Arithmetic and metric properties of Oppenheim continued fraction expansions

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Abstract

We introduce a class of continued fraction expansions called Oppenheim continued fraction (OCF) expansions. Basic properties of these expansions are discussed and metric properties of the digits occurring in the OCF expansions are studied.
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1. Introduction

Over the last twenty years, considerable interests are shown in various continued fraction expansions. Examples of such continued fraction expansions include, for example, the backward continued fraction [1], the fraction with even partial quotients [18] and the Farey-shift [9]. For more details, see [8] and the references therein. Metric properties of these expansions have been well studied. Each of these continued fraction expansions is ergodic and has an infinite and $\sigma$-finite, invariant measure which is absolutely continuous with respect to Lebesgue measure.

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In 2002, Y. Hartono, C. Kraaikamp and F. Schweiger [7] introduced a new continued fraction algorithm with nondecreasing partial quotients, named Engel continued fraction (ECF) expansion. Just as the name suggests, the ECF expansion is originated from the classical Engel series expansion.

Recall that the Engel series expansion is generated by the transformation \( SE : [0, 1) \rightarrow [0, 1) \) given by

\[
S_E(x) := \left( \left\lfloor \frac{1}{x} \right\rfloor + 1 \right) \left( x - \frac{1}{\left\lfloor \frac{1}{x} \right\rfloor + 1} \right), \quad x \neq 0; \quad S(0) := 0,
\]

where \( \lfloor x \rfloor \) denotes the largest integer not exceeding \( x \). While the Engel continued fraction expansion is generated by the transformation \( TE : [0, 1) \rightarrow [0, 1) \) given by

\[
T_E(x) := \left( \frac{1}{\lfloor 1/x \rfloor} \right) \left( \frac{1}{x} - \lfloor 1/x \rfloor \right), \quad x \neq 0; \quad T_E(0) := 0.
\]

For each \( x \in (0, 1) \), the transformation \( S_E \) generates a (unique) series expansion of the form

\[
x = \frac{1}{d_1(x)} + \frac{1}{d_1(x)d_2(x)} + \cdots + \frac{1}{d_1(x)d_2(x)\cdots d_n(x)} + \cdots,
\]

where \( d_n(x) = \lfloor 1/S^{n-1}(x) \rfloor + 1, \ n \geq 1 \), and the digits \( \{d_n(x): \ n \geq 1\} \) satisfy the condition \( 2 \leq d_n(x) \leq d_{n+1}(x) \) for all \( n \geq 1 \). In fact, it was W. Sierpiński [16] who first studied these series expansions in 1911. Metric properties of Engel series expansion were established by P. Erdös, A. Rényi and P. Szüsz [2] and A. Rényi [15]. F. Schweiger [17] showed that \( S_E \) is ergodic and Thaler [20] found a whole family of infinite and \( \sigma \)-finite measure for \( S_E \). Fractal properties of exceptional sets related to the Engel series expansion have been discussed by Y.Y. Liu and J. Wu [12].

For each \( x \in (0, 1) \), the transformation \( T_E \) generates a new type of continued fraction expansion of the form

\[
x = \frac{1}{d_1(x)} + \frac{1}{d_1(x)d_2(x)} + \cdots + \frac{1}{d_1(x)d_2(x)\cdots d_n(x)} + \cdots,
\]

where \( d_n(x) = \lfloor 1/T^{n-1}(x) \rfloor + 1, \ n \geq 1 \), and the digits \( \{d_n(x): \ n \geq 1\} \) satisfy the condition \( 1 \leq d_n(x) \leq d_{n+1}(x) \) for all \( n \geq 1 \). Arithmetic and ergodic properties of \( T_E \) associated to this new continued fraction expansion were studied by Y. Hartono, C. Kraaikamp and F. Schweiger in [7]. They showed that \( T_E \) has no finite invariant measure equivalent to the Lebesgue measure, but has infinitely many \( \sigma \)-finite, infinite invariant measures. Also they showed that \( T_E \) is ergodic with respect to Lebesgue measure. In [11], C. Kraaikamp and J. Wu derived some metric properties of the digits \( \{d_n(x): \ n \geq 1\} \) occurring in such an expansion. They also investigated different kinds of exceptional sets on which the metric properties fail to hold.

Even if the ergodic and metric properties of both Engel series and Engel continued fraction are very similar, F. Schweiger [19] constructed a class of algorithms, with proper choice of a parameter, with increasing digits which have quite different properties from those of Engel case.
In the present paper, we introduce a class of continued fraction expansions, which we call Oppenheim continued fraction (OCF) expansions. The Engel continued fraction (ECF) expansion stands as a special case. In general, there is no longer an associate dynamical system. We will use probability method to study these OCFs. Before introducing the Oppenheim continued fraction expansions, we would like to present Oppenheim series expansions [6].

