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A Compact Space with a Measure That Knows Which Sets Are Homeomorphic*

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We construct a compact homogeneous space bH which has a Borel measure $\bar{\bar{\mu}}$ which knows which sets are homeomorphic: if X and Y are homeomorphic Borel sets then $\bar{\bar{\mu}}(X) = \bar{\bar{\mu}}(Y)$, and, as a partical converse, if X and Y are open and $\bar{\bar{\mu}}(X) = \bar{\bar{\mu}}(Y)$ and X and Y are both compact or both noncompact, then X and Y are homeomorphic. In particular, $\bar{\bar{\mu}}$ is nonzero and invariant under all autohomeomorphisms; it turns out that up to a multiplicative constant $\bar{\bar{\mu}}$ is unique with respect to these properties. bH is constructed as an easy to visualize compactification of a very special subgroup H of the circle group $\mathbf{T}$; the Haar measure μ on $\mathbf{T}$ induces $\bar{\bar{\mu}}$ and also induces a measure $\bar{\mu}$ on H which knows which subsets of H are homeomorphic.

1. INTRODUCTION

Haar measure on a compact group has the nice property of being invariant under left and right translations, as well as under all topological isomorphisms, in particular under inversion, i.e., is invariant under all algebraically significant autohomeomorphisms. Also, Haar measure is unique (up to a multiplicative constant) with respect to being a nonzero Borel measure invariant under (e.g., left) translation [8, 15.8].

The restriction to algebraically significant autohomeomorphisms is essential here: It is easy to see that the circle group $\mathbf{T}$ has an autohomeomorphism under which Haar (= Lebesgue) measure is not invariant. So $\mathbf{T}$ has too many autohomeomorphisms for there to be a nonzero Borel measure which is invariant under all autohomeomorphisms.

There do exist infinite compact spaces which have a nonzero Borel measure invariant under all autohomeomorphisms: simply take one which is *rigid*, i.e., the only autohomeomorphism is the identity. But now the space has not enough autohomeomorphisms for the measure to be unique.

In this paper we construct a compact space which has a unique nonzero Borel measure invariant under all autohomeomorphisms, using only methods

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of classical general topology. In fact, our measure is much better than invariant: it knows which sets are homeomorphic. Apparently our example is so counterintuitive that nobody has asked if something like it can exist.

1.1. EXAMPLE. There exists an infinite compact zero-dimensional homogeneous space bH, which is separable and linearly orderable, and there exists a Borel measure $\bar{\bar{\mu}}$ on bH such that

(1) if X and Y are homeomorphic Borel sets, then $\bar{\bar{\mu}}(X)=\bar{\bar{\mu}}(Y)$;

(2) if X and Y are open subsets of bH, both compact or both noncompact, then

(a) if $\bar{\bar{\mu}}(X)=\bar{\bar{\mu}}(Y)$ then X and Y are homeomorphic, and

(b) if $\bar{\bar{\mu}}(X)\leqslant\bar{\bar{\mu}}(Y)$ then X can be embedded in Y as an open subspace;

(3) up to a multiplicative constant $\bar{\bar{\mu}}$ is the only nonzero Borel measure on bH which is invariant under all autohomeomorphisms.

1.2. *Remarks.* (a) bH cannot have too few autohomeomorphisms because of (3). In fact, since bH is homogeneous it has quite a few autohomeomorphisms.

(b) The fact that bH is a separable linearly orderable space implies that every open set is an F_σ, in particular that the classes of Borel sets and Baire sets coincide. Hence $\bar{\bar{\mu}}$ is regular [7, p. 228].

(c) In Theorem 7.7 we will conclude from (1) and (2) that two open subsets of bH are homeomorphic iff each can be embedded in the other. [Shades of Schröder–Bernstein!]

We construct bH as a compactification of a very special subgroup H [This explains the notation.] of the circle group $\mathbf{T}$. A pleasant feature of bH is that it is easy to visualize. The Lebesgue (=Haar) measure μ on $\mathbf{T}$ induces a Borel measure $\bar{\mu}$ on H which in turn induces $\bar{\bar{\mu}}$. Also, $\bar{\mu}$ acts on H the same way $\bar{\bar{\mu}}$ acts on bH. So H is a very unusual separable metrizable zero-dimensional space.

We now describe what sort of subgroup we need.

1.3. DEFINITION. Let M be a space with a Borel measure μ.

(a) The function f will be called a *compression* of M if f is a homeomorphism with $\mathrm{dom}(f)$ and $\mathrm{range}(f)$ compact subsets of M satisfying $\mu(\mathrm{dom}(f)) > \mu(\mathrm{range}(f))$.

(b) The subspace S of M will be called a *stiff* subset of M if for every compression f of M there is an $x\in S\cap\mathrm{dom}(f)$ such that $f(x)\notin S$. [Every compression pushes some element of S out of S.]

1.4. DEFINITION. If G is a group, then the subset S of G will be called *tiny* if for every countable subset C of G the set $\{x \in G: S \cap (x + C) = \emptyset\}$ is dense in G. [Every countable set can be pushed out of S.]

1.5. DEFINITION. The subset S of the space X will be called *fat* if S intersects every uncountable compact subset of X.

The essential feature of H is that it is a stiff subgroup of T; for technical reasons we also require H to be tiny and fat. We show in Section 4 that **T**, as well as some other groups, to be discussed in Section 3, has a stiff subgroup. This is far from easy, and it should be emphasized that we do NOT use additional axioms like CH. Making the subgroup in addition tiny and fat is easy.

The original motivation for constructing bH has nothing to do with topological measure theory, but is the following question of Monk and Rubin about Boolean algebras, which was raised in an early version of [3]:

(1) Let B be an infinite Boolean algebra. If the Stone space of B is homogeneous (in the topological sense), must B be homogeneous (in the Boolean algebraic sense)?
[A Boolean algebra B is called *homogeneous* if it is isomorphic to $B \restriction b = \{a \in B: a \leqslant b\}$ for every $b \in B\text{-}\{0\}$.] In topological language this becomes:

(2) Let X be an infinite compact zero-dimensional space. If X is homogeneous, is X homeomorphic to every nonempty clopen subspace?
Our example bH answers this question in the negative in a very strong way as it is *incompressible* [=not homeomorphic to any proper subspace]. [In Boolean algebraic terms, the dual algebra B of bH is *Hopfian*, i.e., not isomorphic to any proper quotient. In fact B has a finitely additive measure λ such that for all $a, b \in B$

$$B \restriction a \text{ and } B \restriction b \text{ are isomorphic iff } \lambda(a) = \lambda(b).]$$

We now briefly discuss the converse of (2), i.e.:

(3) Let X be an (infinite) compact zero-dimensional space. If X is homeomorphic to every nonempty clopen subspace, must X be homogeneous?
The Čech–Stone compactification of the rationals gives an easy counterexample. It is amusing to note, however, that under the additional condition that X is first countable the answer is yes, even without compactness, but that bH is a first countable counter example to (2).

2. Preliminaries

Spaces and groups. *Space* means Hausdorff topological space, and *group* means topological group. We always use μ to denote Haar measure on a locally compact Abelian group, and *always* use $+$ to denote the operation on an Abelian group. **R** denotes the reals, **T** denotes the circle group. If λ is a measure, then as usual λ^* and λ_* denote the outer and inner measure, respectively.

Clopen means closed-and-open, and a space is called *zero-dimensional* if the family of its clopen sets is a base. A space X is called *homogeneous* if for every two $x, y \in X$ there is an *autohomeomorphism* [=homeomorphism of X onto itself] which sends x to y. A space is called *linearly orderable* if some linear order on X induces the topology on X the same way the topology of **R** or $[0, 1]$ is induced by $<$.

The following classical result of Lavrentieff ([15], see, e.g., [6, Theorem 4.3.21] for a proof in English) is important in our proof that homeomorphic Borel subsets of H or bH have the same measure.

2.1. Lavrentieff's Lemma. *Let X and Y be completely metrizable spaces, let $A \subseteq X$ and $B \subseteq Y$. If there is a homeomorphism $h: A \to B$ then there are G_δ-subsets $\tilde{A}$ of X and $\tilde{B}$ of Y with $A \subseteq \tilde{A} \subseteq \bar{A}$ and $B \subseteq \tilde{B} \subseteq \bar{B}$ such that h can be extended to a homeomorphism $\tilde{h}: \tilde{A} \to \tilde{B}$.*

Set theory. We follow the usual conventions. A cardinal is an initial ordinal, and an ordinal is the set of smaller ordinals. ω denotes ω_0, and will be thought of as the set of nonnegative integers, while **Z** denotes the set of all integers. If $2 \leqslant n < \omega$, then we think of $n = \{0, \ldots, n-1\}$ as a group, under addition modulo n; for clarity we denote this group by $\mathbf{n}$.

The domain and range of a function f are denoted by $\mathrm{dom}(f)$ and $\mathrm{range}(f)$. We use $f \restriction A$, $f^{\rightarrow}A$ and $f^{\leftarrow}A$ to denote the restriction of f to A and the image or inverse image of A under f, and f^{-1} to denote the inverse map of a bijection f. We emphasize that when we write $f \restriction A$ then it is NOT tacitly understood that $A \subseteq \mathrm{dom}(f)$; put differently, we use $f \restriction A$ to abbreviate $f \restriction (A \cap \mathrm{dom}(f))$. Similar remarks apply to $f^{\rightarrow}A$ and $f^{\leftarrow}A$.

The set of functions $A \to B$ is as usual denoted by ${}^{A}B$, and ${}^{<\omega}B$ denotes $\bigcup_{n<\omega} {}^{n}B$, the set of all finite B-valued sequences. Note that if $2 \leqslant n < \omega$ then ${}^{\omega}\mathbf{n}$, when given the product topology, is a topological group under coordinatewise operations.

We use "countable" for "at most countable."

3. Nice Groups

We are able to construct a stiff subgroup of a group G if $G = {}^{\omega}\mathbf{p}$ for some prime p or if G is nice.

3.1. Definition. Let G be a locally compact Abelian group. For $k \neq 0$ define $h_k: G \to G$ by

$$h_k(x) = k \cdot x.$$

G will be called *nonshrinking* if for all $k \neq 0$ and compact $K \subseteq G$ we have

$$\mu(h_k^{\to} K) \geqslant \mu(K).$$

G will be called *nice* if it is nondiscrete, nonshrinking, second countable and in addition satisfies

$$|h_k^{\leftarrow}\{0\}| \leqslant \omega \qquad \text{for } k \neq 0. \tag{*}$$

3.2. Proposition. *$\mathbf{R}$ and $\mathbf{T}$ are nice.*

▌ It suffices to check that $\mathbf{R}$ and $\mathbf{T}$ are nonshrinking. For $\mathbf{R}$ this is easy, and in fact

$$\mu(h_k^{\to} B) = |k| \cdot \mu(B) \qquad (B \subseteq \mathbf{R} \text{ Borel}). \tag{1}$$

Let $k \neq 0$. Let $\mathscr{A}$ be a disjoint collection consisting of $|k|$ disjoint half-open arcs in $\mathbf{T}$ each of length $2\pi/|k|$, so $\bigcup \mathscr{A} = \mathbf{T}$. Given a compact (or even Borel) $K \subseteq \mathbf{T}$ there is $A \in \mathscr{A}$ with $\mu(A \cap K) \geqslant \mu(K)/|k|$. But from (1) we see $\mu(h_k^{\to}(A \cap K)) = |k| \cdot \mu(A \cap K)$. ▌

We have no characterization of nonshrinking groups or nice groups. The following observations serve to give some insight in these two classes.

