Discrete Mathematics 55 (1985) 161–166
North-Holland

ON THE CHROMATIC NUMBER OF MULTIPLE INTERVAL GRAPHS AND OVERLAP GRAPHS

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Received 22 March 1984

Let $\chi(G)$ and $\omega(G)$ denote the chromatic number and clique number of a graph G. We prove that χ can be bounded by a function of ω for two well-known relatives of interval graphs. Multiple interval graphs (the intersection graphs of sets which can be written as the union of t closed intervals of a line) satisfy $\chi \leq 2t(\omega-1)$ for $\omega \geq 2$. Overlap graphs satisfy $\chi \leq 2^{\omega}\omega^2(\omega-1)$.

1. Introduction

Let $\chi(G)$ and $\omega(G)$ denote the chromatic number and clique number (maximum size of a clique) of a graph G. To avoid trivial cases, we always assume that $\omega(G) \geq 2$. It is well known that interval graphs are perfect, in particular $\chi(G) = \omega(G)$ for every interval graph G. In this paper we study the closeness of χ and ω for two well-known non-perfect relatives of interval graphs: multiple interval graphs and overlap graphs.

Multiple interval graphs are the intersection graphs of sets $A_1, A_2, \ldots, A_n$ such that for all i, $1 \leq i \leq n$, A_i is the union of closed intervals of the real line. If for all i, $1 \leq i \leq n$, A_i is the union of t closed intervals then we speak about t-interval graphs. Multiple interval graphs were introduced by Harary and Trotter in [9]. Relations betwen the packing number and transversal number of multiple intervals were studied in [6]. Obviously, 1-interval graphs are exactly the interval graphs. It is easy to see that 2-interval graphs (or double interval graphs) include another distinguished family of graphs, the circular arc graphs. *Circular-arc graphs* are the intersection graphs of closed arcs of a circle. It is straightforward that $\chi \leq 2\omega$ holds for circular-arc graphs. A conjecture of Tucker states that $\chi \leq \lfloor \frac{3}{2}\omega \rfloor$ for circular-arc graphs [10]. We shall prove that $\chi \leq 2t(\omega-1)$ holds for t-interval graphs (Theorem 1).

Overlap graphs are graphs whose vertices can be put into one-to-one correspondence with a collection of intervals on a line in such a way that two vertices are adjacent if and only if the corresponding intervals intersect but neither contains the other. Overlap graphs can be equivalently defined as intersection graphs of chords of a circle (see [5, Ch. 11.3]). We shall prove that $\chi \leq 2^{\omega}\omega^2(\omega-1)$ holds for overlap graphs (Theorem 2.).

It is worth to mention a generalization of interval graphs where χ cannot be bounded by any function of ω. Roberts introduced in [8] the *d-dimensional box graphs* as the intersection graphs of d-dimensional parallelopipeds having sides parallel to the coordinate axes. The one-dimensional box graphs are the interval graphs. For 2-dimensional box graphs $\chi \leq 4\omega^2 - 3\omega$ was proved by Asplund and Grünbaum in [1]. However, a surprising construction of Burling [2] shows that there are 3-dimensional box graphs with $\omega = 2$ and with an arbitrary large chromatic number.

The determination of the chromatic number of overlap graphs and of circular-arc graphs is an NP-complete problem as proved in [3]. Since circular-arc graphs are special 2-interval graphs, finding the chromatic number of t-interval graphs is also NP-complete for $t \geq 2$. Therefore it is justified to look for approximative polynomial algorithms for coloring overlap graphs and multiple interval graphs. The proof of Theorem 1 gives a polynomial algorithm for coloring a t-interval graph G with at most $2t(\omega - 1)$ colors by finding an indexing $x_1, x_2, \ldots, x_n$ of the vertices of G such that for all i, $1 \leq i \leq n$, $|\{x_j\colon i < j, (x_i, x_j) \in E(G)\}| < 2t(\omega - 1)$. The performance ratio of this algorithm is constant for fixed t. The situation is much worse when the proof of Theorem 2 is formulated as a polynomial algorithm to color overlap graphs since it uses $2^{\omega}\omega^2(\omega - 1)$ colors. It shows however a polynomial coloring algorithm whose performance ratio on overlap graphs of fixed ω does not tend to infinity with the number of vertices. If $\omega = 2$, this algorithm colors triangle free overlap graphs with at most 16 colors. It is