Let \( \gamma_j \) be a sequence of positive rational-valued functions defined on \( \mathbb{N} \setminus \{1\} \) satisfying

\[
\gamma_j(n) \geq \frac{1}{n(n - 1)}, \quad \text{for all } j \geq 1.
\]

For each real number \( 0 < x < 1 \), define the integers \( d_j = d_j(x) \) and the real numbers \( x_j (j = 1, 2, \ldots) \) by the algorithm:

\[
x_1 = x, \quad d_j = \left\lfloor \frac{1}{x_j} \right\rfloor + 1, \quad x_{j+1} = \frac{1}{\gamma_j(d_j)}\left(x_j - \frac{1}{d_j}\right). \quad (1.3)
\]

This leads to the Oppenheim series expansion of \( x \):

\[
1/d_1 + \gamma_1(d_1) \cdot 1/d_2 + \gamma_1(d_1)\gamma_2(d_2) \cdot 1/d_3 + \cdots. \quad (1.4)
\]

Write

\[
h_j(n) = \gamma_j(n)n(n - 1).
\]

If \( h_j \) is integer-valued for all \( j \geq 1 \), the series (1.4) is said to be a restricted Oppenheim series expansion of \( x \).

Here are some special cases which were extensively studied:

- **Lüroth series expansion**: \( h_j(n) = 1 \);
- **Engel series expansion**: \( h_j(n) = n - 1 \);
- **Sylvester series expansion**: \( h_j(n) = n(n - 1) \).

The expansion (1.4) with \( d_j \) defined by (1.3) was first studied by A. Oppenheim [14] who established basic arithmetical properties, including the criteria of rationality of the expansion. The metric theory for Oppenheim series expansion was established by J. Galambos [3–5]. Exceptional sets associated with this expansion were discussed by J. Wu [22,23]. Further information on the Oppenheim series expansion can be found in J. Galambos [3,6], F. Schweiger [18] and W. Vervaat [21].

Now, let us introduce Oppenheim continued fraction expansion.

Let \( \{h_j\}_{j \geq 1} \) be a sequence of nonnegative integer valued functions defined on \( \mathbb{N} \). For any \( x \in (0, 1) \), define the integers \( d_j = d_j(x) \) and the real numbers \( x_j, j = 1, 2, \ldots \), by the algorithm:

\[
x_1 = x, \quad d_j = \left\lfloor \frac{1}{x_j} \right\rfloor, \quad x_{j+1} = \frac{1}{h_j(d_j) + 1}\left(\frac{1}{x_j} - d_j\right). \quad (1.5)
\]

It can be proved that this algorithm leads to a kind of continued fraction expansion of \( x \in (0, 1) \) with the following form

\[ x = \frac{1}{d_1 + \frac{h_1(d_1)+1}{d_2 + \cdots + \frac{h_{n-1}(d_{n-1})+1}{d_n + \cdots + \frac{h_n(d_n)+1}{d_{n+1} + \cdots}}}} \]  

(1.6)

where \( d_n \in \mathbb{N} \) and \( d_{n+1} \geq h_n(d_n) + 1 \) for all \( n \geq 1 \) (see Proposition 2.5).

Here we list some special cases of Oppenheim continued fraction expansion:

- **Regular continued fraction expansion**: \( h_j(n) = 0 \);
- **Engel continued expansion**: \( h_j(n) = n - 1 \);
- **Sylvester continued expansion**: \( h_j(n) = n(n - 1) \).

In this paper, we discuss basic arithmetic properties (Section 2) and study metric properties of the digits \( \{d_n(x) : n \geq 1\} \) occurring in these expansions, including weak and strong large number law and the central limit theorem (Section 3). We also investigate approximation speed (Section 4).

2. OCF expansions

2.1. Some arithmetic properties

In this subsection, we study the arithmetic properties, including the convergence theorem of Oppenheim continued fraction expansions, uniqueness of the expansion, and a general property on the digits.

**Definition 2.1.** A vector of positive integers \( (d_1, d_2, \ldots, d_n) \) is called an admissible vector for the Oppenheim continued fraction expansion if there exists \( x \in (0, 1) \) such that \( d_j(x) = d_j \) for all \( 1 \leq j \leq n \). A sequence \( (d_1, d_2, \ldots, d_n, \ldots) \) is called an admissible sequence if \( (d_1, d_2, \ldots, d_n) \) is an admissible vector for each \( n \geq 1 \).

**Proposition 2.2.** A sequence of positive integers \( (d_1, d_2, \ldots, d_n, \ldots) \) is admissible if and only if for each \( j \geq 1 \),

\[ d_{j+1} \geq h_j(d_j) + 1. \]  

(2.1)

**Proof.** The necessity of (2.1) is obvious by the definition of \( d_j = d_j(x) \). To prove the sufficiency, for each \( n \geq 1 \), we take

\[ x = x_1 = \frac{1}{d_1 + \frac{h_1(d_1)+1}{d_2 + \cdots + \frac{h_{n-1}(d_{n-1})+1}{d_n + \cdots + \frac{h_n(d_n)+1}{d_{n+1} + \cdots}}}}. \]

Since \( \frac{1}{d_1+1} < x_1 < \frac{1}{d_1} \), then \( d_1(x) = d_1 \). Hence, by the algorithm (1.5), we know

\[ x_2 = \frac{1}{d_2 + \frac{h_2(d_2)+1}{d_3 + \cdots + \frac{h_{n-1}(d_{n-1})+1}{d_n + \cdots + \frac{h_n(d_n)+1}{d_{n+1} + \cdots}}}}. \]

By induction, we get \( d_j(x) = d_j \) for all \( 1 \leq j \leq n \). \( \square \)
The following proposition is a characterization of rational numbers.