3.3. Proposition. (a) *There is a nondiscrete locally compact Abelian group such that $|h_k^{\leftarrow}\{0\}| = \omega$ for all $k \neq 0$.*

(b) *If G is a compactly generated locally compact Abelian group, then $|h_k^{\leftarrow}\{0\}| \neq \omega$ for all $k \neq 0$.*

▌ (a) Let $G = H \times \mathbf{R}$, where H is a suitable countable discrete group.

(b) G is topologically isomorphic to $R^k \times Z^n \times F$ for integers k, n and a compact Abelian group F [8, 9.8]. But no compact group is countably infinite. [This proof was suggested to me by Comfort and Reid, who also independently noted that (a) holds.] ▌

3.4. PROPOSITION. *Let G be a compact Abelian group.*

(a) *If G is nonshrinking, then G is divisible.*

(b) *If G is torsion-free and divisible, then G is nonshrinking.*

∎ (b) Each h_k $(k \neq 0)$ will be both an autohomeomorphism and an isomorphism, so it suffices to recall that Haar measure is unique (up to a multiplicative constant) [8, 15.8]. ∎

3.5. *Remarks.* Compactness is essential for (a): consider $\mathbf{2} \times \mathbf{R}$. Also, (b) is a very limited result: There is a group Σ such that a nontrivial compact Abelian group is torsion-free and divisible iff it is topologically isomorphic to some power of Σ. This follows easily from the characterization of compact torsion-free Abelian groups [8, 25.8].

4. CONSTRUCTION OF STIFF SUBGROUPS

Throughout this section G and Z' are defined as follows:

> either G is a nice group and Z' is the set of all nonzero integers (including the negative ones) or $G = {}^{\omega}\mathbf{p}$ for some prime p, and $Z' = \{1, \ldots, p-1\}$.

Also, if $A \subseteq G$ then we define «A» by

$$\text{«}A\text{» is the subgroup of } G \text{ generated by } A.$$

Then the following statements hold:

(A) if F is a subgroup of G and $x \in G$, then

$$\text{«}F \cup \{x\}\text{»} = F \cup (F + \{k \cdot x : k \in Z'\});$$

(B) for all $P \subseteq G$, if $|P| < \mathfrak{c}$ then

$$|\{x \in G : \exists k \in Z' \; \exists p \in P (k \cdot x = p)\}| < \mathfrak{c};$$

(C) for every compact $A \subseteq G$ and $k \in Z'$ we have

$$\mu(\{k \cdot x : x \in A\}) \geqslant \mu(A).$$

The actual construction. Since G is second countable, it has at most $\mathfrak{c}$ compact subsets, and for each nonempty $A \subseteq G$ there are at most $\mathfrak{c}$ continuous functions $A \to G$ ["at most" is redundant]. This enables us to list all compressions [there is at least one, see Remark 4.3] as $\langle f_\alpha : \alpha < \mathfrak{c} \rangle$, and we can list all uncountable compact subsets [there is at least one] as

$\langle K_\alpha : \alpha < \mathfrak{c}\rangle$. Let $\mathscr{B}$ be a countable base for G; then since $|G| = \mathfrak{c}$ we can list $\mathscr{B} \times \{E \subseteq G: |E| \leqslant \omega\}$ as $\langle\langle B_\alpha, E_\alpha\rangle : \alpha < \mathfrak{c}\rangle$.

With transfinite recursion we will pick $x_\alpha, y_\alpha, z_\alpha \in G$ for $\alpha < \mathfrak{c}$ such that if

$$H_\alpha \text{ denotes } «\textstyle\bigcup_{\xi<\alpha}\{x_\xi, y_\xi\}» \qquad (\alpha \leqslant \mathfrak{c}),$$

then

$$x_\xi \in \operatorname{dom}(f_\xi), \qquad \text{but } f_\xi(x_\xi) \notin H_\alpha \ \ (\xi < \alpha \leqslant \mathfrak{c}); \tag{1}$$

$$y_\xi \in K_\xi \qquad (\xi < \mathfrak{c}); \tag{2}$$

$$H_\alpha \cap (z_\xi + E_\xi) = \varnothing \qquad \text{and} \qquad z_\xi \in B_\xi \qquad (\xi < \alpha \leqslant \mathfrak{c}). \tag{3}$$

Then $H_\mathfrak{c}$ will be a subgroup of G which is stiff because of (1), is fat because of (2) and which is tiny because of (3).

We observe that

> if (1) and (3) hold for $\alpha = \gamma + 1$ for each $\gamma < \beta$, then (1) and (3) hold for $\alpha = \beta$ ($\beta \leqslant \mathfrak{c}$).

In other words, if $\beta \leqslant \mathfrak{c}$, and we have constructed x_α, y_α and z_α for $\alpha < \beta$, then we can assume (1) and (3) hold for $\alpha = \beta$.

Now let $\beta < \mathfrak{c}$, and assume we have found x_α, y_α and z_α for $\alpha < \beta$. We have to pick $x_\beta, y_\beta, z_\beta$ such that (1) and (3) hold for $\alpha = \beta + 1$ (and $y_\beta \in K_\beta$). The difficulty is to show that we can pick x_β. So we first assume x_β has been picked, and perform the easy task of picking y_β and z_β.

Picking y_β. Define

$$H' = «H_\beta \cup \{x_\beta\}», \quad \text{and} \quad F' = \{f_\xi(x_\xi) : \xi \leqslant \alpha\} \cup \bigcup_{\xi<\alpha} (z_\xi + E_\xi).$$

We will have picked x_β in such a way that

$$H' \cap F' = \varnothing. \tag{4}$$

We want to show that there is a $y \in K_\beta$ such that

$$«H' \cup \{y\}» \cap F' = \varnothing.$$

As $«H' \cup \{y\}» = H' \cup (H' + \{k \cdot y: k \in Z'\})$, (4) reduces our task to finding $y \in K_\beta$ with

$$(H' + \{k \cdot y: k \in Z'\}) \cap F' = \varnothing.$$

As $|H'| \cdot |F'| < \mathfrak{c}$, it follows from (B) that this equality fails to hold for less than $\mathfrak{c}$ elements y of G. Since $|K_\beta| = \mathfrak{c}$, this shows that we can pick y_β as desired.

Picking z_β. Define

$$H'' = \langle\!\langle H_\beta \cup \{x_\beta, y_\beta\} \rangle\!\rangle.$$

There are at most $|H''| \cdot |E_\beta| < \mathfrak{c}$ elements z of B_β for which

$$H'' \cap (z + E_\beta) \neq \varnothing.$$

Hence there is no difficulty picking $z_\beta \in B_\beta$ such that $H'' \cap (z_\beta + E_\beta) = \varnothing$.

We now come to the interesting part of the construction:

Picking x_β. Define

$$F = \{f_\alpha(x_\alpha) : \alpha < \beta\} \cup \bigcup_{\alpha < \beta} (z_\alpha + E_\alpha).$$

Our inductive hypothesis is that

$$H_\beta \cap F = \varnothing, \tag{5}$$

and our aim is to show that there is $x \in \operatorname{dom}(f)$ such that

$$\langle\!\langle H_\beta \cup \{x\} \rangle\!\rangle \cap (F \cup \{f_\beta(x)\}) = \varnothing. \tag{6}$$

Because of (A) and (5) this is equivalent to the conjunction of

$$f_\beta(x) \notin H_\beta; \tag{7}$$

$$(H_\beta + \{k \cdot x : k \in Z'\}) \cap F = \varnothing; \tag{8}$$

and

$$f_\beta(x) \notin H_\beta + \{k \cdot x : k \in Z'\}. \tag{9}$$

Now G has less than $\mathfrak{c}$ elements x for which (7) or (8) fails to be true. [Use the fact that f_β is an injection for (7) and use (B) for (8).] Therefore we can pick our x once we show that $\operatorname{dom}(f)$ has $\mathfrak{c}$ elements x for which (9) is true. This is the heart of the construction, and we state it as a separate Lemma so as to give the reader time for a break. ∎

4.1. Lemma. *Let f be a compression of G, let D denote* $\operatorname{dom}(f)$, *and let $Y \subseteq G$ have $|Y| < \mathfrak{c}$. Define*

$$S = \{x \in D : f(x) \in \{k \cdot x : k \in Z'\} + Y\}.$$

Then S is small, i.e., $|D - S| = \mathfrak{c}$.

▌ For $k \in Z'$ define

$$S_k = \{x \in D : f(x) \in k \cdot x + Y\},$$

then we can define a continuous

$$s_k : S_k \to Y \qquad \text{by} \qquad s_k(x) = f(x) - k \cdot x.$$

For $C \subseteq Y$ define

$$C^{\leftarrow} = \bigcup_{k \in Z'} s_k^{\leftarrow} C.$$

Note that $Y^{\leftarrow} = S$.

Claim 1. If $C \subseteq Y$ is countable, then $D - C^{\leftarrow}$ is uncountable.

Proof of Claim. We first point out that for all $k, l \in Z'$ and $y, z \in Y$,

$$\text{if } k \neq l \text{ or } y \neq z \text{ then } |s_k^{\leftarrow}\{y\} \cap s_l^{\leftarrow}\{z\}| \leqslant \omega. \tag{1}$$

Indeed, since we may assume $k - l \in Z' \cup \{0\}$ [recall that Z' could be $\{1, ..., p-1\}$], this follows from (B) and the inclusion

$$s_k^{\leftarrow}\{y\} \cap s_l^{\leftarrow}\{z\} \subseteq \{x \in G : (k-l) \cdot x = z - y\}.$$

We next observe that for all $k \in Z'$ and $y \in Y$ we have

$$f^{\rightarrow}(s_k^{\leftarrow}\{y\}) = y + \{k \cdot x : x \in s_k^{\leftarrow}\{y\}\},$$

for if $s_k(x) = y$ then $f(x) = k \cdot x + y$. It follows from (C) that

$$\mu(f^{\rightarrow}(s_k^{\leftarrow}\{y\})) \geqslant \mu(s_k^{\leftarrow}\{y\}) \qquad (k \in Z', y \in Y). \tag{2}$$

Note that the sets involved in (2) are measurable, and in fact compact, since $s_k^{\leftarrow}\{y\} = \{x \in D : f(x) = k \cdot x + y\}$ is closed in the compact set D; this we will also use below. [$D = \operatorname{dom}(f)$ is compact by Definition 1.3a.]

