)$ and of its push-forward to Δ). We also have $h_n \to h$ on $\Delta \setminus \{1\}$. It follows from the classical Hartogs' lemma [27] that $\limsup h_n(1) \leqslant h(1)$. So, $\limsup\langle R, \Phi'\rangle \leqslant \langle R, \Phi\rangle$.

On the other hand, since R_θ is obtained from R by smoothing using an averaging on a group of linear transformations, a coordinate change implies that $\langle R_\theta, \Phi\rangle = \langle R, \Phi_\theta\rangle$ where Φ_θ is obtained from Φ by a similar smoothing. The fact that Φ is defined on $\widetilde{M} \times N'$ guarantees that Φ_θ is horizontal in D. We also have $\Phi_\theta \to \Phi$ when $\theta \to 1$ for the $\mathscr{C}^\infty$ topology. Since h is continuous we deduce that $\langle R, \Phi_\theta\rangle \to \langle R, \Phi\rangle$ when $\theta \to 1$. So, $\langle R, \Phi\rangle = \limsup\langle R, \Phi'\rangle$ when Φ is smooth. In other words, $\langle R, \Phi\rangle := \limsup\langle R, \Phi'\rangle$ defines an extension of R to all Φ in $\mathrm{PSH}_h(D)$. It is clear that the extension does not depend on the choice of $\widetilde{N}$.

For a general current Φ, there are smooth forms Φ_n converging to Φ in $\mathrm{PSH}_h(D)$. Define h_n and h as above. The function h is defined on $\Delta \setminus \{1\}$. The functions h_n are continuous subharmonic, bounded from above and converge to h on $\Delta \setminus \{1\}$. It follows that h can be extended to a subharmonic function on Δ. By Hartogs' lemma, we have

$$h(1) \geqslant \limsup h_n(1) = \limsup\langle R, \Phi_n\rangle.$$

It follows that $h(1) \geqslant \langle R, \Phi\rangle = \limsup\langle R, \Phi'\rangle$.

On the other hand, since h is subharmonic, we have $h(1) = \limsup h(\theta) = \limsup \langle R, \Phi_\theta \rangle$ when $\theta \to 1$. We deduce as above that $h(1) = \langle R, \Phi \rangle$. Since h depends linearly on R and Φ, $\langle R, \Phi \rangle$ depends linearly on R and Φ. We also obtain that $\theta \mapsto \langle R_\theta, \Phi \rangle$ is subharmonic on Δ.

It remains to prove that $\langle R, \Phi \rangle = \limsup \langle R', \Phi \rangle$ with $R' \to R$ in $\mathscr{C}_v^1(M \times \overline{N})$. This property implies that $\langle R, \Phi \rangle$ is independent of the choice of M'. Since $\langle R, \Phi \rangle = \limsup \langle R_\theta, \Phi \rangle$ for $\theta \to 1$, we have $\langle R, \Phi \rangle \leqslant \limsup \langle R', \Phi \rangle$ with $R' \to R$. Now, if (R'_θ) is the structural disc associated to R' and if $h'(\theta) := \langle R'_\theta, \Phi \rangle$, then $h'(\theta) \to h(\theta)$ for $\theta \neq 1$. We deduce from Hartogs' lemma that $h(1) \geqslant \limsup h'(1)$ which implies that $\langle R, \Phi \rangle \geqslant \limsup \langle R', \Phi \rangle$ and completes the proof. $\square$

Remark 2.3. We can consider R as a vertical current and Φ as a horizontal one in appropriate domains $D' \Subset D$ and define $\langle R, \Phi \rangle$ on D' instead of D. We will obtain the same value. Indeed, in order to define (R_θ) we can find smoothings which are adapted for both D and D', see [17] for details.

Remark 2.4. Let R be a current in $\mathscr{C}_v(D)$, S in $\mathscr{C}_h(D)$ and φ a p.s.h. function on D. If φ is integrable with respect to the trace measure $S \wedge \omega^p$ of S then φS defines a current in $\mathrm{PSH}_h(D)$. We deduce from the above results that

$$\langle R \wedge S, \varphi \rangle = \limsup_{\theta \to 1} \langle R_\theta \wedge S, \varphi \rangle = \limsup_{\theta \to 1} \langle R_\theta, \varphi S \rangle = \langle R, \varphi S \rangle.$$

Definition 2.5. Let $\Lambda : \mathscr{C}_v^1(M \times \overline{N}) \to \mathbb{R} \cup \{-\infty\}$ be an upper semi-continuous function which is not identically $-\infty$. We say that Λ is a *p.s.h. super-function* if it is p.s.h. or identically equal to $-\infty$ on each structural variety in $\mathscr{C}_v^1(M \times \overline{N})$, and Λ is *pluriharmonic* if both Λ and $-\Lambda$ are p.s.h., see also [19].

Proposition 2.6. *Let Φ be a real horizontal $(k-p, k-p)$-current on D. If Φ is dd^c-closed, then $R \mapsto \langle R, \Phi \rangle$ defines a pluriharmonic super-function. If Φ is PSH, then $R \mapsto \langle R, \Phi \rangle$ is a p.s.h. super-function.*

Proof. We only have to prove the second assertion. Consider a structural variety $(R_x)_{x \in X}$ as above. Without loss of generality, we can assume that R_x are vertical in $M' \times \widetilde{N}$ and Φ is horizontal in $M \times N'$, see Remark 2.3. We want to prove that $x \mapsto \langle R_x, \Phi \rangle$ is identically equal to $-\infty$ or p.s.h. If Φ is smooth, this was proved in [17, Lemma 2.2]. For the general case, we have

$$\langle R_x, \Phi \rangle = \limsup_{\theta \to 1} \langle R_{x,\theta}, \Phi \rangle = \limsup_{\theta \to 1} \langle R_x, \Phi_\theta \rangle,$$

where $(R_{x,\theta})$ is the particular structural disc constructed as above using the same smoothing for each R_x. We deduce from the regularity of $R_{x,\theta}$ that $\langle R_{x,\theta}, \Phi \rangle$ is locally uniformly bounded on $(x, \theta) \in X \times (\Delta \setminus \{1\})$. Since $\theta \mapsto \langle R_x, \Phi_\theta \rangle$ is p.s.h., it follows from the maximum principle that $\langle R_{x,\theta}, \Phi \rangle$ is locally uniformly bounded from above on $X \times \Delta$. Hence, the upper semi-continuous regularization of $x \mapsto \langle R_x, \Phi \rangle$ is p.s.h. or identically $-\infty$. It is enough to show that $x \mapsto \langle R_x, \Phi \rangle$ is upper semi-continuous.

For every $a \in X$, we have $\limsup_{x\to a}\langle R_x, \Phi_\theta\rangle = \langle R_a, \Phi_\theta\rangle$ for $\theta \neq 1$. Since the functions $\theta \mapsto \langle R_x, \Phi_\theta\rangle$ are subharmonic, we deduce using Hartogs' lemma that $\limsup_{x\to a}\langle R_x, \Phi_\theta\rangle \leqslant \langle R_a, \Phi_\theta\rangle$ for every θ. This implies the result. □

2.5. PB, PC currents and measures

Let T be a vertical current of bidegree (p, p) in $\mathscr{C}_v(D)$. We say that T is *PB* if $\langle T, \Phi\rangle$ is bounded when Φ is in a relatively compact subset of $\mathrm{PSH}_h(D)$. We say that T is *PC* if it can be extended to a continuous linear form on $\mathrm{PSH}_h(D)$ with respect to the topology we have introduced. Observe that this extension coincides with the extension in Proposition 2.2. PC currents are PB. PB and PC horizontal currents of bidegree $(k-p, k-p)$ are defined in the same way. In the case of bidegree $(1, 1)$, PB and PC currents correspond to currents with bounded and continuous local potentials, see also [13,15,16].

A positive measure μ with compact support in D is said to be *PB* if $\langle \mu, \phi\rangle$ is bounded when ϕ are smooth functions in a relatively compact subset in $\mathrm{PSH}(\overline{D})$. Since p.s.h. functions on a neighbourhood of $\overline{D}$ can be approximated by decreasing sequences of smooth ones, μ is PB if and only if p.s.h. functions on a neighbourhood of $\overline{D}$ are μ-integrable. PB measures have no mass on *pluripolar sets*, i.e. sets which are contained in the pole set $\{\phi = -\infty\}$ of a p.s.h. function ϕ. The measure μ is said to be *PC* if it can be extended to a linear continuous form on $\mathrm{PSH}(D)$. Denote by $\langle \mu, \phi\rangle$ the value of this extension on ϕ. Note that by continuity the extension is unique and $\langle \mu, \phi\rangle$ is equal to the usual integral $\langle \mu, \phi\rangle$ of ϕ. Any PC measure is PB.

2.6. Solution of dd^c-equation

We consider the dd^c-equation on D. We will need negative solutions with horizontal or vertical support and with estimates on the mass. The behavior near the rest of the boundary is not important in our study. The following theorem is obtained using classical results. Recall that $d^c := \frac{i}{2\pi}(\bar\partial - \partial)$.

Theorem 2.7. *Let M' and M be convex domains in $\mathbb{C}^p$ such that $M' \Subset M$. Let N' and N'' be convex open sets in $\mathbb{C}^{k-p}$ such that $N'' \Subset N'$. Let Ω be a horizontal positive closed current of bidegree $(k-p+1, k-p+1)$ on $M \times N''$. Then there is a horizontal negative L^1 form Φ of bidegree $(k-p, k-p)$ on $M' \times N'$ such that*

$$dd^c\Phi = \Omega \quad \text{on } M' \times N' \quad \text{and} \quad \|\Phi\|_{M'\times N'} \leqslant c\|\Omega\|_{M\times N''}$$

with $c > 0$ independent of Ω. Moreover, Φ is defined by an integral formula, and depends linearly and continuously on Ω.

In what follows, the solutions of d, $\bar\partial$ or dd^c equations are given by classical integral formulas. Consequently, the linearity, the continuous dependence on data and the estimate on the mass of solutions are satisfied. Therefore, we will focus our attention only on the support of the solutions.

Lemma 2.8. *Let D' and D be convex domains in $\mathbb{C}^k$ with $D' \Subset D$. Let Ω be a positive closed current of bidegree $(k-p+1, k-p+1)$ on D. There is a negative L^1 form Ψ of bidegree $(k-p, k-p)$ on D', smooth out of the support of Ω, such that $dd^c\Psi = \Omega$ on D'.*

Proof. We can assume that D is contained in the ball of center 0 and of radius $1/2$. Define for coordinates (z,ξ) on $\mathbb{C}^k\times\mathbb{C}^k$ the kernel

$$K(z,\xi):=\log\|z-\xi\|\big(dd^c\log\|z-\xi\|\big)^{k-1}.$$

Observe that K is negative when $\|z\|<1/2$, $\|\xi\|<1/2$, and dd^cK is equal to the current of integration on the diagonal of $\mathbb{C}^k\times\mathbb{C}^k$. Let χ be a cut-off function, $0\leqslant\chi\leqslant 1$, with compact support in D, such that $\chi=1$ on a neighbourhood U of $\overline{D}'$. Define

$$\Psi'(z):=\int_\xi \chi(\xi)\Omega(\xi)\wedge K(z,\xi).$$

Hence, Ψ' is a negative L^1 form depending continuously on Ω. If z is outside the support of Ω, then $\Psi'(z)$ is given by an integration outside the singularities of K. So, $\Psi'(z)$ is smooth there.

Let π_1 and π_2 denote the canonical projections of $\mathbb{C}^k\times\mathbb{C}^k$ on its factors. If Ω is smooth we have

$$\Psi'=(\pi_1)_*\big(\pi_2^*(\chi\Omega)\wedge K\big).$$

Since Ω is closed and $dd^cK=[z=\xi]$, we deduce that $\Omega':=dd^c\Psi'-\Omega$ is equal on U to

$$\Omega'=\int_\xi d\chi(\xi)\wedge\Omega(\xi)\wedge d^cK(z,\xi)-\int_\xi d^c\chi(\xi)\wedge\Omega(\xi)\wedge dK(z,\xi)$$
$$+\int_\xi dd^c\chi(\xi)\wedge\Omega(\xi)\wedge K(z,\xi).$$

The last formula is valid for arbitrary Ω by regularization. So, Ω' is defined by integration on $\{d\chi(\xi)\neq 0\}$ where $K(z,\xi)$ is smooth if $z\in U$. It follows that Ω' is smooth. We also have good estimates on $\mathscr{C}^r$ norm of this form on compact subsets of U.