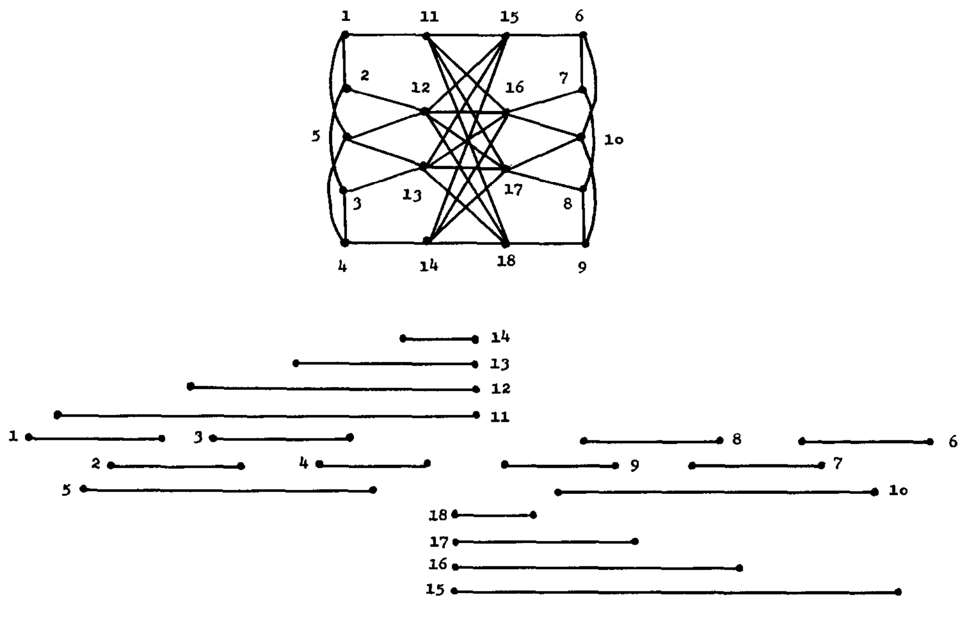

Fig. 1.

possible to color much better in this case. Lehel found an algorithm which colors with at most 5 colors [7]. On the other hand, there are cases when 4 colors are necessary, Fig. 1 displays such a case. We remark that finding ω in circular-arc graphs and in overlap graphs is polynomially solvable [5, p. 193], [4] while the same problem is unsolved for multiple interval graphs.

Applications concerning the chromatic number of circular-arc graphs, overlap graphs and multiple interval graphs can be found in [10, 5, 9].

2. Multiple interval graphs

Theorem 1. *If G is a t-interval graph, $\chi=\chi(G)$, $\omega=\omega(G)$, then*

$$\chi \leqslant 2t(\omega-1) \quad \textit{for } \omega \geqslant 2.$$

Proof. Let G be a t-interval graph with vertices $x_1, x_2, \dots, x_n$. For $x_i \in V(G)$, $A_i(I_i^1, I_i^2, \dots, I_i^t)$ denotes the t-interval corresponding to x_i in a representation of G. We assign a direction to every edge $e=(x_i, x_j) \in E(G)$ as follows: since $A_i \cap A_j \neq \emptyset$, we choose a pair k, m such that $I_i^k \cap I_j^m \neq \emptyset$. If the right endpoint of I_i^k is less than or equal to the right endpoint of I_j^m then e is directed from x_i to x_j. Otherwise e is directed from x_j to x_i. Now the outdegree of each vertex of G is at most $t(\omega-1)$, moreover it is easy to see that all induced subgraphs of G have a vertex of outdegree strictly less than $t(\omega-1)$. We assign new indices to the vertices of G by choosing a vertex of minimum indegree in G as x_1 and defining x_{i+1} as a vertex of minimum indegree in $V(G)-\{x_1, \dots, x_i\}$ for all i, $1 \leqslant i \leqslant n-1$.

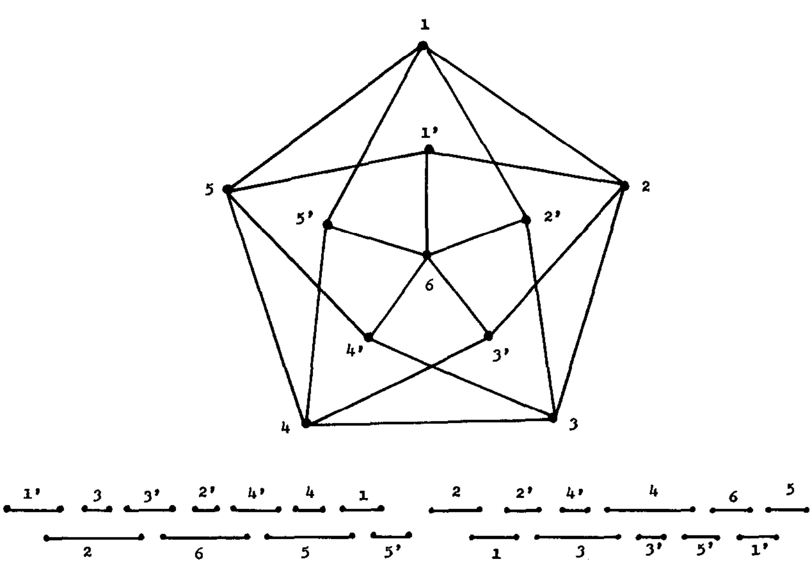

Fig. 2.