Since f is an injection, we see from (1) and (2) that if $C \subseteq Y$ is countable, and if in $\sum_{k,y}$ and $\bigcup_{k,y}$ we assume $k \in Z'$ and $y \in C$, then

$$\begin{aligned}
\mu(C^{\leftarrow}) &= \mu\left(\bigcup_{k,y} s_k^{\leftarrow}\{y\}\right) = \sum_{k,y} \mu(s_k^{\leftarrow}\{y\}) \\
&\leqslant \sum_{k,y} \mu(f^{\rightarrow}(s_k^{\leftarrow}\{y\})) = \mu\left(\bigcup_{k,y} f^{\rightarrow}(s_k^{\leftarrow}\{y\})\right) \\
&= \mu(f^{\leftarrow} C^{\leftarrow}) \leqslant \mu(f^{\rightarrow} D).
\end{aligned}$$

But f is a compression, hence $\mu(D) > \mu(f^{\to}D)$, so that $\mu(D) > \mu(C^{\leftarrow})$. This proves the Claim.

Claim 2. There is a $K \subseteq D$ with $|K| = \mathfrak{c}$ such that $s_k|K$ is an injection for each $k \in K'$.

Once this Claim has been proved, the Lemma follows from the facts that $|Y| < \mathfrak{c}$ and $S = Y^{\leftarrow}$.

The proof of Claim 2 does not require group theoretic or measure theoretic information, and does not depend on the way the s_k's are defined from f (except for the fact that point-inverses are closed in D). Therefore we state the precise statement we use as a separate Lemma, so that the reader gets time for another break. ▮

4.2. LEMMA. *Let M be a separable completely metrizable space. Let Y be a (Hausdorff) space, and let $\mathscr{F}$ be a family of countably many continuous functions such that for each $f \in \mathscr{F}$ we have*

$$\mathrm{dom}(f) \subseteq M, \qquad \mathrm{range}(f) \subseteq Y, \qquad \text{and} \qquad f^{\leftarrow}\{y\} \text{ is closed in } M \qquad (y \in Y).$$

For $C \subseteq Y$ define

$$C^{\leftarrow} = \bigcup_{f \in \mathscr{F}} f^{\leftarrow}C.$$

If there is no countable $C \subseteq Y$ such that $M - C^{\leftarrow}$ is countable, then there is $K \subseteq M$ with $|K| = \mathfrak{c}$ such that $f \restriction K$ is injective for each $f \in \mathscr{F}$.

[K will be homeomorphic to ${}^{\omega}\mathbf{2}$.]
[The special case $\mathscr{F}$ consists of one function f with $\mathrm{dom}(f) = M$ is due to Souslin, cf. [14, p. 437]. Our proof is a complexification of the proof of this special case.]

▮ We find it convenient to know that $\mathrm{id}_M \in \mathscr{F}$. To this end we replace Y by the topological sum of Y and M, and adjoin id_M to $\mathscr{F}$. From now on we work with this new Y and $\mathscr{F}$, and hence with a new set function ${}^{\leftarrow}$ defined as above, which clearly satisfies

$$\text{there is no countable } C \subseteq Y \text{ with } M - C^{\leftarrow} \text{ countable.} \tag{1}$$

For each $f \in \mathscr{F}$ the subspace $\mathrm{dom}(f)$ of M is hereditarily Lindelöf. Since $(\bigcup \mathscr{U})^{\leftarrow} = \bigcup_{U \in \mathscr{U}} U^{\leftarrow}$ for each collection $\mathscr{U}$ of subsets of Y, and since $\mathscr{F}$ is countable, it follows that there is a countable $L \subseteq Y$ such that

$$\bigcup \{\mathrm{Int}_{\mathrm{dom}(f)}(C^{\leftarrow} \cap \mathrm{dom}(f)) \colon C \subseteq Y \text{ countable}, f \in \mathscr{F}\} \subseteq L^{\leftarrow}. \tag{2}$$

Put

$$X = M - L^{\leftarrow}.$$

Claim. For each open U in X and $f \in \mathscr{F}$, if $U \cap \mathrm{dom}(f) \neq \varnothing$ then $|f^{\to} U| \geqslant 2$.

Proof of Claim. Suppose $|f^{\to} U| = 1$, or even $|f^{\to} U| \leqslant \omega$. Let V be open in M with $X \cap V = U$, and let $C = L \cup f^{\to} U$. Then $C^{\leftarrow} \supseteq L^{\leftarrow} \cup U$, hence $C^{\leftarrow} \supseteq V$ since $L^{\leftarrow} \cup X = M$. Since C is countable, it follows from (2) that $V \cap \mathrm{dom}(f) \subseteq L^{\leftarrow}$ hence that $U \cap \mathrm{dom}(f) = \varnothing$ since $U = V - L^{\leftarrow}$.

Since L and $\mathscr{F}$ are countable, and $f^{\leftarrow}\{y\}$ is closed in M for each $f \in \mathscr{F}$ and $y \in Y$, we see that X is a G_δ in M, hence is completely metrizable, as is well known [6, Theorem 4.3.23].

Enumerate $\mathscr{F}$ as $\langle f_n : n < \omega \rangle$ in such a way that each element of $\mathscr{F}$ is listed ω times. We claim that there is an indexed family $\langle U_s : s \in {}^{<\omega}2 \rangle$ of nonempty open subsets, such that the following holds for all $n < \omega$ (if we refer to some complete metric for X, and let $^{-}$ be the closure operator in X, and for $s \in {}^n2$ and $i \in 2$ let $s^\frown i$ be the concatenation of s and i, i.e., $s^\frown i = s \cup \{\langle n, i \rangle\}$):

(a)	$\mathrm{diam}(U_s) < 2^{-n}$	$(s \in {}^n2)$
(b)	$\bar{U}_{s^\frown i} \subset U_s$	$(s \in {}^n2, i \in 2)$
(c)	$U_s \cap U_t = \varnothing$	$(s, t \in {}^n2$ distinct$)$
(d)	$f_n^{\to} U_s \cap f_n^{\to} U_t = \varnothing$	$(s, t \in {}^n2$ distinct$)$.

Then clearly $K = \bigcap_n \bigcup \{U_s : s \in {}^n2\}$ will be as required. For the construction of $U_{s^\frown 0}$ and $U_{s^\frown 1}$ from U_s there are two cases to consider:

Case 1. $U_s \cap \mathrm{dom}(f_{n+1}) \neq 0$.
By the claim there are disjoint open V_0 and V_1 in $\mathrm{ran}(f_{n+1})$ which both intersect $f^{\to} U_s$. Pick $x_i \in f_{n+1}^{\leftarrow} V_i$, and then pick disjoint open U_{s0} containing x_0 and U_{s1} containing x_1 such that for $i < 2$

$$\mathrm{diam}(U_{s^\frown i}) < 2^{-n-1}, \quad \text{and} \quad \bar{U}_{s^\frown i} \subset U_s, \quad \text{and} \quad \mathrm{dom}(f_{n+1}) \cap U_{s^\frown i} \subseteq f_{n+1}^{\leftarrow} V_i.$$

Case 2. $U_s \cap \mathrm{dom}(f_{n+1}) = \varnothing$.
This is even easier. Note that we know that U_s is not a singleton by the claim, since $\mathrm{id}_M \in \mathscr{F}$. ∎

4.3. *Remark.* We point out that H will not include any uncountable closed subset of G. For let F be any uncountable closed subset of G. With a standard construction one finds copies K_0 and K_1 of the Cantor discontinuum ${}^\omega\mathbf{2}$ with

$$K_0 \subseteq F \text{ and } \mu(K_0) = 0, K_1 \subseteq G \text{ and } \mu(K_1) > 0.$$

Consider any homeomorphism $f: K_1 \to K_0$. Then f is a compression, hence $K_0 - H \neq \varnothing$, so that $F \nsubseteq H$.

4.4. *Remark.* We made H a fat subgroup since this is so easy to build in. Actually, we do not need this; it would be sufficient for our purposes to know that H is a *thick* subset of G, i.e., $\mu(K) = 0$ for every compact $K \subseteq G - H$, cf. [7, p. 74]. The same sort of argument as used in Remark 4.3 shows that every stiff subgroup is thick. But I don't find this esthetically pleasing.

It should be pointed out that under CH, the Continuum Hypothesis, a stiff tiny subgroup need not be fat. For let J be any set of measure zero with $0 \notin J$. We indicate how to change the construction of H and get $H \cap J = \varnothing$. Basically, all we do is ignore the y_α's, and have the additional inductive hypothesis

$$H_\alpha \cap J = \varnothing.$$

This does not affect picking the z_β's. In Lemma 4.1 we want to show that D-$(S \cup S') \neq \varnothing$, where

$$\begin{aligned} S' &= \{x \in G: (Y_\beta + \{k \cdot x: k \in Z'\}) \cap J \neq \varnothing\} \\ &= \{x \in G: \{k \cdot x: k \in Z'\} \cap (J + Y_\beta) \neq \varnothing\}. \end{aligned}$$

Since we assume CH we have $|Y| \leqslant \omega$, so from the proof of Claim 1 we see that $\mu(D - S) > 0$. Since $|Y| \leqslant \omega$ we have $\mu(J + Y) = 0$, so we know that D-$(S \cup S') \neq \varnothing$ provided

$$\text{if } M \subseteq G \text{ is Borel, and } \mu(M) = 0, \text{ and } A_k (k \in Z') \text{ is defined by}$$
$$A_k = \{x \in G: k \cdot x \in M\}, \text{ then } \mu(\textstyle\bigcup_{k \in Z'} A_k) = 0. \qquad (*)$$

Now each A_k is Borel since the function $x \to k \cdot x (x \in G)$ is continuous, and (C) implies the first inequality of

$$\mu(A_k) \leqslant \mu(\{k \cdot x: x \in A_k\}) \leqslant \mu(M),$$

while the second is trivial, hence (*) holds. So under CH we can construct a stiff tiny subgroup H of G which misses J. [In fact Martin's Axiom would also do. I didn't check what happens in ZFC.]

5. CONSTRUCTION OF bH

Note. From now on H will denote a stiff tiny fat subgroup of $\mathbf{T}$.

Our construction is motivated by the Alexandroff Double Arrow line A [1, Ex. A_7]. Recall that one can think of A as the space one gets from the closed unit interval $[0, 1]$ by splitting each point of $(0, 1)$ into a left and a right point; the new set gets a linear order (and hence a topology) in the natural way.

We split each $x \in \mathbf{T} - H$ into two points, x^- and x^+. The points of H will not be split, but for convenience each $x \in H$ gets x^- and x^+ as two additional names. For $B \subseteq \mathbf{T}$ we define

$$B^- = \{x^- : x \in B\}, \qquad B^+ = \{x^+ : x \in B\} \qquad \text{and} \qquad B^\pm = B^- \cup B^+.$$

So $bH = \mathbf{T}^\pm$ and $H = H^- = H^+ = H^\pm$.