Since Ω' is closed and smooth, it is classical to obtain smooth solution of the equation $dd^c\Psi''=\Omega'$ with estimates (we first solve a d-equation and then a $\bar\partial$-equation, the method will be described below with details in a situation where more estimates are needed). One checks that $dd^c\Psi=\Omega$ for $\Psi:=\Psi'-\Psi''-c\omega^{k-p}$ where $\omega:=dd^c\|z\|^2$ is the standard Kähler form on $\mathbb{C}^k$ and $c>0$ is large enough in order to guarantee that Ψ is negative on D'. □

Now, we need to control the support of the solution. We shrink slightly M and extend slightly N''. This allows to assume that Ω is defined in $\widetilde{M}\times F$ for some fixed compact set F in N''. Using the previous lemma, we can find Ψ on $M\times N$, smooth outside the support of Ω such that $dd^c\Psi=\Omega$. Let χ be a cut-off function equal to 1 on a neighbourhood of $M\times F$ and equal to 0 near $M\times\partial N''$ and on $M\times(N\setminus N'')$. In particular, $\chi=1$ on the support of Ω and Ψ is smooth on $\{d\chi\neq 0\}$. Define $\Phi_1:=\chi\Psi$ and $\Omega':=dd^c\Phi_1-\Omega$. This is a smooth horizontal closed form of bidegree $(k-p+1,k-p+1)$ with support in $M\times N''$. Moreover, Ω' vanishes near $M\times F$ and has a controlled $\mathscr{C}^r$ norm. We will find a smooth positive solution of the equation $dd^c\Phi_2=\Omega'$ with horizontal support in $M'\times N'$. The current $\Phi:=\Phi_1-\Phi_2$ satisfies Theorem 2.7.

A construction using an integral formula as in the book [5, pp. 37–39 and 61–63] by Bott and Tu implies that there is a real smooth form Ψ which is horizontal in $M \times N''$ such that $d\Psi = \Omega'$ (shrink M and extend N'' if necessary). Of course, it satisfies the desired estimates in $\mathscr{C}^r$ norms. Moreover, we can write $\Psi = \Psi' + \Psi''$ with Ψ' of bidegree $(k-p, k-p+1)$ and Ψ'' of bidegree $(k-p+1, k-p)$ such that $\Psi'' = \overline{\Psi}'$.

Lemma 2.9. *There is a smooth horizontal form Φ' on $M' \times N'$, of bidegree $(k-p, k-p)$, such that $\bar\partial\Phi' = \Psi'$.*

Proof. Recall that we can, in each step of the proof, shrink or extend slightly the considered domains M, N' or N''. This permits to avoid the problem near the boundary and to assume that they are strictly convex with smooth boundary. Since $d\Psi$ is of bidegree $(k-p, k-p)$, we have $\bar\partial\Psi' = 0$. So, using a classical integral formula (see for example [25,34]) we can find a smooth form Φ^* of bidegree $(k-p, k-p)$ on $M \times N$ such that $\bar\partial\Phi^* = \Psi'$. Its support is not necessarily horizontal. So, we have $\bar\partial\Phi^* = 0$ outside the support of Ψ'.

We will apply a result of Andreotti–Grauert [26, p. 109] in order to solve the equation $\bar\partial H = \Phi^*$ on $M' \times (N \setminus N'')$ with H smooth of bidegree $(k-p, k-p-1)$. Let $\widetilde{\chi}$ be a cut-off function equal to 0 on $M' \times N''$ and 1 in a neighbourhood of $M' \times (N \setminus N')$. The form $\widetilde{\chi}H$ is defined on $M' \times N$. It is clear that $\Phi' := \Phi^* - \bar\partial(\widetilde{\chi}H)$ is horizontal in $M' \times N'$ and satisfies $\bar\partial\Phi' = \Psi'$, which completes the proof.

In order to apply the Andreotti–Grauert theorem, i.e. to solve the $\bar\partial$-equation for a $\bar\partial$-closed form of bidegree $(l, k-s)$, $s \geqslant p$, in $M' \times (N \setminus \overline{N}'')$, we only have to prove that $M' \times (N \setminus \overline{N}'')$ satisfies the right convexity property. More precisely, one should construct a smooth exhaustion function ρ on $M' \times (N \setminus \overline{N}'')$ such that $dd^c\rho$ has at every point $p+1$ strictly positive eigenvalues. The domain is completely strictly p-convex in the terminology of [26, p. 65]. We need a much weaker result than Theorem 12.7 in [26].

Let ρ_1 be a smooth strictly convex function on N such that $\rho_1(z) \to \infty$ when $z \to \partial N$ and $N'' = \{\rho_1 < 1\}$. Since M' is strictly convex, we may find an unbounded exhaustion function ρ_0 for M' which is smooth and strictly convex. Define

$$\rho(z) := \rho_0(z') + c\rho_1(z'') + \kappa\big(\rho_1(z'')\big), \quad z = (z', z'') \in M' \times (N \setminus \overline{N}''),$$

with $\kappa(t) := \frac{1}{t-1}$ and $c > 0$ large enough. The function ρ is an exhaustion function on $M' \times (N \setminus \overline{N}'')$. The p eigenvalues of $dd^c\rho$ with respect the variable z' are strictly positive. On the other hand, since

$$i\partial\bar\partial(\kappa \circ \rho_1) = \kappa' \cdot i\partial\bar\partial\rho_1 + \kappa'' \cdot i\partial\rho_1 \wedge \bar\partial\rho_1,$$

and $\kappa''(t) \gg |\kappa'(t)|$ as $t \to 1^+$, $dd^c\rho$ admits, at every point, at least one strictly positive eigenvalue with respect to the variable z''. This completes the proof. $\square$

2.7. End of the proof of Theorem 2.7

Define $\Phi'' := -i\pi(\Phi' - \overline{\Phi}')$. This is a real smooth horizontal form in $M' \times N'$. We have

$$dd^c\Phi'' = \partial\bar\partial(\Phi' - \overline{\Phi}') = \partial\Psi' + \bar\partial\overline{\Psi}' = d\Psi = \Omega'.$$

The smooth form Φ'' is not necessarily positive. We can assume that it has support in $M' \times F$ for some compact subset F of N'. We now construct a horizontal closed form U on $M' \times N'$ of bidegree $(k-p, k-p)$ which is strictly positive on $M' \times F$. Then, the form $\Phi_2 := \Phi'' + cU$, with $c > 0$ large enough, is positive and satisfies $dd^c \Phi_2 = \Omega'$.

For every point $z \in \overline{M}' \times F$ there is a complex plane P of dimension p passing through z which does not intersect $\overline{M}' \times \partial N'$. This plane defines by integration a positive closed $(k-p, k-p)$-current $[P]$. Using a convolution, we obtain by averaging on small perturbations of $[P]$, a smooth positive closed form U_z which is horizontal in $M' \times N'$ and is strictly positive at z. By continuity, such a form is strictly positive in a neighbourhood of z. It is enough to take a finite sum of such forms in order to obtain a form U which is strictly positive on $\overline{M}' \times F$. This completes the proof. □

Remark 2.10. If Ω is a continuous form then $\|\Phi\|_{\mathscr{C}^1(M' \times N')} \leqslant c\|\Omega\|_{\mathscr{C}^0(M \times N'')}$ with a constant $c > 0$ independent of Ω. Indeed, we are using a solution given by a "good" kernel.

3. Horizontal-like maps

In this section we introduce the class of horizontal-like maps, the main dynamical objects of our study, and we give some basic properties.

3.1. Horizontal-like maps and Julia sets

A horizontal-like map f on D is not necessarily defined on the whole domain D but only on a vertical subset $f^{-1}(D)$ of D. It takes values in a horizontal subset $f(D)$ of D. Horizontal-like maps are defined by their graphs Γ as follows [17]. Let pr_1 and pr_2 be the canonical projections of $D \times D$ on its factors.

Definition 3.1. A *horizontal-like map* f on D is a holomorphic map with graph Γ such that

1. Γ is a submanifold of $D \times D$.
2. $\mathrm{pr}_{1|\Gamma}$ is injective; $\mathrm{pr}_{2|\Gamma}$ has finite fibers.
3. $\overline{\Gamma}$ does not intersect $\overline{\partial_v D} \times \overline{D}$ nor $\overline{D} \times \overline{\partial_h D}$.

The last property is equivalent to the fact that the projections of Γ on the first factor M and the last factor N in $D \times D$ are relatively compact. The map $f = \mathrm{pr}_2 \circ (\mathrm{pr}_{1|\Gamma})^{-1}$ is defined on $f^{-1}(D) := \mathrm{pr}_1(\Gamma)$ and its image is equal to $f(D) := \mathrm{pr}_2(\Gamma)$. There exist open sets $M' \Subset M$ and $N' \Subset N$ such that $f^{-1}(D) \subset D_v := M' \times N$ and $f(D) \subset D_h := M \times N'$. We have $\Gamma \subset D_v \times D_h$. This property characterizes horizontal-like maps and we often use it in order to check that a map is horizontal-like. Since Γ is a submanifold of $D \times D$, when z tends to $\partial f^{-1}(D) \cap D$, $f(z)$ tends to $\partial_v D$. When z tends to $\partial f(D) \cap D$, $f^{-1}(z)$ tends to $\partial_h D$. So, the vertical part of $\partial f^{-1}(D)$ is sent into the vertical part of $\partial f(D)$. If g is another horizontal-like map on D, $f \circ g$ is also a horizontal-like map. When $p = k$, we obtain the polynomial-like maps studied in [13].

If $\mathrm{pr}_{2|\Gamma}$ is injective, we say that f is *invertible*. In this case, up to a coordinate change (an exchange of horizontal and vertical directions), $f^{-1} : \mathrm{pr}_2(\Gamma) \to \mathrm{pr}_1(\Gamma)$ is a horizontal-like map. When $k = 2$ and $p = 1$, we obtain the Hénon-like maps [12,20]. In order to simplify the paper, we consider only invertible horizontal-like maps.

Small perturbations of an invertible horizontal map are still horizontal and invertible if one shrinks slightly the domain D. Therefore, it is easy to construct large families of such maps.

Define $f^n := f \circ \cdots \circ f$ (n times) *the iterate of order n of f* and $f^{-n} := f^{-1} \circ \cdots \circ f^{-1}$ (n times) its inverse. Let $\mathscr{K}_+$ (respectively $\mathscr{K}_-$) denote the set of points $z \in D$ such that f^n (respectively f^{-n}) are defined at z for every $n \geqslant 0$. In other words, we have $\mathscr{K}_+ := \bigcap_{n\geqslant 0} f^{-n}(D)$ and $\mathscr{K}_- := \bigcap_{n\geqslant 0} f^n(D)$. It is easy to check that $\mathscr{K}_\pm$ are closed in D; $\mathscr{K}_+$ is vertical and $\mathscr{K}_-$ is horizontal. We call $\mathscr{K}_+$ *the filled Julia set of f* and $\mathscr{K}_-$ *the filled Julia set of f^{-1}*. Their boundaries are called *Julia sets*. Define also $\mathscr{K} := \mathscr{K}_+ \cap \mathscr{K}_-$. This is a compact subset of D. We have $f^{-1}(\mathscr{K}_+) = \mathscr{K}_+$, $f(\mathscr{K}_-) = \mathscr{K}_-$ and $f^{\pm 1}(\mathscr{K}) = \mathscr{K}$, see [17].

3.2. Dynamical degrees, Green currents and equilibrium measure

The operator $f_* := (\mathrm{pr}_{2|\Gamma})_* \circ (\mathrm{pr}_{1|\Gamma})^*$ acts continuously on horizontal currents. If S is a horizontal current or form, so is $f_*(S)$. The operator $f^* := (\mathrm{pr}_{1|\Gamma})_* \circ (\mathrm{pr}_{2|\Gamma})^*$ acts continuously on vertical currents. If R is a vertical current or form, so is $f^*(R)$. The continuity of f^*, f_* for non-invertible maps is treated in [18]. Recall from [17] the following proposition for positive closed currents of the right bidegree.

Proposition 3.2. *The operator $f_* : \mathscr{C}_h(D_v) \to \mathscr{C}_h(D_h)$ is well-defined and continuous. Moreover, there exists an integer $d \geqslant 1$ such that $\|f_*(S)\|_h = d\|S\|_h$ for every $S \in \mathscr{C}_h(D_v)$. The operator $f^* : \mathscr{C}_v(D_h) \to \mathscr{C}_v(D_v)$ is well-defined and continuous. If R belongs to $\mathscr{C}_v(D_h)$, we have $\|f^*(R)\|_v = d\|R\|_v$.*

The integer d is called *the main dynamical degree of f*. In the sequel, it is often denoted by $d(f)$. Note that the previous proposition implies that $d(f) = d(f^{-1})$ and $d(f^n) = d^n$. Consider a vertical subvariety L of dimension $k - p$ in D. The projection $\pi_2 : L \to N$ defines a (ramified) covering. If m is the degree of this covering, the current $[L]$ has slice mass m. We deduce from the previous proposition that $f^{-1}(L)$ is a vertical subvariety of degree md. For $m = 1$, we obtain that d is an integer. There is an analogous picture when we push forward a horizontal subvariety. Note also that the projection of Γ onto the product of the first factor N with the second factor M defines a (ramified) covering of degree d. The following results were proved in [17].