Now $|\{j: i<j,\ (x_i, x_j)\in E(G)\}|<2t(\omega-1)$ for every i, $1\leq i\leq n$. Thus G can be colored with at most $2t(\omega-1)$ colors by the greedy algorithm, coloring the vertices of G sequentially in the order $x_n, x_{n-1}, \ldots, x_1$. $\square$

Theorem 1 is probably far from being the best. However, for $\omega=2$, $t=2$ it is sharp. Figure 2 shows the 4-chromatic Mycielski-graph and its representation by double intervals. This example can be modified easily to see that $\chi(G)\geq 2\omega(G)$ holds with infinitely many ω for suitable 2-interval graphs.

3. Overlap graphs

Theorem 2. *If G is an overlap graph, $\chi=\chi(G)$, $\omega=\omega(G)$, then*

$$\chi\leq 2^{\omega}\omega^2(\omega-1).$$

Proof. Consider an overlap graph G and a representation of G. For $x\in V(G)$, $I(x)$ denotes the interval corresponding to x in the representation of G.

Assume that $G_1, G_2, \ldots, G_m$ denote the connected components of G. For every i, $1\leq i\leq m$, we define $x_i\in V(G_i)$ such that the starting point of $I(x_i)$ is the leftmost in the representation of G_i. Consider the subsets of $V(G_i)$ having the same distance from x_i. These sets for $1\leq i\leq m$ define a partition of $V(G)$ which is called *partition into levels*. The vertex x_i is called the *root of H* if H is a level of the partition such that $H\subset V(G_i)$.

Lemma 1. *Let H be a level of an overlap graph G. Assume that $I(x)\subset\bigcap_{y\in Y}I(y)$ for some $x\in H$, $Y\subset H$. There exists a $z\in V(G)$ such that $(z,x)\in E(G)$ and $(z,y)\in E(G)$ for all $y\in Y$.*

Proof. Let z_1 be the root of level H and consider the shortest path $z_1, z_2, \ldots, z_k, x$ in G between z_1 and x. It is easy to check that $z=z_k$ satisfies the requirements of the lemma. $\square$

Now we define *levels of depth i* in G for $i=1,\ldots,\omega$. The levels of depth one are the levels of G as defined above. If levels of depth i are already defined for some i, $1\leq i<\omega$, the levels of depth $i+1$ are the levels of the subgraphs of G induced by the levels of depth i. If $\mathcal{T}$ is a set of intervals then $\nu(\mathcal{T})$ denotes the maximal number of pairwise disjoint intervals of $\mathcal{T}$.

Lemma 2. *Let H be a level of depth i in the overlap graph G $(1\leq i\leq\omega)$ and let K be a clique of H such that $|K|=\omega-i+1$. The set of intervals $\mathcal{T}(K)=\{I(x)\colon x\in H$, $I(x)\subset\bigcap_{y\in K}I(y)\}$* satisfies $\nu(\mathcal{T}(K))\leq i-1$.

Proof. We prove by induction on i. Assume that $i=1$, K is a clique of H, $|K|=\omega$, $x\in V(H)$ and $I(x)\subset\bigcap_{y\in K} I(y)$. Applying Lemma 1, we find a $z\in V(G)$ such that $I(z)$ overlaps all $I(y)$ for $y\in K$. It is a contradiction since $K\cup z$ is a clique of $\omega+1$ vertices in G. Assume the lemma to be true for some $i<\omega$, we prove it for $i+1$. Let H be a level of depth $i+1$. Now H is a level of H' where H' is a level of depth i. Let K be a clique of H, $|K|=\omega-(i+1)+1=\omega-i$. Assume indirectly that there exist $x_1, x_2, \ldots, x_{i+1}\in H$ such that $I(x_1)$, $I(x_2), \ldots, I(x_{i+1})$ are pairwise disjoint intervals and $\bigcup_{j=1}^{i+1} I(x_j)\subset\bigcap_{y\in K} I(y)$. We may assume that $I(x_{i+1})$ is the rightmost interval among $I(x_1), I(x_2), \ldots, I(x_{i+1})$. Apply Lemma 1 for level H with x_{i+1} in the role of x. The lemma gives a $z\in H'$ such that $z\cup K$ is a clique of H', $|z\cup K|=\omega-i+1$ moreover $\bigcup_{j=1}^{i} I(x_j)\subset\bigcap_{y\in K\cup z} y$ contradicting the inductive hypothesis for i. $\square$