The notion of counterclockwise for $\mathbf{T}$ induces in a natural way a notion of counterclockwise for bH, in which x^- precedes x^+ if x is split. For distinct $x, y \in bH$ we now can define $[x, y]$ and (x, y) to be the closed and open arc (self-explanatory) which runs counterclockwise from x to y; we will also use the self-explanatory notation $[x, y)$. Note that

$$[x^+, x^-] = bH \qquad \text{and} \qquad (x^-, x^+) = \varnothing \qquad \text{for } x \in \mathbf{T} - H.$$

We topologize bH by using the open arcs as a base.

Note. From now on bH will denote the space we just constructed.

Clearly bH is T_1. Since $\mathbf{T} - H$ is dense in $\mathbf{T}$, and since

$$(x^-, y^+) = [x^+, y^-] = bH - (y^-, x^+) \qquad (x, y \in \mathbf{T} - H),$$

it is easy to see that the family

$$\mathscr{A} = \{[x^+, y^-] : x, y \in \mathbf{T} - H\}$$

is a base for bH consisting of clopen arcs. Hence bH is zero-dimensional. Clearly bH is first countable, and H as a subspace of $\mathbf{T}$ coincides with H as a subspace of bH, and is dense in bH.

The reader is invited to test his or her understanding of bH by directly proving that bH is compact. [The fact that bH is a compactification of H explains the notation bH.]

One can define in a natural way a linear order on bH by moving counterclockwise starting at a^+ and finishing at a^-, for some $a \in \mathbf{T} - H$. This order clearly induces the topology of bH, so bH is linearly orderable. As this order is complete, we see again that bH is compact. Furthermore, every open subset is an F_σ. [We sketch the proof for the convenience of the nontopological reader: Every interval is an F_σ since bH is first countable. Every open set is the union of a disjoint collection of intervals; this collection is countable since H is a dense separable subspace.] As is well known and easy to prove (cf. [6, Ex. 3.8A]), it follows that every subspace is

Lindelöf. It also follows that the classes of Baire sets and Borel sets in bH coincide.

For later use, in Theorem 7.1, we make the following easy observation.

5.1. FACT. *$bH - H$ has no uncountable metrizable subspace.*

▮ As $bH - H = (\mathbf{T} - H)^+ \cup (\mathbf{T} - H)^-$, we have, by symmetry, only to prove that if $X \subseteq (\mathbf{T} - H)^+$ is uncountable, then it is not metrizable. Since X is Lindelöf, as just observed, it suffices to show that X is not second countable. The family

$$\{X \cap [s^+, t^-) : s, t \in \mathbf{T} - H\}$$

is a base for X, so if X were second countable, some countable subcollection also is a base, hence there is a countable $C \subseteq \mathbf{T} - H$ such that

$$\mathscr{B} = \{X \cap [s^+, t^-) : s, t \in C\}$$

is a base for X. Pick any $x \in X - C$, then it is easy to see that there is no $B \in \mathscr{B}$ with $x \in B \subseteq [x^+, (x + \pi)^-)$, a contradiction.

[The topological reader will have recognized this as the proof that the Sorgenfrey line S has no uncountable metrizable subspace. Indeed, $(\mathbf{T} - H)^+$ can be embedded in S.] ▮

We conclude this section by proving that bH is homogeneous. There is an obvious way to define the arc length $\bar{\mu}([a^+, b^-])$ of a clopen arc in bH. [Our notation anticipates the fact that $\bar{\mu}([a^+, b^-]) =$ arc length of $[a^+, b^-]$ for clopen arcs.] Clearly,

$$\text{if } [a^+, b^-] \text{ and } [c^+, d^-] \text{ are clopen arcs, then}$$
$$\mu([a^+, b^-]) = \mu([c^+, d^-]) \text{ iff } a - c = b - d.$$

5.2. FACT. *Every two clopen arcs $[a^+, b^-]$ and $[c^+, d^-]$ of bH which have equal arc lengths are homeomorphic. In fact, there is a homeomorphism $h: [a^+, b^-] \to [c^+, d^-]$ satisfying*

$$h^{\rightarrow}(H \cap [a^+, b^-]) = H \cap [c^+, d^-].$$

▮ Since H is a subgroup of $\mathbf{T}$ we can define for every $z \in H$ an autohomeomorphism r_z of bH, which we think of as a rotation, by

$$r_z(x^+) = (x + z)^+ \qquad \text{and} \qquad r_z(x^-) = (x + z)^- \qquad (x \in \mathbf{T}).$$

Clearly r_z preserves arc length. From the existence of the r_z's we see that the

Fact holds in the special case $[c^+, d^-]$ is a rotation of $[a^+, b^-]$, i.e., if there is $z \in H$ such that $\vec{r_z}[a^+, b^-] = [c^+, d^-]$, or, equivalently, if

$$b - a = d - c \qquad \text{and} \qquad c - a \in H.$$

So in order to prove the Fact it suffices to find decreasing neighborhood bases $\langle A_n \rangle_n$ for a^+ and $\langle D_n \rangle_n$ for d^-, consisting of clopen arcs such that

$$A_0 = [a^-, b^+] \qquad \text{and} \qquad D_0 = [c^-, d^+]; \tag{1}$$

$$\bar{\bar{\mu}}(A_n) = \bar{\bar{\mu}}(D_n); \tag{2}$$

and

$$D_n - D_{n+1} \text{ is a rotation of } A_n - A_{n+1}. \tag{3}$$

[Note that $D_n - D_{n+1}$ and $A_n - A_{n+1}$ must be clopen arcs.] In order to guarantee that $\langle A_n \rangle_n$ and $\langle D_n \rangle_n$ are neighborhood bases it suffices to require

$$\bar{\bar{\mu}}(A_{n+1}) \leqslant \tfrac{1}{2} \cdot \bar{\bar{\mu}}(A_n). \tag{4}$$

We show how to construct A_1 and D_1. Since H is dense in $\mathbf{T}$, there is $z \in H$ such that

$$\bar{\bar{\mu}}([(b+z)^+, d^-]) < \tfrac{1}{2} \cdot \bar{\bar{\mu}}([c^+, d^-]).$$

Put

$$A_1 = [a^+, (c-z)^-] \qquad \text{and} \qquad D_1 = [(b+z)^+, d^-].$$

Then $A_1 \subset A_0$, $D_1 \subset D_0$ and

$$A_0 - A_1 = [(c-z)^+, c^-] \qquad \text{and} \qquad D_1 - D_0 = [c^+, (b+z)^-].$$

So (3) and (4) hold for $n = 0$. Continuing this construction one finds the other A_n's and D_n's. ∎

The proof that bH is homogeneous is based on the same idea. Let $x, y \in bH$ be arbitrary. One has to find decreasing neighborhood bases $\langle X_n \rangle_n$ for x and $\langle Y_n \rangle_n$ for y which consist of clopen segments such that

$$\bar{\bar{\mu}}(X_n) = \bar{\bar{\mu}}(Y_n); \tag{5}$$

and

$$X_n - X_{n+1} \text{ and } Y_n - Y_{n+1} \text{ are clopen segments of the same length.} \tag{6}$$

We ensure that $\langle X_n \rangle_n$ and $\langle Y_n \rangle_n$ are neighborhood bases by requiring

$$\bar{\bar{\mu}}(X_{n+1}) < (\tfrac{2}{3}) \cdot \bar{\bar{\mu}}(X_n). \tag{7}$$

[We use $\frac{2}{3}$ instead of $\frac{1}{2}$ in order to avoid difficulties if x (or y) is the exact mid point of X_n or Y_n.] If, at stage n of the construction

$$X_n = [p^+, q^-] \qquad \text{and} \qquad Y_n = [r^+, s^-]$$

for certain p, q, r, s $\mathbf{T} - H$, then one chooses a suitable $z \in H$ such that one can put

$$X_{n+1} = [(p+z)^+, q^-] \qquad \text{or} \qquad X_{n+1} = [p^+, (q-z)^-],$$

and

$$Y_{n+1} = [(r+z)^+, s^-] \qquad \text{or} \qquad Y_{n+1} = [r^+, (s-z)^-].$$

[Note that $p+z, q-z, r+z$ and $s-z$ lie in $\mathbf{T} - H$.] The easy details are omitted.

5.3. *Remark.* It is amusing to note the following consequence of our proof that bH is homogeneous: if $x \in bH - H$ and $y \in H$ then there is an autohomeomorphism h of bH with $h(x) = y$ such that $h^{\leftarrow}H = H - \{y\}$, i.e., h pushes one element of $bH - H$ into H and pushes no element out. Compare with Fact 5.1.

6. Defining Measure on H and bH

Let μ be Haar (= Lebesgue) measure on $\mathbf{T}$, normalized so that

$$\mu(\mathbf{T}) = 2\pi,$$

then μ assigns the arc length to every arc in T.

Every uncountable Borel subset of $\mathbf{T}$, indeed, of every separable completely metrizable space includes a copy of the Cantor Discontinuum, cf. [14, p. 427]. As H is a fat stiff subgroup of $\mathbf{T}$, it follows that $\mathbf{T} - H$ does not include any uncountable Borel set of $\mathbf{T}$. From this it is easy to see that the following defines unambiguously a Borel measure $\bar{\mu}$ on H, cf. [7, p. 75]:

if $B \subseteq H$ is Borel in H, and $B' \subseteq \mathbf{T}$ is Borel in $\mathbf{T}$ with

$B = H \cap B'$, then $\bar{\mu}(B) = \mu(B')$.

Note that $\bar{\mu}$ is not inner regular with respect to compact sets, for H has no uncountable compact subsets by Remark 4.3, but $\bar{\mu}$ is inner regular with respect to closed sets and (dually) is outer regular. Therefore one can also calculate $\bar{\mu}$ using one of the following formulas:

$$\bar{\mu}(B) = \mu^*(B);$$

or

$$\bar{\mu}(B) = \sup\{\mu(\bar{F}) \colon F \text{ closed in } H, F \subseteq B\}. \qquad [\text{Closure in } \mathbf{T}.]$$

We define a Borel measure $\bar{\bar{\mu}}$ on bH by

$$\bar{\bar{\mu}}(B) = \bar{\mu}(H \cap B) \qquad (B \subseteq bH \text{ Borel}).$$

Since the classes of Baire sets and Borel sets in bH coincide, $\bar{\bar{\mu}}$ is a Baire measure. Since every Baire measure on a locally compact space is regular [1, p. 228], it follows that $\bar{\bar{\mu}}$ is regular.

There is an alternative way to construct $\bar{\bar{\mu}}$ directly from μ: Let $\pi \colon bH \to \mathbf{T}$ be the natural projection, i.e.,

$$\pi(x^+) = \pi(x^-) = x \qquad (x \in \mathbf{T}).$$

Then π is continuous. We have the following observation.