Theorem 3.3. *Let f be an invertible horizontal-like map on $D = M \times N$, d its main dynamical degree and $\mathscr{K}_\pm$, $\mathscr{K}$ the filled Julia sets as above. Let R and S be smooth forms in $\mathscr{C}_v^1(D)$ and $\mathscr{C}_h^1(D)$ respectively. Then $d^{-n}(f^n)^*(R)$ (respectively $d^{-n}(f^n)_*(S)$) converge to a current T_+ in $\mathscr{C}_v^1(D)$ (respectively T_- in $\mathscr{C}_h^1(D)$) which does not depend on R (respectively S) and $d^{-2n}(f^n)^*(R) \wedge (f^n)_*(S)$ converge to the probability measure $\mu := T_+ \wedge T_-$. The current T_+ (respectively T_-) is supported on the Julia set $\partial\mathscr{K}_+$ (respectively $\partial\mathscr{K}_-$) and is invariant under $d^{-1}f^*$ (respectively under $d^{-1}f_*$). The measure μ is invariant under f^*, f_* and is supported on $\partial\mathscr{K}_+ \cap \partial\mathscr{K}_-$.*

The current T_+ (respectively T_-) is *the Green current* associated to f (respectively f^{-1}). The measure μ is called *the equilibrium measure* of f.

Theorem 3.4. *With the notation of the previous theorem, the topological entropy of f on $\mathscr{K}$ is equal to* $\log d$ *and μ is a measure of maximal entropy* $\log d$.

The notion of entropy will be recalled in Section 5. We now introduce *the other dynamical degrees* of f. Recall that the open sets $M' \Subset M$ and $N' \Subset N$ are chosen so that $f^{-1}(D) \subset M' \times N$ and $f(D) \subset M \times N'$. So, the restriction of f to $M' \times N'$ is also horizontal-like. For every $0 \leqslant s \leqslant p$, let

$$d_s^+ = d_s(f) := \limsup_{n\to\infty} \Big\{ \sup_S \big\| (f^n)_* S \big\|_{M' \times N} \Big\}^{1/n},$$

the supremum being taken over all positive closed horizontal currents S of bidegree $(k-s, k-s)$ on $D' = M' \times N'$ such that $\|S\|_{D'} = 1$. For every $0 \leqslant s \leqslant k-p$, define

$$d_s^- = d_s(f^{-1}) := \limsup_{n\to\infty} \Big\{ \sup_R \big\| (f^n)^* R \big\|_{M \times N'} \Big\}^{1/n},$$

the supremum being taken over all positive closed vertical currents R of bidegree $(k-s, k-s)$ on $D' = M' \times N'$ such that $\|R\|_{D'} = 1$. In the sequel we will write for short

$$\delta_+ := d_{p-1}^+ \quad \text{and} \quad \delta_- := d_{k-p-1}^-.$$

These are the dynamical degrees which have to be compared to d.

Lemma 3.5. *The dynamical degrees do not depend on the choice of the particular convex domains M' and N'. Moreover, we have $d_0^+ = d_0^- = 1$ and $d_p^+ = d_{k-p}^- = d$.*

Proof. Let M'' and N'' be convex open sets such that $M'' \Subset M' \Subset M$, $N'' \Subset N' \Subset N$ and $f^{-1}(D) \subset M'' \times N$, $f(D) \subset M \times N''$. If in the previous definition, we replace M' by M'' and N' by N'', we obtain δ_s^+ and δ_s^-. It is enough to prove that $\delta_s^+ = d_s^+$ and $\delta_s^- = d_s^-$. We prove the first equality; the second one is obtained in the same way. Let $\widetilde{S}$ be a horizontal positive closed current of bidegree $(k-s, k-s)$ on $M'' \times N''$. Since f is horizontal-like, $f_*(\widetilde{S})$ is horizontal in $M \times N''$ and there is a constant $A > 0$ independent of $\widetilde{S}$ such that $\|f_*(\widetilde{S})\|_{M' \times N'} \leqslant A\|\widetilde{S}\|_{M'' \times N''}$. In particular, if S is horizontal in $M' \times N'$ then we have $\|(f^n)_* S\|_{M' \times N'} \leqslant A\|(f^{n-1})_* S\|_{M'' \times N''}$ for $n \geqslant 2$. If, moreover, $\|S\|_{M' \times N'} = 1$, then $S' := f_*(S)$ is horizontal in $M'' \times N''$ with bounded mass. Therefore,

$$\begin{aligned} d_s^+ &= \limsup_{n\to\infty} \Big\{ \sup_S \big\| (f^n)_* S \big\|_{M' \times N'} \Big\}^{1/n} \\ &\leqslant \limsup_{n\to\infty} \Big\{ \sup_S \big\| (f^{n-1})_* S \big\|_{M'' \times N''} \Big\}^{1/n} \\ &= \limsup_{n\to\infty} \Big\{ \sup_S \big\| (f^{n-2})_* S' \big\|_{M'' \times N''} \Big\}^{1/n} \leqslant \delta_s^+. \end{aligned}$$

For S horizontal in $M'' \times N''$ with $\|S\|_{M'' \times N''} = 1$, define also $S' := f_*(S)$. Then S' is horizontal in $M' \times N'$ with bounded mass and we have

$$\begin{aligned}\delta_s^+ &= \limsup_{n\to\infty}\Big\{\sup_S \big\|\big(f^n\big)_* S\big\|_{M''\times N''}\Big\}^{1/n}\\ &\leqslant \limsup_{n\to\infty}\Big\{\sup_S \big\|\big(f^n\big)_* S\big\|_{M'\times N'}\Big\}^{1/n}\\ &= \limsup_{n\to\infty}\Big\{\sup_S \big\|\big(f^{n-1}\big)_* S'\big\|_{M'\times N'}\Big\}^{1/n} \leqslant d_s^+.\end{aligned}$$

This implies the first part of the lemma.

Since f_* preserves the mass of positive measures on $f^{-1}(D)$, we obtain that $d_0^+ \leqslant 1$. If S is a probability measure on $\mathscr{K}$ then $(f^n)_*(S)$ is also a probability on $\mathscr{K}$. So, $d_0^+ = 1$. We obtain in the same way that $d_0^- = 1$.

Assume that S is of bidegree $(k-p, k-p)$. By definition of slices, we have $\|S\|_h \lesssim \|S\|_{M'\times N}$ and as we already discussed in Section 2, $\|S\|_{M'\times N} \lesssim \|S\|_h$. So,

$$d^n \lesssim \big\|\big(f^n\big)_* S\big\|_{M'\times N} \lesssim d^n,$$

which implies that $d_p^+ = d$. We obtain in the same way that $d_{k-p}^- = d$. □

3.3. Action on super-functions

We reduce slightly D and assume that f is defined in a neighbourhood of $\overline{D}$. Let Φ be a current in $\mathrm{PSH}_h(D)$ and Λ the super-function associated to Φ defined on $\mathscr{C}_v^1(M \times \overline{N})$, i.e. $\Lambda(R) := \langle R, \Phi\rangle$, see Proposition 2.6. The following lemma is useful in our calculus.

Lemma 3.6. *The function $R \mapsto \Lambda(d^{-1} f^*(R))$ is the super-function associated to $d^{-1} f_*(\Phi)$. In other words, we have*

$$\big\langle f^*(R), \Phi\big\rangle = \big\langle R, f_*(\Phi)\big\rangle$$

for $R \in \mathscr{C}_v(D)$ and $\Phi \in \mathrm{PSH}_h(D)$.

Proof. Let Λ' denote the function $R \mapsto \Lambda(d^{-1} f^*(R))$ and Λ'' the super-function associated to $d^{-1} f_*(\Phi)$. It is clear that $\Lambda'(R) = \Lambda''(R)$ for R smooth. We have to prove this equality for general R.

Let $\mathscr{R}$ be the current in $\Delta \times D$ associated to the structural disc (R_θ) constructed in Section 2. If $F : \Delta \times f^{-1}(D) \to \Delta \times f(D)$ is the map given by $F(\theta, z) := (\theta, f(z))$, one can check that the current $d^{-1}F^*(\mathscr{R})$ defines a structural disc (R'_θ) with $R'_\theta = d^{-1} f^*(R_\theta)$. Since Λ is p.s.h., $\Lambda'(R_\theta) = \Lambda(R'_\theta)$ is subharmonic on $\theta \in \Delta$. The super-function Λ'' is also subharmonic on the disc (R_θ) and coincide with Λ' at R_θ with $\theta \neq 1$ because R_θ is smooth for $\theta \neq 1$. Hence, Λ' and Λ'' coincide also at $R_1 = R$, that is, $\Lambda'(R) = \Lambda''(R)$. □

3.4. Product maps

Let f_i be horizontal-like maps on $D_i = M_i \times N_i$. Define the product map $F(x_1, x_2) := (f_1(x_1), f_2(x_2))$ on $D_1 \times D_2$. Up to a permutation of coordinates, we can identify $D_1 \times D_2$ to $(M_1 \times M_2) \times (N_1 \times N_2)$. One checks easily that F is a horizontal-like map on this domain. If

d_i denote the main dynamical degree of f_i, the main degree of F is $d_1 d_2$. We can deduce from Theorem 3.3 the following properties. If $T_{i,\pm}$ are the Green currents associated to $f_i^{\pm 1}$, the Green currents associated to $F^{\pm 1}$ are $T_{1,+} \otimes T_{2,+}$ and $T_{1,-} \otimes T_{2,-}$. If μ_i are the equilibrium measures of f_i, the equilibrium measure of F is $\mu_1 \otimes \mu_2$. In what follows, we will use the product F of the horizontal-like maps $f_1 := f$ and $f_2 := f^{-1}$ defined on $D = M \times N$ as above. In this case, we have $M_1 = N_2 = M$ and $M_2 = N_1 = N$; the Green currents of F and F^{-1} are $T_+ \otimes T_-$ and $T_- \otimes T_+$. We can perturb F in order to obtain new families of examples.

3.5. About the hypothesis on dynamical degrees

The hypothesis we need in this paper is that the main dynamical degree is larger than the other dynamical degrees. The following proposition shows that the family of the maps f satisfying this condition is open.

Proposition 3.7. *Let f be a horizontal-like map on $D = M \times N$ with the main dynamical degree d as above and $D' := M' \times N'$ a domain such that $D' \Subset D$ and that $D \setminus D'$ is small enough. Then every small perturbation f_ϵ of f is a horizontal-like map on D' of the same main dynamical degree d. If the dynamical degree of order s of f is strictly smaller than d, then the dynamical degree of order s of f_ϵ satisfies the same property.*

Proof. It is clear that f_ϵ is horizontal-like on D'. Since d can be interpreted as the degree of a covering, the main dynamical degree of f_ϵ is also d. Let d_s^+ and $d_{\epsilon,s}^+$ denote the dynamical degrees of order s of f and f_ϵ. Fix a constant δ such that $d_s^+ < \delta < d$ and a domain $D'' = M'' \times N''$ in D' such that $D' \setminus D''$ is small enough. So, f and f_ϵ restricted to D'' are horizontal-like. Consider a horizontal positive closed $(k-s, k-s)$-current S of mass 1 in D''. By Lemma 3.5, there is an integer n_0 independent of S such that the mass of $(f^{n_0})_* S$ on D' is smaller than $\delta^{n_0}/2$. If ω denotes the standard Kähler form on $\mathbb{C}^k$, we have since S is supported on D'' and $f^{-n_0}(D'') \subset M'' \times N$

$$\left\| \left(f^{n_0}\right)_* S \right\|_{D''} = \int_{D''} \left(f^{n_0}\right)_* S \wedge \omega^s = \int_{f^{-n_0}(D'') \cap D''} S \wedge \left(f^{n_0}\right)^* \omega^s.$$

If f_ϵ is close enough to f, $(f_\epsilon^{n_0})^* \omega^s - (f^{n_0})^* \omega^s$ is a small form on $f^{-n_0}(D') \cap D'$ and $f_\epsilon^{-n_0}(D'') \cap D'' \subset f^{-n_0}(D') \cap D'$. Hence,

$$\left\| \left(f_\epsilon^{n_0}\right)_* S \right\|_{D''} \leqslant \left\| \left(f^{n_0}\right)_* S \right\|_{D'} + \int_{f_\epsilon^{-n_0}(D'') \cap D''} S \wedge \left[\left(f_\epsilon^{n_0}\right)^* \omega^s - \left(f^{n_0}\right)^* \omega^s\right].$$

It follows that $\|(f_\epsilon^{n_0})_* S\|_{D''} \leqslant \delta^{n_0}$. The estimate is independent of S and implies by iteration that $\|(f_\epsilon^n)_* S\|_{D''} \lesssim \delta^n$ for $n \geqslant 1$ uniformly on S. Hence, $d_{\epsilon,s}^+ \leqslant \delta < d$. We get a similar results for f_ϵ^{-1} and its dynamical degrees. $\square$

4. Convergence theorems

In this section we will give several quantitative versions of Theorem 3.3 under the hypothesis that the main dynamical degree d is strictly larger than the degrees $\delta_+ := d_{p-1}^+$ and $\delta_- := d_{k-p-1}^-$.