Corollary. *Let H be a level of depth ω in the overlap graph G. The set of intervals $\mathcal{T}_x=\{I(y)\colon y\in H,\ I(y)\subset I(x)\}$ satisfies $\nu(\mathcal{T}_x)\leqslant\omega-1$.*

Lemma 3. *If H is a level of depth ω in G, then $\chi(H)\leqslant\omega^2(\omega-1)$.*

Proof. Let $H'\subset H$ be the set of vertices corresponding to maximal intervals of $\{I(x)\colon x\in H\}$. Clearly H' induces a perfect graph in G since its complement has a transitive orientation (disjointness is a transitive relation between intervals). Obviously, $\omega(H')\leqslant\omega(G)=\omega$, therefore, we can color H' with $1,2,\ldots,\omega$. To each $x\in H-H'$ we assign a color class $c(x)$ of H' in such a way that $I(x)\subset I(y)$ for some $y\in H'$ of color class $c(x)$. Consider the vertex set $H_i=\{x\in H-H'\colon c(x)=i\}$ $(1\leqslant i\leqslant\omega)$. We shall define a good coloring of H by using colors $c_1^i, c_2^i, \ldots, c_{(\omega-1)\omega}^i$ on the vertices of H_i. It is enough to show that $H_i(y)=\{x\in H_i\colon I(x)\subset I(y)\}$ has a good coloring with these $(\omega-1)\omega$ colors for all fixed $y\in H'$ and fixed i, $1\leqslant i\leqslant\omega$. Using the corollary, the set of intervals $\mathcal{T}_i(y)=\{I(x)\colon x\in H_i(y)\}$ satisfies $\nu(\mathcal{T}_i(y))\leqslant\omega-1$ for all $y\in H'$ and for all i, $1\leqslant i\leqslant\omega$. The perfectness of interval graphs (Gallai's theorem) implies that $\mathcal{T}_i(y)=\bigcup_{j=1}^{\omega-1}\mathcal{T}_i^j(y)$ where $\mathcal{T}_i^j(y)$ contains pairwse intersecting intervals. The set $H_i^j(y)$ corresponding to the intervals of $\mathcal{T}_i^j(y)$ induces a perfect graph since the overlap graph of pairwise intersecting intervals is a so-called permutation graph [5, p. 250]. Therefore we can define a good coloring of the vertices of $H_i^j(y)$ with colors $c_{(j-1)\omega+1}^i, c_{(j-1)\omega+2}^i, \ldots, c_{j\omega}^i$. $\square$

Now the proof of Theorem 2 can be finished by the following argument. First, it is clear that a graph can be colored with 2c colors if its levels can be colored with c colors. The truth of this statement follows from the following fact: if $(x, y)\in E(G)$ then either both x and y belong to the same level of G or x and y belong to two consecutive levels of G having the same root. Repeating the same argument, G can be colored with 4c colors provided that G's levels of depth two can be colored with c colors. Continuing this argument, we see that G can be colored with $2^{\omega}c$ colors if G's levels of depth ω can be colored with c colors. Since the levels of depth ω can be colored with $c = \omega^2(\omega - 1)$ colors by Lemma 3, an overlap graph G can be colored with $2^{\omega}\omega^2(\omega - 1)$ colors. $\square$

References

[1] E. Asplund and B. Grünbaum, On a coloring problem, Math. Scand. 8 (1960) 181–188.
[2] J.P. Burling, On coloring problems of families of prototypes, Ph.D. Thesis, University of Colorado Boulder, CO (1965).
[3] M.R. Garey, D.S. Johnson, G.L. Miller and C.H. Papadimitriou, The complexity of coloring circular arcs and chords, SIAM J. Algebraic Discrete Methods 1 (1980) 216–227.
[4] F. Gavril, Algorithms for a maximum clique and a maximum independent set of a circle graph, Networks 3 (1973) 261–273.
[5] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs (Academic Press, New York, 1980).
[6] A. Gyárfás and J. Lehel, A Helly-type problem in trees, Colloq. Math. Soc. János Bolyai 4 (1970) 571–584.
[7] J. Lehel, Personal communication (1983).
[8] F.S. Roberts, On the boxicity and cubicity of a graph, in: Recent Progress in Combinatorics (Academic Press, New York, 1969) 301–310.
[9] W.T. Trotter, Jr. and F. Harary, On double and multiple interval graphs, J. Graph Theory 3 (1979) 205–211.
[10] A. Tucker, Coloring a family of circular-arcs, SIAM J. Appl. Math. Mech. 3 (1975) 493–502.