6.1. FACT. *The family of all Borel sets of bH is precisely the family*

$$\mathscr{B} = \{B \subseteq bH \colon \pi^{\to}B \text{ is Borel in } \mathbf{T} \text{ and } |(\pi^{\leftarrow}\pi^{\to}B) - B| \leqslant \omega\}.$$

▮ Denote the family of Borel sets of bH by $\mathscr{B}'$. Note that $\mathscr{B}$ is a σ-algebra. Since clearly every clopen interval belongs to $\mathscr{B}$, and since every open set in bH is the union of countably many clopen intervals, we have $\mathscr{B}' \subseteq \mathscr{B}$. Now $\mathscr{B}$ is the σ-algebra of subsets of bH generated by $\mathscr{B}'' \cup \{\{x\} \colon x \in bH\}$, where

$$\mathscr{B}'' = \{\pi^{\leftarrow}B \colon B \subseteq \mathbf{T} \text{ is Borel in } \mathbf{T}\}.$$

Clearly $\mathscr{B}'' \subseteq \mathscr{B}'$. But also $\{x\} \in \mathscr{B}'$ for all $x \in bH$ since bH is first countable. Hence $\mathscr{B} \subseteq \mathscr{B}'$. ▮

So one could also define a Borel measure $\bar{\bar{\mu}}$ on bH by

$$\bar{\bar{\mu}}(B) = \mu(\pi^{\to}B) \qquad (B \in \mathscr{B}),$$

and this measure is regular since μ is, for

$$\mathscr{B}' = \{B \subseteq bH \colon \text{there is a countable } C \subseteq B \text{ such that } \pi^{\to}(B - C) \text{ is Borel in } \mathbf{T} \text{ and } \pi^{\leftarrow}\pi^{\to}(B - C) = B - C\},$$

and $\pi^{\leftarrow}K$ is compact for all compact $K \subseteq \mathbf{T}$.

We leave it to the reader to verify that our two definitions of $\bar{\bar{\mu}}$ agree.

7. Why $\bar{\mu}$ and $\bar{\bar{\mu}}$ Know Which Sets Are Homeomorphic

The following theorem implies (1) of Example 1.1.

7.1. Theorem. (a) *If A and B are homeomorphic subspaces of H then*

$$\bar{\mu}_*(A) \leqslant \bar{\mu}^*(B).$$

(b) *If X and Y are homeomorphic subspaces of bH then $\bar{\bar{\mu}}(X)_* \leqslant \bar{\bar{\mu}}^*(Y)$.*

▌ We first prove (a). Assume there are subspaces A and B of H with $\bar{\mu}_*(A) > \bar{\mu}^*(B)$, yet there is a homeomorphism $h: A \to B$. Since μ is inner regular with respect to closed sets we may assume that A is closed in H, hence

$$A = H \cap \bar{A}, \qquad \text{so that } \mu(\bar{A}) = \bar{\mu}(A). \tag{1}$$

From the definition of $\bar{\mu}$ we see that $\mu^*(B) = \bar{\mu}^*(B)$, so we can pick a G_δ B' in $\mathbf{T}$ with

$$B \subseteq B' \qquad \text{and} \qquad \bar{\mu}^*(B) = \mu(B'). \tag{2}$$

From Lavrentieff's Lemma 2.1 we see that there are G_δ's $\tilde{A}$ and $\tilde{B}$ in $\mathbf{T}$ and a homeomorphism $\tilde{f}: \tilde{A} \to \tilde{B}$ such that

$$A \subseteq \tilde{A} \subseteq \bar{A}, \qquad \text{and} \qquad B \subseteq \tilde{B} \subseteq \bar{B}, \qquad \text{and } \tilde{f} \text{ extends } f. \tag{3}$$

Since $\tilde{f}^{\leftarrow}(\tilde{B} \cap B')$ is a G_δ in $\tilde{A}$, hence in $\mathbf{T}$, we see from (2) that we can make sure that

$$\mu(\tilde{B}) = \bar{\mu}^*(B). \tag{4}$$

From (1) and (3) we get $\mu(\tilde{A}) = \bar{\mu}(A) > \bar{\mu}^*(B)$, hence $\tilde{A}$ has a compact subset K with $\mu(K) > \mu^*(B)$. As $f^{\rightarrow}K \subseteq \tilde{B}$ we see from (4) that $\tilde{f} \restriction K$ is a compression. Since H is a stiff subgroup of $\mathbf{T}$ it follows that there is $x \in H \cap K$ with $\tilde{f}(x) \notin H$. But $H \cap K \subseteq H \cap \bar{A} = A$, hence this contradicts the fact that $\tilde{f}$ extends f.

We next prove (b). Let X and Y be subspaces of bH. Assume there is a homeomorphism $g: X \to Y$. Since $bH - H$ has no uncountable metrizable subset, by Fact 5.1, the sets

$$C_0 = \{x \in H \cap X: g(x) \notin H\} \qquad \text{and} \qquad C_1 = \{x \in H \cap Y: g^{-1}(x) \notin H\}$$

are countable, so if we put

$$A = H \cap X - C_0 \qquad \text{and} \qquad B = H \cap Y - C_1,$$

then $g^{\rightarrow}A = B$, hence A and B are homeomorphic, hence

$$\bar{\mu}_*(A) \leqslant \bar{\mu}^*(B).$$

Let $F \subseteq bH$ be closed and $U \subseteq bH$ be open when $F \subseteq X$ and $Y \subseteq U$. Since C_0 and C_1 are countable we have

$$\bar{\bar{\mu}}(F) = \bar{\bar{\mu}}(F - C_0) = \bar{\mu}((F - C_0) \cap H) \leqslant \mu_*(A),$$

and

$$\bar{\bar{\mu}}(U) = \bar{\mu}(H \cap U) \geqslant \bar{\mu}^*(H \cap Y) = \bar{\mu}^*(B),$$

hence $\bar{\bar{\mu}}_*(X) \leqslant \bar{\bar{\mu}}^*(Y)$, as required. ∎

7.2. Corollary. *Each of bH and H has ¢ pairwise nonhomeomorphic clopen subpaces and ¢ pairwise nonhomeomorphic dense open subspaces.*

∎ Fix $s \in \mathbf{T} - H$. Then $\{[s^+, t^-]: t \in \mathbf{T} - H\}$ is a family of ¢ clopen subspaces of bH no two of which have equal measure. Next, for each $m \in (0, 2\pi]$ one can find a dense open subset U with $\bar{\bar{\mu}}(U) = m$; this we leave to the reader. Moreover, bH does not have more than ¢ open sets, e.g., since the ¢ clopen arcs are a base, and bH is hereditarily Lindelöf, so that each open set is the union of countably many clopen arcs.

The proof for H is the same. ∎

In the following theorem a *rotation* of bH is an autohomeomorphism of the form $r_z (z \in H)$, defined by

$$r_z(x^+) = (x + z)^+, \qquad \text{and} \qquad r_z(x^-) = (x + z)^- \qquad (x \in \mathbf{T}).$$

7.3 Theorem. *Let λ be a Borel measure on bH such that*

> *$\lambda(bH) = 2\pi$ and $\lambda(A) = \lambda(r^{\rightarrow}A)$ for every clopen arc A in bH and every rotation r of bH.*

Then $\lambda = \bar{\bar{\mu}}$.

[Note that this is (formally) much stronger than (3) of Example 1.1.]

∎ Since Baire sets and Borel sets are the same in bH, λ is outer regular [7, p. 228], hence it suffices to prove that $\lambda(U) = \bar{\bar{\mu}}(U)$ for every open $U \subseteq bH$. Since every subspace of bH is Lindelöf, and the clopen arcs are a base, every open set U is the union of some countable family of clopen arcs; it is easy to transform this family into a disjoint family of clopen arcs whose union still is U. This reduces our task to proving

$$\text{if } A \text{ is a clopen arc then } \lambda(A) = \bar{\bar{\mu}}(A). \tag{1}$$

Let N be the *positive* integers. We first prove the following weak version of (1):

if A is a clopen arc, and $\bar{\mu}(A) = 2\pi/k$ for some $k \in N$, then $\lambda(A) = \bar{\mu}(A)$. (2)

Fix $k \in N$. Denote $\exp(2\pi i/k)$ by $\mathbf{k}^{-1}$ for short. Since every finite set can be pushed into $\mathbf{T} - H$ there is $s \in \mathbf{T}$ such that

$$s + j \cdot \mathbf{k}^{-1} \notin H \qquad \text{for } 0 \leqslant j < k.$$

As $s + k \cdot \mathbf{k}^{-1} = s$, we see that

$$\mathscr{A} = \{[(s + j \cdot \mathbf{k}^{-1})^+, (s + (j+1) \cdot \mathbf{k}^{-1})^-] : 0 \leqslant j < k\}$$

is a disjoint cover of bH by clopen arcs each of which is a rotation of $[s^+, (s + \mathbf{k}^{-1})^-]$. It follows that

$$\lambda([s^+, (s + \mathbf{k}^{-1})^-]) = 2\pi/k.$$

If A is any clopen arc in bH with $\bar{\mu}(A) = 2\pi/k$, then there is $t \in \mathbf{T} - H$ with $t + \mathbf{k}^{-1} \in \mathbf{T} - H$ such that $A = [t^+, (t + \mathbf{k}^{-1})^-]$. If $t - s \in H$ then A is a rotation of $[s^+, (s + \mathbf{k}^{-1})^-]$, so $\lambda(A) = 2\pi/k$. If $t - s \notin H$ then this argument won't work. However, from the proof that clopen arcs of the same length are homeomorphic we see that $[s^+, (s + \mathbf{k}^{-1})^-)$ is the union of a disjoint sequence $\langle C_n \rangle_n$ of clopen arcs and $(t^+, (t + \mathbf{k}^{-1})^-]$ is the union of a disjoint sequence $\langle D_n \rangle_n$ of clopen arcs such that C_n is a rotation of D_n for $n < \omega$. Therefore $\lambda(A) = 2\pi/k$. This completes the proof of (2).