We will see that this hypothesis is natural and is satisfied for large families of maps. A similar condition was considered in the context of polynomial-like maps, see [13].

4.1. Convergence towards the Green currents

We will use the PSH horizontal currents as test "forms." The above solution of the dd^c-equation allows to write such a test current as the sum of a PSH current with good estimates and a dd^c-closed one. We obtain in particular the following result.

Theorem 4.1. *Let f be an invertible horizontal-like map on $D = M \times N$ and d, δ_+ its dynamical degrees as above. Assume that $d > \delta_+$. Then the Green current T_+ of f is PC.*

We first consider the dd^c-closed test currents. The following result shows that in this case, without any hypothesis on the dynamical degrees, the convergence is exponentially fast and uniform.

Proposition 4.2. *Let $\mathscr{H}$ be a compact family of currents in $\mathscr{H}_h(D)$. Then there are constants $A_0 > 0$ and $\lambda_0 > 1$ such that*

$$\left|\left\langle d^{-n}\left(f^n\right)^* R - T_+, \Psi\right\rangle\right| \leqslant A_0 \lambda_0^{-n}$$

for all $R \in \mathscr{C}_v^1(M' \times N)$, $\Psi \in \mathscr{H}$ and $n \geqslant 0$.

Proof. Reducing D allows to assume that R is in $\mathscr{C}_v^1(M' \times \widetilde{N})$ and $\mathscr{H}$ is compact in $\mathscr{H}_h(\widetilde{M} \times N')$. There is a constant $A' > 0$ such that $|\langle d^{-n}(f^n)^* R, \Psi\rangle| \leqslant A'$ for all $R \in \mathscr{C}_v^1(D)$, $\Psi \in \mathscr{H}$ and $n \geqslant 0$. This follows from Proposition 2.1 since $(R, \Psi) \mapsto \langle R, \Psi\rangle$ is continuous. If Ψ' is in $\mathscr{C}_h^1(\widetilde{M} \times N')$, we have and $\langle T_+, \Psi'\rangle = 1$ and $\langle d^{-n}(f^n)^* R, \Psi'\rangle = 1$ for every $R \in \mathscr{C}_v^1(D)$. By adding to Ψ a multiple of Ψ', we can assume that $\langle T_+, \Psi\rangle = 0$ and we only need to prove the estimate under this assumption. Assume also for simplicity that $A' = 1$.

Denote by Λ_Ψ the super-function $\Lambda_\Psi(R) := \langle R, \Psi\rangle$ and $L := d^{-1} f^*$ the linear operator from $\mathscr{C}_v^1(D)$ into $\mathscr{C}_v^1(M' \times \widetilde{N})$. Since T_+ is invariant, we have $\Lambda_\Psi \circ L^n(T_+) = 0$. Let $\mathscr{F}$ denote the set of pluriharmonic super-functions Λ on $\mathscr{C}_v^1(M \times \overline{N})$ such that $\Lambda(T_+) = 0$ and $\|\Lambda\|_\infty \leqslant 1$. Then, by Lemma 3.6 and the assumption that $A' = 1$, $\Lambda_\Psi \circ L^n$ belongs to $\mathscr{F}$ for $n \geqslant 1$ and we have

$$\left\langle d^{-n}\left(f^n\right)^* R - T_+, \Psi\right\rangle = \Lambda_\Psi \circ L^n(R).$$

So, by induction, it is enough to show that $\|\Lambda \circ L\|_\infty \leqslant 1/\lambda_0$ for Λ in $\mathscr{F}$ and for some constant $\lambda_0 > 1$.

Assume that no constant λ_0 satisfies the above condition. Then there are $\Lambda \in \mathscr{F}$ and $R' \in \mathscr{C}_v^1(M' \times \widetilde{N})$ such that $|\Lambda(R')|$ is as close to 1 as we want. Recall that as in Section 2 we can construct a structural disc τ' (respectively τ) such that $\tau'(1) = R'$ (respectively $\tau(1) = T_+$). Moreover, $\tau'(0)$, $\tau(0)$ are equal to a fixed current R_0. These discs are parametrized by a fixed neighbourhood Δ of $[0, 1]$. By Harnack's inequality applied to the non-vanishing harmonic function $1 - \Lambda \circ \tau'$ on Δ, $|\Lambda(R_0)|$ is close to 1. Applying again the Harnack's inequality to $1 - \Lambda \circ \tau$, we deduce that $|\Lambda(T_+)|$ is close to 1. This contradicts the definition of $\mathscr{F}$. $\square$

Proof of Theorem 4.1. Fix a constant δ such that $\delta_+ < \delta < d$. Consider a test current Φ in a fixed compact set of $\mathrm{PSH}_h(D)$. Define $\Omega_0 := dd^c\Phi$ and $\Omega_n := (f^n)_*\Omega_0$. The currents Ω_n are positive of bidegree $(k-p+1, k-p+1)$ and by definition of δ_+, we have $\|\Omega_n\|_{M'\times N} \leqslant A\delta^n\|\Omega_0\|$ with $A>0$ independent of Φ. By Theorem 2.7 applied to M'' and M', there are negative horizontal L^1 forms Φ_n such that $dd^c\Phi_n = \Omega_n$ with $\|\Phi_n\|_{M''\times N} \lesssim \delta^n$. Then, $\delta^{-n}\Phi_n$ belong to a fixed compact set of $\mathrm{PSH}_h(M''\times N)$. Define $\Psi_0 := \Phi - \Phi_0$ and $\Psi_n := f_*(\Phi_{n-1}) - \Phi_n$ for $n \geqslant 1$. We have $dd^c\Psi_n = 0$ and since f_* is continuous, $\delta^{-n}\Psi_n$ belong to some compact set in $\mathscr{H}_h(M''\times N)$.

Fix a current R in $\mathscr{C}_v^1(D)$. We can assume that M'' is chosen so that R is supported on $M''\times N$. We have since $\Phi = \Psi_0 + \Phi_0$

$$\begin{aligned}\big\langle d^{-n}\big(f^n\big)^*R, \Phi\big\rangle &= \big\langle d^{-n}\big(f^n\big)^*R, \Psi_0\big\rangle + \big\langle d^{-n+1}\big(f^{n-1}\big)^*R, d^{-1}f_*(\Phi_0)\big\rangle \\ &= \big\langle d^{-n}\big(f^n\big)^*R, \Psi_0\big\rangle + \big\langle d^{-n+1}\big(f^{n-1}\big)^*R, d^{-1}\Psi_1\big\rangle \\ &\quad + \big\langle d^{-n+1}\big(f^{n-1}\big)^*R, d^{-1}\Phi_1\big\rangle.\end{aligned}$$

By induction and using the identity $f_*(\Phi_n) = \Psi_{n+1} + \Phi_{n+1}$, we obtain

$$\begin{aligned}\big\langle d^{-n}\big(f^n\big)^*R, \Phi\big\rangle &= \sum_{0\leqslant j\leqslant n-1} \big\langle d^{-n+j}\big(f^{n-j}\big)^*R, d^{-j}\Psi_j\big\rangle + \big\langle R, d^{-n}f_*(\Phi_{n-1})\big\rangle \\ &= \sum_{0\leqslant j\leqslant n} \big\langle d^{-n+j}\big(f^{n-j}\big)^*R, d^{-j}\Psi_j\big\rangle + \big\langle R, d^{-n}\Phi_n\big\rangle. \qquad (1)\end{aligned}$$

Now assume that R is smooth and let $n\to\infty$. The estimate on $\|\Phi_n\|$ implies that the last term tends to 0. Recall that $\delta^{-n}\Psi_n$ belong to a compact set in $\mathscr{H}_h(M''\times N)$ and that $\delta < d$. On the other hand, by Theorems 3.3, $d^{-n+j}(f^{n-j})^*R$ tends to T_+ when $n-j\to\infty$. Proposition 4.2 and Lebesgue's convergence theorem, applied to the series in the identity (1), imply that for Φ smooth

$$\langle T_+, \Phi\rangle = \Big\langle T_+, \sum_{j\geqslant 0} d^{-j}\Psi_j\Big\rangle. \qquad (2)$$

Observe that the last sum is pluriharmonic and depends continuously on Φ in $\mathrm{PSH}_h(D)$. It follows from Proposition 2.1 that the right-hand side of the last identity depends continuously on Φ. So, T_+ is a PC current and the identity (2) holds for all Φ in $\mathrm{PSH}_h(D)$. □

The following propositions give the speed of convergence towards the Green current.

Proposition 4.3. *Let f be as in Theorem 4.1 with $d > \delta_+$. Let $\mathscr{P}_v$ be a compact family of currents in $\mathscr{C}_v^1(D)$ and $\mathscr{D}_h$ a compact family of test currents in $\mathrm{PSH}_h(D)$. Then, there exist constants $A>0$ and $\lambda>1$ such that*

$$\big\langle d^{-n}\big(f^n\big)^*R - T_+, \Phi\big\rangle \leqslant A\lambda^{-n}$$

for all $R\in\mathscr{P}_v$, $\Phi\in\mathscr{D}_h$ and $n\geqslant 0$.

Proof. Observe that when Φ belongs to a compact family in $\mathrm{PSH}_h(D)$, $\delta^{-n}\Phi_n$ and $\delta^{-n}\Psi_n$ belong to compact families in $\mathrm{PSH}_h(M' \times N)$ and in $\mathscr{H}_h(M' \times N)$ for some $M' \Subset M$. It follows from the identities (1) and (2) that $\langle d^{-n}(f^n)^*R - T_+, \Phi\rangle$ is equal to

$$\sum_{0\leqslant j\leqslant n} d^{-j}\big\langle d^{-n+j}\big(f^{n-j}\big)^* R - T_+, \Psi_j\big\rangle - \sum_{j\geqslant n+1} d^{-j}\langle T_+, \Psi_j\rangle + d^{-n}\langle R, \Phi_n\rangle.$$

Proposition 4.2 implies that $|\langle d^{-n+j}(f^{n-j})^*R - T_+, \Psi_j\rangle| \lesssim \lambda_0^{-n+j}\delta^j$. We also deduce from Proposition 2.1 applied to $M' \times N$ instead of D, that $|\langle T_+, \Psi_j\rangle| \lesssim \delta^j$. Since Φ_n is negative, the last term in the previous sum is negative. This implies the desired estimate for $1 < \lambda < \min(\lambda_0, d/\delta)$. $\square$

Proposition 4.4. *Let f be as in Theorem* 4.1 *with $d > \delta_+$. Let $\mathscr{P}'_v$ be a bounded family of PB currents in $\mathscr{C}^1_v(D)$ and $\mathscr{D}_h$ a compact family of test currents in* $\mathrm{PSH}_h(D)$. *Then, there exist constants $A > 0$ and $\lambda > 1$ such that*

$$\big|\big\langle d^{-n}\big(f^n\big)^* R - T_+, \Phi\big\rangle\big| \leqslant A\lambda^{-n}$$

for all $R \in \mathscr{P}'_v$, $\Phi \in \mathscr{D}_h$ and $n \geqslant 0$.

Proof. As in Proposition 4.3, it is enough to estimate $|\langle R, \Phi_n\rangle|$. We have $|\langle R, \Phi_n\rangle| \lesssim \delta^n$ since R belongs to a bounded family of PB currents in $\mathscr{C}^1_v(M' \times N)$ for some $M' \Subset M$. This implies the proposition. $\square$

Remark 4.5. In Propositions 4.3 and 4.4, the condition $d > \delta_+$ is superflous if the mass of $d^{-n}(f^n)_*(dd^c\Phi)$ decreases to 0 exponentially and uniformly on $\Phi \in \mathscr{D}_h$ when n goes to infinity. We will use this observation in the proof of Theorem 5.1.

The following result gives a strong ergodic property for the action of f on vertical currents.

Theorem 4.6. *Let f be an invertible horizontal-like map as above with $d > \delta_+$. Then $d^{-n}(f^n)^*R$ converge to T_+ uniformly on $R \in \mathscr{C}^1_v(D)$. In particular, T_+ is the unique current in $\mathscr{C}^1_v(D)$ which is invariant under $d^{-1}f^*$.*

Proof. Since smooth horizontal test forms are generated by the PSH ones, it is enough to test smooth PSH horizontal forms. Using identity (1) for Φ smooth, we only have to show that $d^{-n}\langle R, \Phi_n\rangle$ tend to 0 uniformly on R. Recall that Φ_n is negative, so $d^{-n}\langle R, \Phi_n\rangle$ is negative. For simplicity, we reduce the size of D and we replace R by $d^{-1}f^*(R)$. So, we can assume that f is defined in a neighbourhood $\widehat{D} = \widehat{M} \times \widehat{N}$ of $\widetilde{D} = \widetilde{M} \times \widetilde{N}$ and that R, Φ_n are vertical or horizontal on $M'' \times \widehat{N}$ and $\widehat{M} \times N''$ respectively. Recall that the convex sets $\widetilde{M}$, $\widehat{M}$, $\widetilde{N}$ and $\widehat{N}$ are chosen so that $M \Subset \widetilde{M} \Subset \widehat{M}$ and $N \Subset \widetilde{N} \Subset \widehat{N}$. We can also assume that the $\mathscr{C}^1$ norm of f^{-1} on $\widehat{D}$ is bounded by a constant $A > 0$.

Assume by contradiction that there is an increasing sequence (n_i) such that $\langle R_i, \Phi_{n_i}\rangle \leqslant -2cd^{n_i}$ for some positive constant $c > 0$ and some sequence (R_i) in $\mathscr{C}^1_v(M' \times \widehat{N})$. Let $(R_{i,\theta})_{\theta\in\Delta}$ denote the structural discs associated to R_i as in Section 2. Define $\varphi_i(\theta) := \delta^{-n_i}\langle R_{i,\theta}, \Phi_{n_i}\rangle$ with

$\delta_+ < \delta < d$. The properties of $R_{i,\theta}$ and of Φ_n imply that φ_i belong to a compact family of subharmonic functions on Δ. It is then classical that for every compact subset K of Δ there are constants $C > 0$ and $\alpha > 0$ such that $\|e^{-\alpha\varphi_i}\|_{L^1(K)} \leqslant C$, see e.g. [27].

The currents $R_{i,\theta}$ are obtained by smoothing of R. Using a coordinate change, we obtain that

$$\langle R_{i,\theta}, \Phi_{n_i}\rangle = \langle R_i, \Phi_{n_i,\theta}\rangle$$

where $\Phi_{n_i,\theta}$ is a smoothing of Φ_{n_i}. With the notation in Section 2, we have

$$\Phi_{n_i,\theta} := \int (h_{a,b,\theta})^*(\Phi_{n_i})\, d\nu(a,b).$$

Since the family $h_{a,b,\theta}$ is holomorphic and $h_{a,b,1} = \mathrm{id}$, we obtain (see also [17, Lemma 2.7])

$$\|\Phi_{n_i,\theta} - \Phi_{n_i}\|_{\infty,D} \lesssim |\theta - 1|\|\Phi_{n_i}\|_{\mathscr{C}^1(\widetilde{D})}$$

for θ close to 1. On the other hand, the $\mathscr{C}^1$ norm of f^{-n} is bounded by A^n, hence Theorem 2.7 and Remark 2.10 imply that

$$\|\Phi_n\|_{\mathscr{C}^1(\widetilde{D})} \lesssim \left\|\left(f^n\right)_*\left(dd^c\Phi\right)\right\|_{\mathscr{C}^1(\widehat{D})} \lesssim A^{2kn}.$$

Therefore, $\|\Phi_{n_i,\theta} - \Phi_{n_i}\|_{\infty,D} \lesssim |\theta - 1|A^{2kn_i}$ and since the mass of R_i is bounded

$$\left|\langle R_i, \Phi_{n_i,\theta} - \Phi_{n_i}\rangle\right| \lesssim |\theta - 1|A^{2kn_i}.$$

Hence, for θ in a disc of center 1 and of radius $\simeq A^{-2kn_i}$, we have $\langle R_{i,\theta}, \Phi_{n_i}\rangle \leqslant -cd^{n_i}$ and then $\varphi_i(\theta) \leqslant -cd^{n_i}\delta^{-n_i}$. This contradicts the above uniform integrability of $e^{-\alpha\varphi_i}$. □

4.2. *Convergence towards the equilibrium measure*

The main result in this section is the following property of the equilibrium measure.

Theorem 4.7. *Let f be an invertible horizontal-like map as above with $d > \delta_+$ and $d > \delta_-$. Then the equilibrium measure μ of f is PC.*

Proof. By Theorem 4.1, T_+, T_- are PC on D and also on $D' := M' \times N'$. If φ is a p.s.h. function on D, φ is locally integrable with respect to the trace measure $T_- \wedge \omega^p$ of T_-. Hence, φT_- defines a PSH horizontal current. Moreover, the fact that T_- is PC implies that $\varphi \mapsto \varphi T_-$ is continuous on $\varphi \in \mathrm{PSH}(D)$ with values in $\mathrm{PSH}_h(D)$. Indeed, if Θ is a smooth vertical (p,p)-form, then $\varphi \mapsto \langle\Theta, \varphi T_-\rangle$ is continuous since it is upper semi-continuous when Θ is positive and continuous when Θ is positive closed. In general, $\pm\Theta$ can be written as differences of a positive form and a positive closed one. Using the PC property of T_+ and the identity $\langle\mu, \varphi\rangle = \langle T_+, \varphi T_-\rangle$, see Remark 2.4, we obtain that $\langle\mu, \varphi\rangle$ depends continuously on φ. Therefore, μ is PC. □

We can now prove estimates on the speed of convergence towards the equilibrium measure.

Proposition 4.8. *Let f be as in Theorem 4.7 with $d > \delta_+$ and $d > \delta_-$. Let $\mathscr{P}_v$ (respectively $\mathscr{P}_h$) be a compact family of currents in $\mathscr{C}_v^1(D)$ (respectively in $\mathscr{C}_h^1(D)$). Then, there exist constants $A > 0$ and $\lambda > 1$ such that*

$$\langle d^{-2n}(f^n)^* R \wedge (f^n)_* S - \mu, \varphi\rangle \leqslant A\lambda^{-n}$$

for all $R \in \mathscr{P}_v$, $S \in \mathscr{P}_h$, φ p.s.h. on D with $|\varphi| \leqslant 1$ and $n \geqslant 0$.

Proof. Since $\mu = T_+ \wedge T_-$, we can write $\langle d^{-2n}(f^n)^* R \wedge (f^n)_* S - \mu, \varphi\rangle$ as the sum of the following two integrals

$$\langle d^{-2n}(f^n)^* R \wedge (f^n)_* S - d^{-n}(f^n)^* R \wedge T_-, \varphi\rangle = \langle d^{-n}(f^n)_* S - T_-, \varphi d^{-n}(f^n)^* R\rangle$$

and

$$\langle d^{-n}(f^n)^* R \wedge T_- - T_+ \wedge T_-, \varphi\rangle = \langle d^{-n}(f^n)^* R - T_+, \varphi T_-\rangle.$$

Since R is in a compact family in $\mathscr{C}_v^1(D)$, $d^{-n}(f^n)^* R$ belong also to a compact family in $\mathscr{C}_v^1(D)$ independent of $n \geqslant 0$. Indeed, their supports are controlled. Hence, for $|\varphi| \leqslant 1$, $\varphi d^{-n}(f^n)^* R$ belong to a compact family in $\mathrm{PSH}_v(D)$. By Proposition 4.3 applied to f^{-1}, the first integral is $\lesssim \lambda^{-n}$ for some $\lambda > 1$. Since φT_- belongs to a compact family in $\mathrm{PSH}_h(D)$, the second integral is also $\lesssim \lambda^{-n}$ for some $\lambda > 1$. The proposition follows. □

Proposition 4.9. *Let f be as in Theorem 4.7 with $d > \delta_+$ and $d > \delta_-$. Let $\mathscr{P}_v$ (respectively $\mathscr{P}_h$) be a bounded family of PB currents in $\mathscr{C}_v^1(D)$ (respectively in $\mathscr{C}_h^1(D)$). Then, there exist constants $A > 0$ and $\lambda > 1$ such that*

$$\left|\langle d^{-2n}(f^n)^* R \wedge (f^n)_* S - \mu, \varphi\rangle\right| \leqslant A\lambda^{-n}$$

for all $R \in \mathscr{P}_v$, $S \in \mathscr{P}_h$, φ p.s.h. on D with $|\varphi| \leqslant 1$ and $n \geqslant 0$.

Proof. We proceed as in the proof of Proposition 4.8 using Proposition 4.4 instead of Proposition 4.3. □

5. Properties of the equilibrium measure

In this section, we prove two important properties of the equilibrium measure for horizontal-like maps with large main dynamical degree.

5.1. Decay of correlations

It was proved in [17] that the equilibrium measure is mixing for a general invertible horizontal-like map. Under our hypothesis on dynamical degrees, we have the following result.

Theorem 5.1. *Let f be an invertible horizontal-like map as above with $d > \delta_+$ and $d > \delta_-$. Then the equilibrium measure μ of f is exponentially mixing. More precisely, for all test functions ϕ of class $\mathscr{C}^\alpha$ and ψ of class $\mathscr{C}^\beta$ on D with $0 < \alpha, \beta \leqslant 2$, the following estimate holds*

$$\left|\langle \mu, (\phi \circ f^n)\psi\rangle - \langle \mu, \phi\rangle\langle \mu, \psi\rangle\right| \leqslant A_{\alpha,\beta}\lambda^{-n\alpha\beta}\|\phi\|_{\mathscr{C}^\alpha}\|\psi\|_{\mathscr{C}^\beta}$$

where $A_{\alpha,\beta} > 0$ is a constant independent of ϕ, ψ, n and $\lambda > 1$ is a constant independent of α, β, ϕ, ψ, n.

Recall that the measure μ is mixing means that the left-hand side of the above inequality tends to 0 when n goes to infinity. It follows from the theory of interpolation between Banach spaces [36] that the previous inequality for general α, β is deduced from the case where $\alpha = \beta = 2$, see [11] for details. In the case of Hénon-like maps, i.e. $k = 2$, we have $\delta_+ = \delta_- = 1$. So, the hypothesis in the previous theorem is automatically satisfied and we obtain the following corollary.

Corollary 5.2. *Let f be a Hénon-like map. Then the equilibrium measure of f is exponentially mixing.*

Proof of Theorem 5.1. We only have to consider the case where $\alpha = \beta = 2$. Define

$$I_n(\phi, \psi) := \big\langle \mu, \big(\phi \circ f^n\big)\psi \big\rangle - \langle \mu, \phi \rangle \langle \mu, \psi \rangle.$$

Observe that since $I_{n+1}(\phi, \psi) = I_n(\phi \circ f, \psi)$, it is enough to consider the case where n is even. Note also that since μ is invariant, $I_n(\phi, \psi) = 0$ when ϕ or ψ is constant.

Near $\operatorname{supp}(\mu)$ we can write ϕ and ψ as differences of functions which are strictly p.s.h. on a neighbourhood of $\overline{D}$. So, we can assume that $dd^c\phi \geqslant dd^c\|z\|^2$, $dd^c\psi \geqslant dd^c\|z\|^2$ and that ϕ, ψ have $\mathscr{C}^2$ norms bounded by a fixed constant. This allows to fix a constant $A > 0$ large enough such that $(\phi(z) + A)(\psi(z') + A)$ and $(-\phi(z) + A)(\psi(z') - A)$ are p.s.h. on (z, z') in D^2. We have to bound from above

$$I_{2n}(\phi, \psi) = I_{2n}(\phi + A, \psi + A)$$

and

$$-I_{2n}(\phi, \psi) = I_{2n}(-\phi + A, \psi - A).$$

We will consider the first quantity, the proof for the second one is similar. For that purpose, we will apply Proposition 4.3 and Remark 4.5 to the product F of the horizontal-like maps f and f^{-1} defined in Section 3.

Define $\varphi(z, z') := (\phi(z) + A)(\psi(z') + A)$. Let Δ denote the diagonal of $D \times D$ and $[\Delta]$ the current of integration on Δ. We have since μ is invariant

$$I_{2n}(\phi + A, \psi + A) = \big\langle \mu, \big(\phi \circ f^n + A\big)\big(\psi \circ f^{-n} + A\big)\big\rangle - \langle \mu, \phi + A \rangle \langle \mu, \psi + A \rangle.$$

Lifting these integrals to $D \times D$ and using the identity

$$d^{-2} F^*(T_+ \otimes T_-) = d^2 F_*(T_+ \otimes T_-) = T_+ \otimes T_-,$$

we obtain that $I_{2n}(\phi + A, \psi + A)$ is equal to

$$\begin{aligned}
&\big\langle (T_+\otimes T_-)\wedge[\Delta], \varphi\circ F^n\big\rangle - \langle \mu\otimes\mu, \varphi\rangle \\
&\quad = \big\langle (T_+\otimes T_-)\wedge d^{-2n}\big(F^n\big)_*[\Delta], \varphi\big\rangle - \big\langle (T_+\wedge T_-)\otimes(T_+\wedge T_-), \varphi\big\rangle \\
&\quad = \big\langle (T_+\otimes T_-)\wedge d^{-2n}\big(F^n\big)_*[\Delta], \varphi\big\rangle - \big\langle (T_+\otimes T_-)\wedge(T_-\otimes T_+), \varphi\big\rangle \\
&\quad = \big\langle d^{-2n}\big(F^n\big)_*[\Delta] - T_-\otimes T_+, \varphi(T_+\otimes T_-)\big\rangle.
\end{aligned}$$

The current $[\Delta]$ is not horizontal but $F_*[\Delta]$ is horizontal. So, we can apply Proposition 4.3 and Remark 4.5 for F^{-1}.