Now let $[x^+, y^-]$ be any clopen arc, and let $k \in \mathbf{N}$ be arbitrary. There is an $n \in \mathbf{N}$ such that

$$x + i \cdot \mathbf{k}^{-1} \in [x, y]_{\mathbf{T}} \qquad \text{if } 0 \leqslant i < n \text{ but} \qquad x + n \cdot \mathbf{k}^{-1} \notin [x, y]_{\mathbf{T}},$$

where $[x, y]_{\mathbf{T}}$ denotes the arc in $\mathbf{T}$ that runs counterclockwise from x to y. Since H is a tiny subgroup of $\mathbf{T}$, there is $z \in \mathbf{T}$ such that

$$x + z + i \cdot \mathbf{k}^{-1} \notin H \quad (0 \leqslant i \leqslant n), \text{ and} \quad x + z + n \cdot \mathbf{k}^{-1} \in (y, x + n \cdot \mathbf{k}^{-1})_{\mathbf{T}}.$$

Then

$$\bigcup\{[(x + z + i \cdot \mathbf{k}^{-1})^+, (x + z + (i+1) \cdot \mathbf{k}^{-1})^-] : 1 \leqslant i \leqslant n - 2\}$$
$$\subseteq [x^+, y^-] \subseteq \bigcup\{[(x + z + i \cdot \mathbf{k}^{-1})^+, (x + z + (i+1) \cdot \mathbf{k}^{-1})^-] : 0 \leqslant i \leqslant n - 1\}.$$

It now follows from (2) that

$$|\lambda([x^+, y^-]) - \bar{\bar{\mu}}([x^+, y^-])| < 2 \cdot 2\pi/k,$$

hence $\lambda([x^+, y^-]) = \bar{\bar{\mu}}([x^+, y^-])$ since k was arbitrary. ∎

One can prove the following theorem in exactly the same way, using the second part of Fact 5.2.

7.4. THEOREM. *Let λ be an outer regular Borel measure on H such that $\lambda(H) = 2\pi$, and $\lambda(x + A) = \lambda(A)$ for every $x \in H$ and every clopen $A \subseteq H$ (or: every clopen arc A in H).*
Then $\lambda = \bar{\mu}$. ∎

In the proof that bH is homogeneous we showed that

every two clopen arcs in bH of the same length are homeomorphic. $(*)$

We now improve this as follows.

7.5. THEOREM. (a) *If X and Y are two clopen subspaces of bH then X and Y are homeomorphic iff $\bar{\bar{\mu}}(X) = \bar{\bar{\mu}}(Y)$.*

(b) *If X and Y are two noncompact open subspaces of bH then X and Y are homeomorphic iff $\bar{\bar{\mu}}(X) = \bar{\bar{\mu}}(Y)$.*

∎ In both cases necessity follows from Theorem 7.1. We only prove the sufficiency of (b); the proof of the sufficiency of (a) is similar (and easier).

Let X and Y be noncompact open subspaces of bH with $\bar{\bar{\mu}}(X) = \bar{\bar{\mu}}(Y)$. Since X and Y are Lindelöf, each is the union of some countable family of clopen arcs. Since X and Y are not compact, one can use this family to find pair-wise disjoint sequences $\langle [p_n^+, q_n^-]: n < \omega \rangle$ and $\langle [x_n^+, y_n^-]: n < \omega \rangle$ of (nonempty) clopen segments such that

$$X = \bigcup_n [p_n^+, q_n^-] \qquad \text{and} \qquad Y = \bigcup_n [x_n^+, y_n^-].$$

Define sequences $\langle r_n \rangle_n$ and $\langle z_n \rangle_n$ in $\mathbf{T}$ as follows:

$$r_0 = z_0 = 0, \quad \text{and} \quad r_{n+1} = r_n + q_n - p_n, \quad \text{and} \quad z_{n+1} = z_n + y_n - x_n.$$

Since every countable set can be pushed into $\mathbf{T} - H$ there is $a \in \mathbf{T}$ such that

$$a + r_n \notin H \qquad \text{and} \qquad a + z_n \notin H \qquad (n \in \omega).$$

Put

$$X' = \bigcup_n [(a + r_n)^+, (z + r_{n+1})^-], \quad \text{and} \quad Y' = \bigcup_n [(a + z_n)^+, (a + z_{n+1})^-].$$

From (*) we see that X' is homeomorphic to X and Y' is homeomorphic to Y'.

So it remains to prove that $X' = Y'$. Indeed, since the $[p_n^+, q_n^-]$'s and $[x_n^+, y_n^-]$'s are nonempty, there are $b, c \in \mathbf{T}$ such that

$$X' = [a^+, b^-) \qquad \text{and} \qquad Y' = [a^+, c^-).$$

As $\bar{\bar{\mu}}(X') = \bar{\bar{\mu}}(Y')$, we must have $b = c$. ∎

7.6. Corollary. *If X and Y are open subspaces of bH, then X and Y are homeomorphic iff each of X and Y can be embedded into the other.*

∎ Any continuous image of an open subset of bH must be Borel, being σ-compact, hence mutual embeddability implies $\bar{\bar{\mu}}(X) = \bar{\bar{\mu}}(Y)$, by Theorem 7.1, and implies that X and Y are either both compact or both noncompact. For if U is open but noncompact, then $\bar{\bar{\mu}}(K) < \bar{\bar{\mu}}(U)$ for every compact subset K of U. ∎

This is nontrivial because of Corollary 7.2.

7.7. Theorem. *If X and Y are open subspaces of H, then X and Y are homeomorphic iff $\bar{\mu}(X) = \bar{\mu}(Y)$.*

∎ Necessity follows from Theorem 7.1. Sufficiency is a mere corollary to the previous proof: Let X and Y be open subspaces of H with $\bar{\mu}(X) = \bar{\mu}(Y)$. There are open subspaces $X^{\cdot}$ and $Y^{\cdot}$ of bH with $X = H \cap X^{\cdot}$ and $Y = H \cap Y^{\cdot}$. Then

$$\bar{\bar{\mu}}(X^{\cdot}) = \bar{\bar{\mu}}(Y^{\cdot}).$$

We may assume that both $X^{\cdot}$ and $Y^{\cdot}$ are noncompact: if necessary one simply removes a point of $X^{\cdot} - H$ and $Y^{\cdot} - H$. As pointed out in Fact 5.2, if A and B are clopen segments of bH of the same length, then there is a homeomorphism $h: A \to B$ with $h^{\to}(H \cap A) = H \cap B$. This shows that when proving that $X^{\cdot}$ and $Y^{\cdot}$ are homeomorphic, one proves simultaneously that X and Y are homeomorphic. ∎

7.8. Corollary. *If X and Y are open subspaces of H, then X and Y are homeomorphic iff eeach of X and Y can be embedded as an open subspace into the other.*

∎ The same argument as Corollary 7.6. Note that we need the "as an open subspace" to ensure that the homeomorphs of X in Y and Y in X are Borel sets of H. [I did not investigate whether this can be avoided. The argument in Section 9 shows that an open subspace of bH can be homeomorphic to a nonmeasurable subspace.] ∎

The final results of this section deal with the values $\hat{\mu}$ and $\bar{\hat{\mu}}$ take on. We obtain two interesting corollaries.

7.9. THEOREM. *Let $\langle X, \hat{\mu} \rangle$ denote either $\langle bH, \mu \rangle$ or $\langle H, \mu \rangle$. If U is open in X, then for each $m \in [0, \hat{\mu}(U)]$ there is an open $V \subseteq U$ which is closed in U such that $\hat{\mu}(V) = m$.*

▮ We prove this for $\langle bH, \bar{\hat{\mu}} \rangle$. We have seen that every open set in bH is the union of a collection of disjoint clopen arcs, hence we only have to consider clopen arcs. So consider a clopen arc $[s^+, t^-]$ in bH, and let $m \in [0, \bar{\hat{\mu}}([s^+, t^-])]$. If $m = 0$ [or if $m = \bar{\hat{\mu}}([s^+, t^-])$] then we have nothing to do, so assume $m \neq 0$, and pick $p \in [s^+, t^-]$ such that $\mu([s^+, t)) = m$. Next, using the fact that H is tiny, choose $x \in \mathbf{T}$ such that

$$\{x + s, x + p, x + t\} \cap H = \varnothing.$$

Then $[(x+s)^+, (x+t)^-]$ is a clopen arc, homeomorphic to $[s^+, t^-]$ which has a clopen subset of measure m, namely, the clopen arc $[(x+s)^+, (x+p)^-]$. As homeomorphisms of one Borel subset of $\mathbf{T}$ onto another preserve measure, it follows that $[s^+, t^-]$ has a clopen subset of measure m. ▮

7.10. COROLLARY. *Let X and Y be open subspaces of bH, such that X is compact if Y is. Then the following are equivalent:*

(1) $\bar{\hat{\mu}}(X) \leqslant \bar{\hat{\mu}}(Y)$;

(2) *X can be embedded into Y; and*

(3) *X can be embedded into Y as a clopen subspace.*

▮ (3) → (2): Trivial.

(2) → (1): This follows from Theorem 7.1, since every homeomorph of X will be Borel, being σ-compact.

(1) → (3): Remove a subset from Y which is clopen in Y and has measure $\bar{\hat{\mu}}(Y) - \bar{\hat{\mu}}(X)$. Remove one more point if Y is compact but X is not. Now apply Theorem 7.5. ▮

7.11. COROLLARY. *Let X and Y be open subspaces of H. Then $\hat{\mu}(X) \leqslant \hat{\mu}(Y)$ iff X can be embedded as an open subspace in Y.*

▮ Same proof. [See the remark in the proof of Corollary 7.8.] ▮

7.12. *Remark.* Our proofs, with the exception of the proofs of Theorem 7.1 and Corollary 7.2, used the "geometry" of bH and H, i.e., the clopen arcs, in an essential way. I did not investigate whether or not the statements about H can be proved without using this geometry. [But see

Remark 7.13.] It is clear that we do have a similar geometry available for stiff fat tiny subgroups of **R**. Hence if S is such a group, all results about H also hold for S provided one reformulates Theorem 7.4 as follows: $\bar{\mu}$ is unique, up to a multiplicative constant, with respect to the property that $\bar{\mu}(x+A)=\bar{\mu}(A)$ ($x \in S$, and B a Borel set of S). In particular, we see that the following are equivalent for an open subspace U of S;

(1) $\bar{\mu}(U)=\infty$;

(2) *U and S are homeomorphic*; *and*

(3) *S is the union of a family of finitely many homeomorphs of U. Consequently in S the notions "of finite measure" and "of infinite measure" are purely topological.*

7.13. *Remark.* The statements in Corollary 7.2 and Theorem 7.9 about the clopen subsets of bH depend in an essential way on the geometry of bH, and so does the fact that open sets in bH are F_σ's. I convinced myself that for a prime $p \geqslant 3$, if G denotes a stiff fat subgroup of ${}^\omega\mathbf{p}$ then one can find a compactification cG of G (by splitting each point of ${}^\omega\mathbf{p}-G$ into p points) which has a Borel measure $\bar{\bar{\mu}}$ such that if $\bar{\bar{\mu}}(G)=1$ then

$$\{\bar{\bar{\mu}}(K): K \subseteq cG \text{ is clopen}\}=\{k \cdot p^{-n}: n<\omega, 0 \leqslant k \leqslant p^n\},$$

and such that the analogue of (1) holds. Also, cG is homogeneous. However, cG is not perfectly normal, i.e., has an open subset which is not an F_σ. This makes cG less interesting than bH.