For Remark 4.5, we need to show that the mass of

$$d^{-2n}\big(F^n\big)^*\big[dd^c\varphi\wedge(T_+\otimes T_-)\big] = dd^c\big(\varphi\circ F^n\big)\wedge(T_+\otimes T_-)$$

decreases exponentially (we reduce the size of D if necessary). We have

$$\big\|dd^c\big(\varphi\circ F^n\big)\wedge(T_+\otimes T_-)\big\|_{D^2} = \int_{D^2}\big(dd^c\|z\|^2 + dd^c\|z'\|^2\big)^{k-1}\wedge dd^c\big(\varphi\circ F^n\big)\wedge(T_+\otimes T_-).$$

In the last wedge-product, T_+ depends only on z and T_- depends only on z'. Then we expand

$$dd^c\big(\varphi\circ F^n\big) = dd^c\big[\big(\phi\big(f^n(z)\big)+A\big)\big(\psi\big(f^{-n}(z')\big)+A\big)\big].$$

In this product, the terms containing mixed derivatives

$$d\phi\big(f^n(z)\big)\wedge d^c\psi\big(f^{-n}(z')\big) \quad\text{and}\quad d^c\phi\big(f^n(z)\big)\wedge d\psi\big(f^{-n}(z')\big)$$

vanish when wedged with $(dd^c\|z\|^2 + dd^c\|z'\|^2)^{k-1}\wedge(T_+\otimes T_-)$ by bidegree consideration. This, combined with the fact that φ and ψ are bounded, implies that

$$\big\|dd^c\big(\varphi\circ F^n\big)\wedge(T_+\otimes T_-)\big\|_{D^2} \lesssim \big\|dd^c\big(\phi\circ f^n\big)\wedge T_+\big\|_D + \big\|dd^c\big(\psi\circ f^{-n}\big)\wedge T_-\big\|_D.$$

We have

$$dd^c\big(\phi\circ f^n\big)\wedge T_+ = d^{-n}\big(f^n\big)^*(dd^c\phi\wedge T_+)$$

and

$$dd^c\big(\psi\circ f^{-n}\big)\wedge T_- = d^{-n}\big(f^n\big)_*(dd^c\psi\wedge T_-).$$

Since $d>\delta_+$ and $d>\delta_-$, the masses of these currents decrease exponentially. This completes the proof. $\square$

Remark 5.3. We can prove the converse of Theorem 4.1: *if the current T_+ of f is PB then $d>\delta_+$*. This will allow to prove that F satisfies also the hypothesis on dynamical degrees if $d>\delta_+$ and $d>\delta_-$, hence we can apply directly Proposition 4.3. However, the proof requires a long development on the notion of super-functions introduced in Section 2, and we prefer to avoid it here, see also [19].

5.2. Lyapounov exponents

We will show that when the main dynamical degree of f is larger than the other ones, the measure μ is hyperbolic: it admits p strictly positive and $k-p$ strictly negative Lyapounov exponents. We follow the approach by de Thélin [9].

Recall that the measure μ is mixing and is supported on the filled Julia set $\mathscr{K} := \mathscr{K}_+ \cap \mathscr{K}_-$ which is compact in D, see [17]. Using the theory of Oseledec and Pesin [32], we can decompose the tangent space of $\mathbb{C}^k$ at μ-almost every point x into a direct sum of vector subspaces $T_x = \bigoplus_{i=1}^m E_{i,x}$ with the following properties:

- The integer m and the dimension of each $E_{i,x}$ do not depend on x.
- The decomposition $T_x = \bigoplus_{i=1}^m E_{i,x}$ is unique and depends in a measurable way on x.
- The vector bundle $E_{i,x}$ is *invariant* under f, that is, the differential Df of f defines an isomorphism between $E_{i,x}$ and $E_{i,f(x)}$.
- The decomposition $T_x = \bigoplus_{i=1}^m E_{i,x}$ has a *tempered distortion*. More precisely, if I and J are disjoint subsets of $\{1,\dots,m\}$, define $E_{I,x} := \bigoplus_{i\in I} E_{i,x}$ and $E_{J,x} := \bigoplus_{i\in J} E_{i,x}$. Then, the angle $\measuredangle(E_{I,f^n(x)}, E_{J,f^n(x)})$ between $E_{I,f^n(x)}$ and $E_{J,f^n(x)}$ satisfies
$$\lim_{n\to\pm\infty} \frac{1}{n} \log\sin\measuredangle(E_{I,f^n(x)}, E_{J,f^n(x)}) = 0.$$
- There are distinct real numbers λ_i independent of x such that
$$\lim_{n\to\pm\infty} \frac{1}{|n|} \log \frac{\|Df^n(v)\|}{\|v\|} = \pm\lambda_i$$
uniformly on v in $E_{i,x}\setminus\{0\}$.

The constants λ_i are *the Lyapounov exponents* of μ. *The multiplicity* of λ_i is the dimension of $E_{i,x}$. So, μ admits k Lyapounov exponents counted with multiplicities.[2] The Lyapounov exponents of f^n are $n\lambda_i$ even for n negative. When there is no zero Lyapounov exponent, μ is said to be *hyperbolic*.

Theorem 5.4. *Let f be an invertible horizontal-like map as above with dynamical degrees d, d_s^+ and d_s^-. Define $\widetilde{\delta}_+ := \max_{s\leqslant p-1} d_s^+$ and $\widetilde{\delta}_- := \max_{s\leqslant k-p-1} d_s^-$. If $\widetilde{\delta}_+ < d$, then μ admits p strictly positive Lyapounov exponents larger than or equal to $\frac{1}{2k}\log(d/\widetilde{\delta}_+)$. If $\widetilde{\delta}_- < d$, then μ admits $k-p$ strictly negative ones which are smaller than or equal to $-\frac{1}{2k}\log(d/\widetilde{\delta}_-)$.*

We prove the first assertion. The second one is treated in the same way using f^{-1} instead of f. We will need the following lemmas where ω_v denotes the restriction to $M''\times N$ of the standard Kähler form ω on $\mathbb{C}^k$. Define $\widetilde{d}_q^+ := \max_{s\leqslant q} d_s^+$ for $0\leqslant q\leqslant p-1$.

Lemma 5.5. *Let δ be a constant strictly larger than $\widetilde{d}_q^+$. Then there exists a constant $C>0$ such that for all positive closed current S of bidegree $(k-q,k-q)$ supported on $M\times N'$ we have*

[2] If f is considered as a real map, the multiplicity of λ_i is $2\dim E_{i,x}$ and μ has $2k$ Lyapounov exponents; this is the reason for the coefficients $\frac{1}{2}$ in Theorem 5.4.

$$\int S \wedge \left(f^{n_1}\right)^* \omega_v \wedge \cdots \wedge \left(f^{n_q}\right)^* \omega_v \leqslant C\delta^{n_1} \|S\|_D$$

for all integers $n_1 \geqslant \cdots \geqslant n_q \geqslant 0$.

Proof. We prove the lemma by induction on q. Clearly, the lemma is valid for $q = 0$. Suppose it holds for the rank $q-1$. This, applied to f restricted to $D_v := M' \times N$ and to $S' := (f^{n_q})_* S \wedge \omega$, implies that

$$\int_{D_v} S' \wedge \left(f^{n_1-n_q}\right)^* \omega_v \wedge \cdots \wedge \left(f^{n_{q-1}-n_q}\right)^* \omega_v \leqslant C\delta^{n_1-n_q} \|S'\|_{D_v}.$$

By definition of d_s^+, there is a constant $c > 0$ such that

$$\|S'\|_{D_v} = \left\| \left(f^{n_q}\right)_* S \right\|_{D_v} \leqslant c\delta^{n_q} \|S\|_D.$$

Consequently,

$$\int_{D_v} S' \wedge \left(f^{n_1-n_q}\right)^* \omega_v \wedge \cdots \wedge \left(f^{n_{q-1}-n_q}\right)^* \omega_v \leqslant C\delta^{n_1} \|S\|_D$$

for some constant $C > 0$. The left-hand side of the last inequality is equal to

$$\int_{f^{-n_q}(D_v)} S \wedge \left(f^{n_1}\right)^* \omega_v \wedge \cdots \wedge \left(f^{n_q}\right)^* \omega_v.$$

This implies the lemma for rank q. Note that the last integral does not change if we replace $f^{-n_q}(D_v)$ by D since $(f^{n_q})^*\omega_v$ is supported on $f^{-n_q}(D_v)$. $\square$

Let Γ_n denote the graph of $(f, \ldots, f^n)$, i.e. the set of points $(x, f(x), \ldots, f^n(x))$ in D^{n+1}. We will use the standard Kähler metric ω_n in $D^{n+1} \subset \mathbb{C}^{k(n+1)}$. If Π_j, with $0 \leqslant j \leqslant n$, denote the projections from D^{n+1} onto its factors D, we have $\omega_n = \sum \Pi_j^*(\omega)$. Let π_j denote the restriction of Π_j to Γ_n and $\mathrm{vol}_n(S)$ the mass of $\pi_0^*(S)$ on $\bigcap_{0\leqslant j\leqslant n} \pi_j^{-1}(M'' \times N)$.

Lemma 5.6. *Let δ be a constant strictly larger than $\widetilde{d}_q^+$. Then there exists a constant $C > 0$ such that for all positive closed current S of bidegree $(k-q, k-q)$ supported on $M \times N'$ we have $\mathrm{vol}_n(S) \leqslant C\delta^n \|S\|_D$.*

Proof. Observe that f^j can be identified with $\pi_j \circ \pi_0^{-1}$. This allows to write $\mathrm{vol}_n(S)$ as the following sum of $(n+1)^q$ integrals

$$\mathrm{vol}_n(S) = \left\langle \pi_0^*(S), \left(\sum \pi_j^*(\omega_v)\right)^q \right\rangle = \sum_{0\leqslant n_j \leqslant n} \int S \wedge \left(f^{n_1}\right)^* \omega_v \wedge \cdots \wedge \left(f^{n_q}\right)^* \omega_v.$$

Lemma 5.5 applied to a constant $\delta' > \widetilde{d}_q^+$ implies that $\mathrm{vol}_n(S) \leqslant C' n^q \delta'^n \|S\|$ for some constant $C' > 0$. We obtain the result by choosing a δ' smaller than δ. □

A subset A of D is said to be (n, ϵ)-*separated* if f^j is defined on A with $f^j(A) \subset D'' := M'' \times N''$ for $0 \leqslant j \leqslant n$ and for every distinct points a, b in A the distance between $f^j(a)$ and $f^j(b)$ is larger than ϵ for at least one j with $0 \leqslant j \leqslant n$. Define for a subset X of D the *topological entropy* of f on X by

$$h_X(f) := \sup_{\epsilon>0} \limsup_{n\to\infty} \frac{1}{n} \log \max \#\big\{A \subset X,\ A\ (n, \epsilon)\text{-separated}\big\}.$$

We have the following version of the Gromov's inequality, see also [9,14,24].

Proposition 5.7. *Let δ be a constant strictly larger than $\widetilde{d}_q^+$ with $q \leqslant p - 1$. Let X be a horizontal subvariety of dimension q of D. Then for every $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that every (n, ϵ)-separated subset in X contains at most $C_\epsilon \delta^n$ points. In particular, we have $h_X(f) \leqslant \log \widetilde{d}_q^+$.*

Proof. We can choose N' such that X is contained in $M \times N'$. We can also assume that ϵ is small enough. So, X defines a horizontal positive closed current $[X]$ of bidegree $(k - q, k - q)$. Lemma 5.6 applied to M' instead of M'', implies that the volume of $\pi_0^{-1}(X)$ in $\bigcap_{0\leqslant j\leqslant n} \pi_j^{-1}(M' \times N)$ is smaller than $C\delta^n$ for some constant $C > 0$.