Note that an amusing property of cG is that since it has no clopen subset of measure $\frac{1}{2}$, no clopen subset of cG is homeomorphic to its complement. I doubt that this curiosity warrants inclusion of the construction of cG and the verification of its properties.

8. On Homogeneous Zero-Dimensional Separable Metrizable Spaces

Let S be a stiff fat tiny subgroup of **R**. In this section we briefly compare S and H with each other and with more familiar nondiscrete homogeneous zero-dimensional separable metrizable spaces, namely, with

> **Q**, the rationals, or **P**, the irrationals (homeomorphic to ${}^\omega\omega$),
> or ${}^\omega\mathbf{2}$, the Cantor discontinuum, or $\omega \times {}^\omega\mathbf{2}$.

We use M to denote any of these four spaces.

We first differentiate H and S from the four M's.

M and $M \times M$ are homeomorphic [in two of the four cases M is homeomorphic to ${}^\omega M$], but H is not homeomorphic to $H \times H$ and S is not homeomorphic to $S \times S$. The first statement is easy to prove: H is not homeomorphic to any proper closed subspace, by Theorem 7.1. This argument doesn't work for S. If S and $S \times S$ were homeomorphic, then S would be the union of a family $\mathscr{K}$ consisting of ¢ pairwise disjoint *closed* (in S) copies of S, hence $\bar{\mu}(K) = \infty$ for all $K \in \mathscr{K}$. But then $\mu(\bar{K}) = \infty$ (closure in $\mathbf{R}$) for $K \in \mathscr{K}$, and $|\bar{K} \cap \bar{L}| \leqslant \omega$ for distinct $K, L \in \mathscr{K}$ since S is fat. This contradicts μ being σ-finite.

Up to homeomorphism M has one (if M is $\mathbf{P}$ or $\mathbf{Q}$) or two (if M is ${}^{\omega}\mathbf{2}$ or $\omega \times {}^{\omega}\mathbf{2}$) nonempty open subspaces. But both H and S have ¢ mutually nonhomeomorphic clopen subspaces by Corollary 7.2.

We next differentiate H and S.

M and S are homeomorphic to a proper clopen subspace (use Remark 7.12 for S), but H is not homeomorphic to a proper clopen subspace by Theorem 7.1. [In fact H is not homeomorphic to any nondense subspace of itself. Note that H is homeomorphic to a proper subspace, by Theorem 7.7. This suggests the question of whether there is a homogeneous zero-dimensional separable metrizable space that is not homeomorphic to *any* proper subspace. In Appendix 1 we show that this is not the case.]

This way of differentiating H and S used a property shared by all four M's. We conclude this section by twice differentiating H and S with the use of a property shared by some but not all of our M's.

If X is one of H, $\mathbf{P}$, $\mathbf{Q}$ or ${}^{\omega}\mathbf{2}$, and U is a nonempty open subset of X, then there are finitely many open homeomorphs of U in X which cover X. In fact there is a finite set F of autohomeomorphisms of X such that $\{f^{\rightarrow}U : f \in F\}$ covers X. [Compare with the notion of total boundedness for topological groups.] Indeed, if X is $\mathbf{P}$, $\mathbf{Q}$ or ${}^{\omega}\mathbf{2}$ choose a nonempty clopen V in X with $V \subseteq U$, and observe that there is an autohomeomorphism f of X with $f^{\rightarrow}V = X - V$; and if X is H choose a nonempty clopen arc in X which is included in U and has arc length $2\pi/k$ for some k. On the other hand, if Y is one of $\omega \times {}^{\omega}2$ and S, then Y has a nonempty open subset U such that there is no collection of finitely many open homeomorphs of U that covers Y. [Note that there is a countable such collection since Y is Lindelöf and homogeneous.] If $Y = \omega \times {}^{\omega}2$ let U be a nonempty compact open subset of Y, and if $Y = S$ let U be a nonempty open subset of Y of finite measure, see Remark 7.12.

For our second differentiation we simply observe that if X is one of S, $\mathbf{P}$, $\mathbf{Q}$ or $\omega \times {}^{\omega}\mathbf{2}$, then

> the following are equivalent for an open $U \subseteq X$: U is homeomorphic to X, and: there are finitely many open homeomorphs of U in X which cover X,

but this is not true if X is one of H and ${}^{\omega}\mathbf{2}$.

8.1. *Remark.* In [2] a compactification γX of a space X is called *topological* if every autohomeomorphism h of X admits a continuous extension $\gamma h: \gamma X \to \gamma X$; note that γh is an autohomeomorphism of γX since γh^{-1} also exists. Examples are βX and, for locally compact X, the one-point compactification αX.

It is shown in [2, Proposition 3] that if X is noncompact and strongly zero-dimensional, then βX is the only topological compactification of X provided

$$\text{every nonempty clopen subspace is homeomorphic to } X. \tag{1}$$

This applies to $\mathbf{P}$, and $\mathbf{Q}$, among others. The existence of rigid zero-dimensional metrizable spaces, see Section 13, shows that some condition like (1) is necessary, and suggests the question of whether (1) can be replaced by

$$X \text{ is homogeneous, Lindelöf and nowhere locally compact.} \tag{2}$$

Note that (1) implies that X is nowhere locally compact, and that $\omega \times {}^{\omega}2$ shows that one must have this condition since it has 2 (and no more, see [2, Ex. 2]) topological compactifications, namely, $\alpha(\omega \times {}^{\omega}2)$ and $\beta(\omega \times {}^{\omega}2)$. The condition that X be Lindelöf also is essential: if D is an uncountable discrete space, then $\alpha(D \times \beta\mathbf{Q})$ and $\beta(D \times \mathbf{Q}) = \beta(D \times \beta\mathbf{Q})$ are two topological compactifications.

We here use S to answer the question in the negative: a moment's reflection shows that if

$$Y = \bigcup \{\mathrm{Cl}_{\beta S} U: U \text{ is a clopen subset of } S \text{ with finite measure}\}$$

then αY is a topological compactification of S, but $\alpha Y \neq \beta S$ since $|\beta Y - Y| \geqslant 2$. The proof of the result from [2], quoted above, shows that S has no other topological compactifications, and that βH is the only topological compactification of H, for one can replace (1) by

$$\begin{array}{l}\text{for each nonempty clopen } U \subseteq X \text{ there is a finite set } F \text{ of} \\ \text{autohomeomorphism of } X \text{ such that } \bigcup_{f \in F} f^{\to} U = X.\end{array} \tag{3}$$

9. Doubling Destroys Measurability

Let μ be Lebesgue measure on $\mathbf{R}$. It is well known [7, p. 64] that if $t: \mathbf{R} \to \mathbf{R}$ is a linear transformation of the form

$$t(x) = a \cdot x \qquad (a \in \mathbf{R}\text{-}\{0\}),$$

then $t^{\to}A$ is Borel (measurable) if A is Borel (measurable), and

$$\mu(t^{\to}A) = |a| \cdot \mu(A). \tag{1}$$

If S is a subgroup of $\mathbf{R}$ then $t|S$ need not be a transformation $S \to S$ for all a, but it will be for $a \in \mathbf{Z}\text{-}\{0\}$. In this section we show that the analogone of (1) can dramatically fail.

Let S be a stiff fat tiny subgroup of $\mathbf{R}$, and use μ to define a Borel measure $\bar{\mu}$ on S. This measure is outer regular, and is translation and inversion invariant. Define

$$t: S \to S \qquad \text{by } t(x) = 2 \cdot x,$$

and let

$$I = [0, 1] \cap S.$$

Claim. $\bar{\mu}(I) = 1, \bar{\mu}^*(t^{\to}I) = 2 \qquad$ but $\bar{\mu}_*(t^{\to}I) = 0$.

That $\bar{\mu}(I) = 1$ follows from the definition of $\bar{\mu}$.

There is a G_δ-subset G of $\mathbf{R}$ with $t^{\to}I \subseteq G \subseteq [0, 2]$ and $\bar{\mu}^*(t^{\to}I) = \mu(G)$. Then $I \subseteq t^{\leftarrow}G \subseteq [0, 1]$, hence $[0, 1] - t^{\leftarrow}G$ is countable since S intersects every uncountable closed subset of $\mathbf{R}$. It follows that $\mu(t^{\leftarrow}G) = 1$, hence $\bar{\mu}^*(t^{\to}I) = \mu(G) = 2 \cdot \mu(t^{\leftarrow}G) = 2$.

Assume $\bar{\mu}_*(t^{\to}I) > 0$. Then there is a compact F in $\mathbf{R}$ with $\mu(F) > 0$ such that $F \cap S \subseteq t^{\to}I$, so $t^{\leftarrow}(F \cap S) \subseteq I \subseteq S$. But $t^{-1} \restriction F$ is a compression since $\mu(F) > 0$ and $\mu(t^{\leftarrow}F) = \frac{1}{2} \cdot \mu(F)$. So there is $x \in F \cap S$ with $t^{-1}(x) \notin S$, contradicting $t^{\leftarrow}(F \cap S) \subseteq S$. Hence $\bar{\mu}_*(I) = 0$.

10. H and bH Have $2^{\mathfrak{c}}$ Variants

For each stiff tiny fat subgroup H' of $\mathbf{T}$ the construction of Section 5 yields a compact zero-dimensional space bH which can act as Example 1.1. We now show how a simple modification of the construction of Section 4 yields a family $\mathscr{H}$ consisting of $2^{\mathfrak{c}}$ stiff tiny fat subgroups of $\mathbf{T}$ such that bH' and bH'' are nonhomeomorphic for every two distinct $H', H'' \in \mathscr{H}$, and such that the members of $\mathscr{H}$ are pairwise nonhomeomorphic.

Let H' and H'' be two stiff subgroups of $\mathbf{T}$ such that bH' and bH'' are homeomorphic. Let $h: bH' \to bH''$ be a homeomorphism. Since no uncountable subspace of $bH' - H''$ or $bH' - H''$ is metrizable by Fact 5.1, there must be countable $C' \subseteq H'$ and $C'' \subseteq H''$ such that $H' - C'$ and $H'' - C''$ are homeomorphic.

On the family $\mathscr{S}$ of all subspaces of $\mathbf{T}$ the relation $\approx$, defined by

$X \approx X'$ iff there are countable $C \subseteq X$ and $C' \subseteq X'$ such that $X - C$ and $X' - C'$ are homeomorphic,

is an equivalence relation. One easily checks that each equivalence class has cardinality (exactly) $\mathfrak{c}$. This reduces our task to finding $2^{\mathfrak{c}}$ stiff subgroups in $\mathbf{T}$. [Of course this does not give us any insight as to why our bH''s are pairwise nonhomeomorphic: from an intuitive point of view they are all the same.]

10.1. THEOREM. *Let G be a nice group, or let G be ${}^{\omega}\mathbf{p}$ for some prime p. Then G has $2^{\mathfrak{c}}$ stiff tiny fat subgroups.*

▮ We indicate the change that should be made in Section 4. Together with the x_α's, y_α's and z_α's we will construct certain w_α's. We redefine H_α as

$$H_\alpha = \langle\!\langle \bigcup_{\xi<\alpha} \{w_\alpha, x_\alpha, y_\alpha\} \rangle\!\rangle.$$

For $A \subseteq \mathfrak{c}$ we will define a stiff tiny fat subgroup $H(A)$ of G by

$$H(A) = \langle\!\langle \bigcup_{\xi<\mathfrak{c}} \{x_\xi, y_\xi\} \cup \{w_\alpha : \alpha \in A\} \rangle\!\rangle.$$

We make the $H(A)$'s pairwise distinct by requiring

$$\text{for all } \alpha < \mathfrak{c} \text{ and } A \subseteq \mathfrak{c} \text{ we have } w_\alpha \in H(A) \text{ iff } \alpha \in A.$$

The easiest way to build this in is to require that for all $\alpha < \mathfrak{c}$

$$I_\alpha = \bigcup_{\xi<\alpha} \{w_\alpha, x_\alpha, y_\alpha\}$$

is independent [i.e., there are no nontrivial inequalities of the form $c_1 u_1 + \cdots c_n u_n = 0$, with $u_i \in I_\alpha$ $(1 \leqslant i \leqslant n)$], and elements of I_α which have different names are distinct. To obtain this we make sure that when picking $w_\alpha, x_\alpha, y_\alpha$ and z_α (in this order) at stage α we have

$$\{k \cdot w_\alpha : k \in Z'\} \cap H_\alpha = \varnothing;$$

$$\{k \cdot x_\alpha : k \in Z'\} \cap \langle\!\langle H_\alpha \cup \{w_\alpha\} \rangle\!\rangle = \varnothing;$$

and

$$\{k \cdot y_\alpha : k \in Z'\} \cap \langle\!\langle H_\alpha \cup \{w_\alpha, x_\alpha\} \rangle\!\rangle = \varnothing.$$

Since this rules out less than $\mathfrak{c}$ candidates for w_α, x_α and y_α this causes no difficulties. ▮

10.2. *Remark.* Call two spaces X_0 and X_1 *totally different* if one cannot find nonempty open $U_i \subseteq X_i$ $(i < 2)$, such that U_0 and U_1 are homeomorphic. We observe that the relation $\sim$, defined by

> $X \sim X'$ if there are nonempty open $U \subseteq X$ and $U' \subseteq X'$ and countable $C \subseteq U$ and $C' \subseteq U'$ such that $U - C$ and $U' - C'$ are homeomorphic,

is an equivalence relation on the class of all homogeneous subspaces of $\mathbf{T}$, and that each equivalence class has cardinality $\mathfrak{c}$. Hence the argument preceding Theorem 10.1 shows that bH in fact has $2^{\mathfrak{c}}$ pairwise totally different variants.

10.3. *Remark.* In his thesis [17] Maurice constructs a family of ω_1 pairwise nonhomeomorphic (infinite) homogeneous compact linearly orderable spaces, two of which are separable, and asks if there are any other (infinite) homogeneous compact linearly orderable spaces [17, p. 9]. In [18] he constructs another family of ω_1 pairwise nonhomeomorphic (infinite) homogeneous compact linearly orderable spaces, one of which is separable.

Our results imply that there are in fact $2^{\mathfrak{c}}$ pairwise nonhomeomorphic separable homogeneous compact linearly orderable spaces. The number $2^{\mathfrak{c}}$ is best possible here, even if we omit "separable." For an infinite homogeneous compact linearly orderable space is first countable [17, p. 40], hence has cardinality (precisely) $\mathfrak{c}$ [17, p. 40], hence has weight at most $\mathfrak{c}$. But there are only $2^{\mathfrak{c}}$ compact spaces of weight at most $\mathfrak{c}$.

[It should be pointed out that we do not need stiff tiny fat subgroups of $\mathbf{T}$ for this. For our proof that bH is homogeneous only used the fact that H is a dense proper subgroup of $\mathbf{T}$ (such a group has dense complement), and $\mathbf{T}$ has $2^{\mathfrak{c}}$ dense proper subgroups.]

11. Questions

This paper suggests a couple of questions which are listed below so as to make clear what is left open. I have not seriously worked at them. I did not formulate the questions in such a way that I think that yes is the more likely answer.

11.1. Question. *Does there exist an infinite compact connected group such that*

(a) *every autohomeomorphism of G preserves Haar measure*; *or*

(b) *every autohomeomorphism of G that leaves the identity fixed is either an isomorphism or an anti-isomorphism*; *or*

(c) *if $*$ is any group operation on the underlying space of G which* *also makes G a topological group, and which has the same identity element, then either $x*y = xy$ $(x, y \in G)$ or $x*y = yx(x, y \in G)$.*

We are really interested in (a) of course; we mention (b) and (c) since (c) → (b) → (a). The reason for asking for a connected G is that in the opposite case there is no such G: For let G be an infinite compact zero-dimensional group. Then (the underlying space of) G is homeomorphic to ${}^{\kappa}\mathbf{2}$ for $\kappa = w(G)$, the weight of G [8, 9.15]. Every measure on ${}^{\kappa}\mathbf{2}$ that is invariant under all autohomeomorphisms of ${}^{\kappa}\mathbf{2}$ must be equal to the ordinary product measure, as can be seen from considering the measure of basic clopen sets. But the product measure on ${}^{\kappa}\mathbf{2}$ is not invariant under all autohomeomorphisms.

11.2. QUESTION. *Does every nondiscrete second countable locally compact abelian group have a stiff (tiny fat) subgroup?*

One could ask a similar question for groups G which are not necessarily Abelian. If it is compact one can use Definition 1.3, but if G is not compact, or, more precisely, if G is not unimodular, then one could consider left-compressions (self-explanatory), or one could call a function a compression if it is either a left-compression or a right-compression.

11.3. QUESTION. *Does every homogeneous zero-dimensional separable metrizable space admit the structure of a topological group?*[1]

Without zero-dimensional the answer is in the negative: the Hilbert cube is homogeneous [11], but has the fixed point property.

11.4. QUESTION. *Is there a satisfactory characterization of nonshrinking* (see Definition 3.1) *groups?*

12. APPENDIX 1: COMPRESSIBILITY

The following proposition was referred to in Section 8.

PROPOSITION. *If X is an infinite homogeneous zero-dimensional separable metrizable space, then X is not incompressible.*

▌ If X has an isolated point it must be discrete. So assume X has no isolated points. If X is compact it must be homeomorphic to the Cantor discontinuum. So assume X is not compact.

Let $\langle x_n : n < \omega \rangle$ list some closed discrete subset of X, with $x_m \neq x_n$

[1] After this paper was completed I found a counterexample.

whenever $m < n < \omega$. Let $\langle U_n : n < \omega \rangle$ list some discrete clopen family with $x_n \in U_n (n < \omega)$, and $U_m \cap U_n = \varnothing$ whenever $m < n < \omega$.

For $n < \omega$ let h_n be an autohomeomorphism of X with $h_n(x_n) = x_{n+1}$.

With recursion on k select clopen neighborhoods $B(n, k)$ of x_n for $n < k$ such that

$$B(k, k) \subseteq U_k;$$

$$B(n, k+1) \subseteq B(n, k);$$

$$\operatorname{diam}(B(n, k)) < 2^{-n};$$

and

$$h_n^{\to} B(n, k) = B(n+1, k) \qquad (n < k).$$

Now use the h_n's and $B(n, k)$'s to find a homeomorphism h of X onto $X - \{x_0\}$ which

> is the identity outside $\bigcup_n B(n, n)$, maps $B(n, n+1)$ onto $B(n+1, n+1)$, and maps $B(n, n) - B(n, n+1)$ onto $B(0, n) - B(0, n+1)$. ∎

13. APPENDIX 2: HISTORY

In our construction of H we used Lavrentieff's Lemma to construct a space with only "a few" autohomeomorphisms. This is not the first such use of Lavrentieff's Lemma, but earlier constructions aimed at getting a rigid space, i.e., a space with only one autohomeomorphism, the identity.

Kuratowski [12] was the first (as far as I know) to use Lavrentieff's Lemma to construct a rigid subspace of $\mathbf{R}$, in 1925. An unusual feature of his example is that it is meager (= first category), and that this fact is essential in the proof that it is rigid.

In 1932 A. Lindenbaum [16] claimed without proof that

> there is a family of $2^{\mathfrak{c}}$ subspaces of $\mathbf{R}$, none of which can be embedded into another, (1)

and

> there is a family of $2^{\mathfrak{c}}$ subspaces of $\mathbf{R}$, none of which is a continuous image of another. (2)

This was verified in 1947 by Kuratowski [13] and Sierpiński [19], who used Lavrentieff's Lemma and a similar lemma of Lavrentieff concerning extensions of continuous functions which are not necessarily homeomorphisms. Kuratowski proves that the same $\mathscr{F}$ can be used in (1) and (2); a minor change in his proof yields this result:

THEOREM. *There is a family* $\mathscr{F}$ *of* $2^{\mathfrak{c}}$ *subspaces of* $\mathbf{R}$ *such that*

(a) *if* $X, Y \in \mathscr{F}$ *are distinct, then* $|X - Y| = \mathfrak{c}$,

(b) *if* $X \in \mathscr{F}$, *and* $U \subseteq X$ *is open and nonempty, then* $|U| = \mathfrak{c}$,

(c) *if* $X, Y \in \mathscr{F}$ *(not necessarily distinct), if* $G \subseteq X$ *is a* G_δ, *and if* $f: G \to \mathbf{R}$ *is continuous and satisfies* $|f^{\to}G| = \mathfrak{c}$ *and* $G \cap f^{\to}G = \varnothing$, *then* $|(f^{\to}G) - Y)| = \mathfrak{c}$.

Note that this family witnesses (1) and (2), and has in addition the property that each member is (much more than) rigid, by (b). [Kuratowski states (c) only for $G = X$ and $X \neq Y$.] As it stands, the theorem is essentially due to de Groot [4], who proved it in 1959, unaware of the earlier papers.

The first explicit constructions of an infinite compact zero-dimensional rigid space were given by Jónsson [9] and Katětov [10] in 1951. Jónsson's example is quite big. Katětov's example is βX for some countable rigid space in which each point is the limit of a nontrivial convergent sequence in X; as X is Lindelöf, no point of $\beta X - X$ can be the limit of a nontrivial convergent sequence, as is well known, hence βX is rigid because X is. This argument shows that Kuratowski's 1925 paper implicitly contains an infinite compact zero-dimensional rigid space, for one can use his rigid space for X.

The first explicit construction of an infinite first countable compact zero-dimensional rigid space (that even is linearly orderable) was given by de Groot and Maurice [5] in 1968. They use a certain rigid dense subspace of the closed unit interval, and use a modification of the Alexandroff Double Arrow construction which inspired our construction of bH. Again, they could have started from Kuratowski's 1925 example.

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