Consider an (n, ϵ)-separated subset A of X. For every a in A denote by B_a the ball of center $(a, f(a), \ldots, f^n(a))$ and of diameter ϵ in D^{n+1}. Since A is (n, ϵ)-separated, these balls are disjoint. Since ϵ is small and the center of B_a is in $\bigcap_{0\leqslant j\leqslant n} \pi_j^{-1}(D'')$, these balls are contained in $\bigcap_{0\leqslant j\leqslant n} \pi_j^{-1}(M' \times N)$. It follows that the total volume of $B_a \cap \pi_0^{-1}(X)$ is bounded by $C\delta^n$. On the other hand, an inequality of Lelong [30] says that the volume of $B_a \cap \pi_0^{-1}(X)$ is bounded from below by a constant depending only on ϵ. Hence, the number of the balls B_a is $\lesssim \delta^n$. This implies that $\#A \lesssim \delta^n$ and completes the proof. □

Recall that it is proved in [17] that μ is of maximal entropy $\log d$. This also holds for f^{-1} since the main dynamical degree of f^{-1} is also equal to d. Let $B_{-n}(x, \epsilon)$ denote the *Bowen* $(-n, \epsilon)$-*ball* with center x, i.e. the set of the points y such that $f^{-j}(y)$ is defined and $\|f^{-j}(y) - f^{-j}(x)\| \leqslant \epsilon$ for $0 \leqslant j \leqslant n$. *The entropy $h(\mu)$ for f^{-1}* can be obtained by the following Brin–Katok formula [6]

$$h(\mu) := \sup_{\epsilon>0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu\big(B_{-n}(x, \epsilon)\big)$$

for μ-almost every x. So, for every $\theta > 0$, there are positive constants C, ϵ and a Borel set Σ_0 with $\mu(\Sigma_0) > 3/4$ such that $\mu(B_{-n}(x, 6\epsilon)) \leqslant Ce^{-n(\log d - \theta)}$ for $x \in \Sigma_0$ and $n \geqslant 0$.

Proof of Theorem 5.4. Assume in order to reach a contradiction that μ admits at least $k - p + 1$ Lyapounov exponents strictly smaller than $\frac{1}{2k} \log(d/\widetilde{\delta}_+)$. Let $q \leqslant p - 1$ be an integer and $\lambda < \frac{1}{2k} \log(d/\widetilde{\delta}_+)$ a positive constant such that μ admits exactly $k - q$ Lyapounov exponents strictly smaller than λ and the other ones are larger than or equal to $\frac{1}{2k} \log(d/\widetilde{\delta}_+)$. We are going to construct a complex subspace F of dimension q, contradicting the estimate in Proposition 5.7, i.e. with too many (n, ϵ)-separated points.

Fix a positive constant θ such that $\theta \ll \lambda$ and $\theta \ll \frac{1}{2k}\log(d/\widetilde{\delta}_+) - \lambda$. By Oseledec–Pesin theory (replacing f by an iterate f^n and θ, λ, d, $\widetilde{\delta}_+$ by $n\theta$, $n\lambda$, d^n, $\widetilde{\delta}_+^n$ if necessary), we can assume that there is a decomposition $T_x = E_x \oplus F_x$ for μ almost every x with the following properties:

– E_x and F_x are vector spaces of dimension $k-q$ and q respectively.
– The vector bundles E_x and F_x are f-invariant.
– There is a Borel set $\Sigma \subset \mathscr{K}$ with $\mu(\Sigma) \geqslant 1/2$ and a constant $\eta > 0$ such that

$$\|Df^{-1}(v)\| \geqslant e^{-\lambda}\|v\|, \qquad \|Df^{-1}(u)\| \leqslant e^{-\lambda-7\theta}\|u\|, \qquad \measuredangle(E_{f^{-n}(x)}, F_{f^{-n}(x)}) \geqslant \eta e^{-n\theta}$$

for $v \in E_x$, $u \in F_x$, $x \in \Sigma$ and $n \geqslant 0$.

We now identify each T_x with $\mathbb{C}^k$ and consider x as the origin. Fix coordinate systems on E_x and F_x so that the associated distances coincide with the distances induced by the standard metric on $\mathbb{C}^k$. On $T_x = E_x \oplus F_x$ we use the coordinate system induced by the fixed coordinates on E_x and F_x. We call it *dynamical coordinate system*. Note that the angle between E_x and F_x, with respect to the standard coordinates, might be small and in this case there is a big distorsion of the dynamical coordinates with respect to the standard ones.

Fix a positive constant c small enough, $c \ll \eta$ and $c \ll \epsilon$ where ϵ is the constant associated to θ as above. Let $B_{x_{-n}}$ denote the (small) ball of radius $ce^{-n(\lambda+7\theta)}$ of center $x_{-n} := f^{-n}(x)$ in $E_{x_{-n}}$. We are interested in graphs in $T_{x_{-n}} = E_{x_{-n}} \oplus F_{x_{-n}}$ of holomorphic maps over $B_{x_{-n}}$.

Claim 1. *For every $x \in \Sigma$ there are holomorphic maps $h_n : B_{x_{-n}} \to F_{x_{-n}}$ with graph $V_{x_{-n}}$ such that $h_n(0) = 0$, $\|Dh_n\| \leqslant e^{-4n\theta}$ and f sends $V_{x_{-n-1}}$ into $V_{x_{-n}}$.*

The proof of this claim is by induction. For $n = 0$, it is enough to choose $h_0 = 0$. We will obtain $V_{x_{-n}}$ as an open set in $f^{-1}(V_{x_{-n+1}})$. Consider the map f^{-1} on a small neighbourhood of x_{-n+1} with image in a neighbourhood of x_{-n}. In dynamical coordinates for $T_{x_{-n+1}}$ and $T_{x_{-n}}$ we can write

$$f^{-1}(z) = l(z) + r(z) \quad \text{with } l = (l', l'') \text{ and } r = (r', r'')$$

where $l(z)$ is the linear part of f, i.e. the differential Df^{-1} at x_{-n+1}, and $r(z)$ is the rest which is of order $\geqslant 2$ with respect to z.

We have $l' : E_{x_{-n+1}} \to E_{x_{-n}}$ and $l'' : F_{x_{-n+1}} \to F_{x_{-n}}$. We also have $\|l'(z')\| \geqslant e^{-\lambda}\|z'\|$ for $z' \in E_{x_{-n+1}}$ and $\|l''(z'')\| \leqslant e^{-(\lambda+7\theta)}\|z''\|$ for $z'' \in F_{x_{-n+1}}$. In the standard coordinates, the derivatives of f^{-1} are bounded. Taking into account the distortions of dynamical coordinates, we have $\|Dr(z)\| \leqslant Ae^{6n\theta}\|z\|$ with $A > 0$ independent of c, n, θ. Now, consider two points $z = (z', z'')$ and $w = (w', w'')$ in $E_{x_{-n+1}} \oplus F_{x_{-n+1}}$ which are contained in $V_{x_{-n+1}}$. So, $\|z\|$ and $\|w\|$ are smaller than $2ce^{-(n-1)(\lambda+7\theta)}$. Write $\widetilde{z} := (\widetilde{z}', \widetilde{z}'') = f^{-1}(z)$ and $\widetilde{w} := (\widetilde{w}', \widetilde{w}'') = f^{-1}(w)$. We deduce from the estimates on l', Dr and Dh_{n-1} that

$$\begin{aligned}
\|\widetilde{z}' - \widetilde{w}'\| &\geqslant \|l'(z') - l'(w')\| - \|r'(z) - r'(w)\| \\
&\geqslant e^{-\lambda}\|z' - w'\| - 2Ae^{6n\theta}ce^{-(n-1)(\lambda+7\theta)}\|z - w\| \\
&\geqslant e^{-\lambda}\|z' - w'\| - 4Ae^{6n\theta}ce^{-(n-1)(\lambda+7\theta)}\|z' - w'\|.
\end{aligned}$$

Hence, $\|\widetilde{z}' - \widetilde{w}'\| \geqslant e^{-(\lambda+\theta)}\|z' - w'\|$ since c, θ are small and $\theta \ll \lambda$. It follows that $f^{-1}(V_{x_{-n+1}})$ is a graph of a holomorphic map h_n over an open set B of $E_{x_{-n}}$. The last estimate for $w' = 0$ implies that B contains the ball $B_{x_{-n}}$.

On the other hand, we have

$$\begin{aligned}\|\widetilde{z}'' - \widetilde{w}''\| &\leqslant \big\|l''(z'') - l''(w'')\big\| + \big\|r''(z) - r''(w)\big\| \\ &\leqslant e^{-(\lambda+7\theta)}\|z'' - w''\| + 2Ae^{6n\theta}ce^{-(n-1)(\lambda+7\theta)}\|z - w\| \\ &\leqslant e^{-(\lambda+7\theta)}e^{-4(n-1)\theta}\|z' - w'\| + 4Ae^{6n\theta}ce^{-(n-1)(\lambda+7\theta)}\|z' - w'\|.\end{aligned}$$

Therefore, $\|\widetilde{z}'' - \widetilde{w}''\| \leqslant e^{-4n\theta}\|\widetilde{z}' - \widetilde{w}'\|$ since $\theta \ll \lambda$ and c is small. It follows that $\|Dh_n\| \leqslant e^{-4n\theta}$ and this finishes the proof of the claim.

Note that all the constructed graphs are small and contained in a small neighbourhood $\mathscr{U}$ of the filled Julia set $\mathscr{K}$. We now come back to the standard metric on $\mathbb{C}^k$. Let F'_x denote the orthogonal of E_x. We use coordinate systems on F'_x which induce the standard metric. Let $B'_{x_{-n}}$ denote the ball of center 0 and of radius $c'e^{-n(\lambda+10\theta)}$ in $E_{x_{-n}}$ with $c' > 0$ small enough. We claim that $V_{x_{-n}}$ contains some flat graph $V'_{x_{-n}}$.

Claim 2. *For every $x \in \Sigma$, $V_{x_{-n}}$ contains the graph $V'_{x_{-n}}$ of a holomorphic map $h'_n : B'_{x_{-n}} \to F'_{x_{-n}}$ such that $h'_n(0) = 0$ and $\|Dh'_n\| \lesssim e^{-n\theta}$.*

With the considered coordinates on $E_{x_{-n}}$, $F_{x_{-n}}$ and $F'_{x_{-n}}$, denote by $\tau : E_{x_{-n}} \oplus F_{x_{-n}} \to E_{x_{-n}} \oplus F'_{x_{-n}}$ the linear map of coordinate change. Since the angle between $E_{x_{-n}}$ and $F_{x_{-n}}$ is larger than $\eta e^{-n\theta}$, we can write $\tau = (\tau', \tau'')$ with $\|\tau'(z) - z'\| \lesssim e^{n\theta}\|z''\|$ and $\|\tau''(z)\| \leqslant \|z''\|$ for $z = (z', z'')$ in $E_{x_{-n}} \oplus F_{x_{-n}}$. Claim 2 is proved using analogous estimates as in Claim 1 where we replace f^{-1} by τ. We will not give the details here.

We continue the proof of Theorem 5.4. Let A be a subset of $\Sigma \cap \Sigma_0$ such that the balls $B_{-n}(x, 3\epsilon)$ with centers $x \in A$ are disjoint. We choose A maximal satisfying this property. So, the balls $B_{-n}(x, 6\epsilon)$ with centers $x \in A$ cover $\Sigma \cap \Sigma_0$. Since $\mu(\Sigma \cap \Sigma_0) \geqslant 1/4$ and $\mu(B_{-n}(x, 6\epsilon)) \leqslant Ce^{-n(\log d - \theta)}$, A contains at least $(4C)^{-1}e^{n(\log d - \theta)}$ points. Consider the graphs $V_{x_{-n}}$ and $V'_{x_{-n}}$ constructed above for $x \in A$. Since the balls $B_{-n}(x, 3\epsilon)$ are disjoint, the set of x_{-n} are $(n, 3\epsilon)$-separated. Claim 1 implies that the diameter of $V_{x_{-n}}$ is smaller than ϵ. So, if we replace each x_{-n} by a point x'_{-n} in $V_{x_{-n}}$ the resulting set is always (n, ϵ)-separated.

Let Π be an orthogonal projection of $\mathbb{C}^k = \mathbb{C}^p \times \mathbb{C}^{k-p}$ onto a subspace E of dimension $k - q$. If E is a product of a subspace of $\mathbb{C}^p$ with $\mathbb{C}^{k-p}$, then the fibers of Π which are close enough to $\mathscr{K}$ (in particular the fibers which intersect $\mathscr{U}$) are horizontal in D. This property holds for the projection on any small perturbation of E. So, we can choose a finite number of projections $\Pi_1, \ldots, \Pi_N$ on $E_1, \ldots, E_N$ satisfying this property, and a constant $\theta_0 > 0$ such that any subspace F of dimension q in $\mathbb{C}^k$ has an angle $\geqslant \theta_0$ with at least one of E_i. We deduce from Claim 2 that for each of the considered graphs $V'_{x_{-n}}$, the volume of $\Pi_i(V'_{x_{-n}})$ is $\geqslant c''e^{-2n(k-q)(\lambda+10\theta)}$ for at least one projection Π_i with a fixed constant $c'' > 0$. Choose an i such that this property holds for at least $N^{-1}\#A$ graphs $V'_{x_{-n}}$. Since $\#A \geqslant (4C)^{-1}e^{n(\log d - \theta)}$, the sum of the volumes of $\Pi_i(V_{x_{-n}})$ is $\gtrsim e^{n(\log d - \theta) - 2n(k-q)(\lambda+10\theta)}$. Hence, there is a fiber F of Π_i which intersects $\gtrsim e^{n(\log d - \theta) - 2n(k-q)(\lambda+10\theta)}$ graphs $V_{x_{-n}}$. It follows that F contains an (n, ϵ)-separated subset of $\gtrsim e^{n(\log d - \theta) - 2n(k-q)(\lambda+10\theta)} \geqslant e^{n(\log \widetilde{\delta}_+ + \theta)}$ points since $\theta \ll \frac{1}{2k}\log(d/\widetilde{\delta}_+) - \lambda$. This contradicts Proposition 5.7 for $X = F$ since $\widetilde{\delta}_+ \geqslant \widetilde{d}_q^+$, and finishes the proof of Theorem 5.4. $\square$

Remark 5.8. The above bound $\frac{1}{2k}\log(d/\widetilde{\delta}_+)$ can be replaced by the infimum of the numbers $\frac{1}{2(k-q)}\log(d/\widetilde{d}_q^+)$ for $q \leqslant p-1$.

Remark 5.9. The fact that we are in the holomorphic setting is used only in Proposition 5.7 in order to get an estimate on the topological entropy on analytic manifolds of dimension q. The result still holds for real $\mathscr{C}^{1+\alpha}$ horizontal-like maps (i.e. non-uniformly hyperbolic horshoes) with an ergodic invariant measure with compact support. We only need that the entropy of the measure is strictly larger than the entropy on vertical subspaces of dimension $\leqslant k-p-1$ and horizontal manifolds of dimension $\leqslant p-1$, see also Newhouse, Buzzi and de Thélin [9,31].

6. Examples and open problems

Consider a polynomial automorphism f of $\mathbb{C}^k$. We extend f to a birational map on the projective space $\mathbb{P}^k$. Let I_+ and I_- denote the indeterminacy sets of f and f^{-1}. They are in the hyperplane at infinity $L_\infty := \mathbb{P}^k \setminus \mathbb{C}^k$ and we assume that they are non-empty. When I_+ and I_- have empty intersection, f is said to be *regular*. This class of automorphisms was introduced and studied in [35]. In dimension $k=2$, they are the Hénon type maps and any polynomial automorphism of positive entropy is conjugated to a regular automorphism.

There is an integer p such that $\dim I_+ = k-p-1$ and $\dim I_- = p-1$. If d_+ and d_- denote the algebraic degrees of f and f^{-1}, we have $d_+^p = d_-^{k-p}$. At infinity we have $f(L_\infty \setminus I_+) = I_-$ and $f^{-1}(L_\infty \setminus I_-) = I_+$. Define the *filled Julia sets* by

$$\mathscr{K}_+ := \left\{z \in \mathbb{C}^k,\ \left(f^n(z)\right)_{n\geqslant 0} \text{ bounded in } \mathbb{C}^k\right\}$$

and

$$\mathscr{K}_- := \left\{z \in \mathbb{C}^k,\ \left(f^{-n}(z)\right)_{n\geqslant 0} \text{ bounded in } \mathbb{C}^k\right\}.$$

These sets are invariant under f^{-1}, f and satisfy $\overline{\mathscr{K}}_+ = \mathscr{K}_+ \cup I_+$, $\overline{\mathscr{K}}_- = \mathscr{K}_- \cup I_-$. One associates to f and f^{-1} the following functions, called *Green functions*

$$G^+(z) := \lim_{n\to\infty} d_+^{-n}\log^+\left\|f^n(z)\right\| \quad \text{and} \quad G^-(z) := \lim_{n\to\infty} d_-^{-n}\log^+\left\|f^{-n}(z)\right\|,$$

where $\log^+ := \max(\log, 0)$. These functions are continuous p.s.h. on $\mathbb{C}^k$. It follows from [15, Proposition 2.4] that G^+ and G^- are Hölder continuous. They satisfy $G^+ \circ f = d_+ G^+$ and $G^- \circ f^{-1} = d_- G^-$. It is shown in [19] that the *Green currents*

$$T_+ := \left(dd^c G^+\right)^p \quad \text{and} \quad T_- := \left(dd^c G^-\right)^{k-p}$$

are, up to a multiplicative constant, the unique positive closed currents of bidegrees (p,p) and $(k-p,k-p)$ with support in $\mathscr{K}_+$ and $\mathscr{K}_-$ respectively. These currents are invariant: $f^*(T_+) = d_+^p T_+$ and $f_*(T_-) = d_-^{k-p} T_-$. Note that to prove the uniqueness we do not assume invariance.

The family of regular automorphisms is large but for simplicity we restrict to the case where the indeterminacy sets I_+ and I_- are linear. In what follows, we assume that

$$I_+ = \{z_0 = z_1 = \cdots = z_p = 0\} \quad \text{and} \quad I_- = \{z_0 = z_{p+1} = \cdots = z_k = 0\}$$

where $[z_0 : \cdots : z_k]$ denotes the homogeneous coordinates of $\mathbb{P}^k$, $\mathbb{C}^k$ is identified to the chart $\{z_0 = 1\}$ and the hyperplane at infinity L_∞ is given by the equation $z_0 = 0$. The following proposition allows to apply the results in the previous sections to the small (possibly transcendental) perturbations of f and proves Corollary 1.2.

Proposition 6.1. *Let f be a regular polynomial automorphism of $\mathbb{C}^k$ as above. Let B_s^R denote the ball of center 0 and of radius R in $\mathbb{C}^s$. Then, if R is large enough, any holomorphic map f_ϵ on $B_p^R \times B_{k-p}^R$, close enough to f, is horizontal-like with the main dynamical degree $d = d_+^p = d_-^{k-p}$. Moreover, d is strictly larger than the other dynamical degrees associated to f_ϵ and f_ϵ^{-1}.*

Proof. By Proposition 3.7, it is enough to check that f restricted to $B_p^R \times B_{k-p}^R$ is a horizontal-like map of main dynamical degree d which is strictly larger than the other dynamical degrees. Write, using the coordinates $(z_1, \ldots, z_k)$ of $\mathbb{C}^k$

$$f = (f', f'') \quad \text{with } f' = (f_1, \ldots, f_p) \text{ and } f'' = (f_{p+1}, \ldots, f_k).$$

Since $f(L_\infty \setminus I_+) = I_-$, the equation of I_- implies that the components of f'' have degree $\leqslant d_+ - 1$ and the components of f' have degree d_+. Moreover, if f_j^+ denotes the homogeneous part of degree d_+ of f_j, the equation of I_+ implies that $f_1^+ = \cdots = f_p^+ = 0$ only when $z_1 = \cdots = z_p = 0$. The restriction of f to I_- defines an endomorphism of algebraic degree d_+.

Since R is large, it follows that $\|f'(z)\| > R$ for z in the vertical boundary of $B_p^R \times B_{k-p}^R$. Hence, $f^{-1}(B_p^R \times B_{k-p}^R)$ does not intersect the vertical boundary of $B_p^R \times B_{k-p}^R$. In the same way, we show that $f(B_p^R \times B_{k-p}^R)$ does not intersect the horizontal boundary of $B_p^R \times B_{k-p}^R$. This proves that f restricted to $B_p^R \times B_{k-p}^R$ is horizontal-like. In order to avoid confusion, let us denote by $\overline{f}$ the horizontal-like map on $D := B_p^R \times B_{k-p}^R$ associated to f.

Since $\overline{\mathscr{K}}_+ = \mathscr{K}_+ \cup I_+$, the equation of I_+ implies that $\mathscr{K}_+$ restricted to D is vertical. The restriction of T_+ to D is vertical and invariant under $d^{-1}\overline{f}^*$. So, the main dynamical degree of $\overline{f}$ is equal to d. It remains to check that the other dynamical degrees are strictly smaller than d.

Fix an $\alpha > 0$ small enough so that $\overline{f}^{-1}(D) \subset B_p^{R-2\alpha} \times B_{k-p}^R$ and $\overline{f}(D) \subset B_p^R \times B_{k-p}^{R-2\alpha}$. So $\overline{f}$ is horizontal-like on $D' := B_p^{R-\alpha} \times B_{k-p}^{R-\alpha}$ and on $D'' := B_p^{R-2\alpha} \times B_{k-p}^{R-2\alpha}$. Consider the family $\mathscr{Q}_h$ of horizontal positive closed currents of bidegree $(k-s, k-s)$ and of mass 1 in D'' with $s \leqslant p-1$. We will show that the mass of $(\overline{f}^n)_* S$ on D'' for $S \in \mathscr{Q}_h$, is of order $O(d_+^s)$. This implies that the dynamical degree d_s^+ of $\overline{f}$ is $\leqslant d_+^s$ and then is strictly smaller than d. The proof is analogous for the degrees d_s^- associated to $\overline{f}^{-1}$.

Observe that $S' := \overline{f}_*(S)$ is horizontal in D' and has bounded mass. Let $\omega_{\mathrm{FS}} := dd^c H$, with $H := \log(1 + \|z\|^2)^{1/2}$, be the Fubini–Study form on $\mathbb{P}^k$. Since the standard Kähler form on $\mathbb{C}^k$ and ω_{FS} are comparable in compact sets of $\mathbb{C}^k$, it is enough to estimate the mass of $\omega_{\mathrm{FS}}^s \wedge (\overline{f}^n)_* S$ on D''. We have

$$\int_{D''} \omega_{\mathrm{FS}}^s \wedge (\overline{f}^n)_* S = \int_{\overline{f}^{-n+1}(D'')} (f^{n-1})^* \omega_{\mathrm{FS}}^s \wedge S' \leqslant \int_{D'} (f^{n-1})^* \omega_{\mathrm{FS}}^s \wedge S' \tag{3}$$

since $\overline{f}^{-n+1}(D'') \subset B_p^{R-2\alpha} \times B_{k-p}^R$ and $\operatorname{supp}(S') \subset B_p^R \times B_{k-p}^{R-\alpha}$. It was shown in [35] that $d_+^{-n} \log^+ \|f^n(z)\|$ converge locally uniformly to G^+. We deduce easily that $d_+^{-n} H \circ f^n$ converge also locally uniformly to G^+. It follows from the theory of intersection of currents, see [7,23] that the family of currents

$$d_+^{-sn}\left(f^n\right)^*\omega_{\mathrm{FS}}^s \wedge S = d_+^{-sn}\left(dd^c H \circ f^n\right)^s \wedge S$$

is relatively compact. Hence, the integrals in (3) are $\lesssim d_+^{sn}$ and the mass of $(\overline{f}^n)_*S$ on D'' is $\lesssim d_+^{sn}$. This completes the proof. □

Remark 6.2. The restriction of $\mathscr{K}_+$, $\mathscr{K}_-$, T_+ and T_- to $D = B_p^R \times B_{k-p}^R$ coincide with the filled Julia sets and the Green currents constructed for $\overline{f}$. Note that in the context of horizontal-like maps, T_+ is not the unique positive closed (p, p)-current with support in $\mathscr{K}_+$. For the horseshoes, this current can be decomposed into currents of integration on vertical submanifolds of D.

Many questions have to be considered in the context of horizontal-like maps even when we assume that the condition on the dynamical degrees is satisfied. We refer to the paper by Dujardin [20] for the case of dimension 2, see also [2,12].

Question 6.3. Let f be an invertible horizontal-like map as above. Is the sequence $(d_s^+)_{0\leqslant s\leqslant p}$ of dynamical degrees of f increasing?

Question 6.4. Let f be an invertible horizontal-like map as above. Is the Green current T_+ laminar? More precisely, is it decomposable into currents of integration on complex manifolds, not necessarily closed, in D?

We refer to [8,10] for recent results on laminar currents in higher dimension. The following problems are also open for regular polynomial automorphisms.

Question 6.5. Is the equilibrium measure μ the intersection in the geometrical sense of T_+ and T_-? More precisely, is it possible to decompose T_+ and T_- into currents of integration on complex manifolds and to obtain μ as an average on the intersections of such manifolds?

Question 6.6. Are saddle periodic points equidistributed with respect to μ? It is not difficult to show that there are d^n periodic points of period n counted with multiplicities.

Question 6.7. Is the Hausdorff dimension of μ positive? Is there a relation between this dimension and the Lyapounov exponents of μ?

In the case of regular polynomial automorphisms, since $\mu = (dd^c G^+)^p \wedge (dd^c G^-)^{k-p}$ and G^+, G^- are Hölder continuous, μ gives no mass to sets of small Hausdorff dimension, see e.g. [35, Théorème 1.7.3].

We refer to Dupont [21], Ledrappier and Young [29] and the references therein for analogous problems in other contexts.

The dependence of Lyapounov exponents on the map can be studied following the works by Bassanelli and Berteloot [1] and Pham [33].

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