**Proposition 2.3.** A number \( x \in (0, 1) \) has a finite Oppenheim continued fraction expansion (i.e., \( x_j = 0 \) for some \( j \geq 1 \)) if and only if \( x \in \mathbb{Q} \).

**Proof.** By the expansion of \( x \) in (1.6), it is necessary that \( x \in \mathbb{Q} \) if \( x \) has a finite expansion. Suppose now \( x \) is rational. By the algorithm, we know if \( x_j \neq 0 \), then \( x_j \) is rational and \( 0 < x_j < 1 \), hence \( x_j := \frac{a_j}{b_j} = \frac{a_j}{d_j \cdot a_j + a_{j+1}} \) where \( 0 \leq a_{j+1} < a_j \) and \( d_j = [b_j/a_j] \geq 1 \). Thus by the algorithm, we have

\[
x_{j+1} = \frac{1}{h_j(d_j) + 1} \left( \frac{1}{x_j} - \frac{1}{d_j} \right) = \frac{1}{h_j(d_j) + 1} \frac{a_{j+1}}{a_j} := \frac{a_{j+1}}{b_{j+1}}.
\]

Because \( a_{j+1} < a_j \), then this procedure will stop at finite steps, that is to say, \( x_j = 0 \) for some \( j \). \( \Box \)

The proof of the following proposition is omitted, since it is quite straightforward. For a proof the interested reader is referred to Section 1 in [10] where a similar result has been obtained for a class of continued fractions.

**Proposition 2.4.** Let \( \{a_n\}, \{b_n\} \) be two sequences of positive numbers. Let \( \{p_n\}_{n \geq 1} \) and \( \{q_n\}_{n \geq 1} \) be the sequences recursively defined by

\[
\begin{align*}
p_0 &= 0, & p_1 &= 1, & p_n &= b_n \cdot p_{n-1} + a_{n-1} p_{n-2}, & \text{for } n \geq 2, & \quad (2.2) \\
q_0 &= 1, & q_1 &= b_1, & q_n &= b_n \cdot q_{n-1} + a_{n-1} q_{n-2}, & \text{for } n \geq 2. & \quad (2.3)
\end{align*}
\]

Then one has

(i) \( p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \prod_{j=1}^{n-1} a_j. \)

(ii) \( \frac{1}{b_1 + \cdots + a_{n-1}/b_n} = \frac{p_n}{q_n}. \)

(iii) For \( \{p_n/q_n\}_{n \geq 1} \), even-numbered terms are strictly increasing and odd-numbered strictly decreasing, moreover, every even-numbered term is less than every odd-numbered one.

As for the usual continued fractions, we get the convergents obtained through finite truncation: for each \( n \geq 1 \), let

\[
Q_n(d_1, \ldots, d_n) = \frac{1}{d_1 + \frac{h_1(d_1)+1}{d_2 + \cdots + \frac{h_{n-1}(d_{n-1})+1}{d_n}}}. 
\]

For simplicity and convenience, we denote \( Q_n(d_1, \ldots, d_n; 0) = Q_n(d_1, \ldots, d_n) \) and for any \( 0 < x \leq 1 \), define \( Q_n(d_1, \ldots, d_n; x) \) by replacing \( d_n \) in \( Q_n(d_1, \ldots, d_n) \) by \( d_n + x \). From Proposition 2.4, applied to \( b_n = d_n \) and \( a_n = h_n(d_n) + 1 \), we get

\[
Q_n(d_1, \ldots, d_n) = \frac{p_n}{q_n}.
\]
If $d_j = d_j(x)$, we call $\frac{p_n(x)}{q_n(x)}$ the $n$th convergent of $x$ in its Oppenheim continued fraction expansion.

The sequence of convergent converges to the number from which it is generated, as the following proposition shows.

**Proposition 2.5.** For every $x \in (0, 1)$, we have

$$\lim_{n \to +\infty} \frac{p_n(x)}{q_n(x)} = x. \quad (2.4)$$

**Proof.** If $x$ is rational, we conclude (2.4) by Proposition 2.2. Now let $x$ be irrational. By Proposition 2.4(ii), one has

$$x = \frac{p_n(x) + (h_n(d_n(x)) + 1) \cdot x_{n+1} \cdot p_{n-1}(x)}{q_n(x) + (h_n(d_n(x)) + 1) \cdot x_{n+1} \cdot q_{n-1}(x)}. \quad (2.5)$$

Hence following from Proposition 2.4(i), we get

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| = \frac{\prod_{j=1}^{n} (h_j(d_j(x)) + 1) \cdot x_{n+1}}{q_n(x) (q_n(x) + (h_n(d_n(x)) + 1) \cdot x_{n+1} \cdot q_{n-1}(x))} \leq \frac{\prod_{j=1}^{n} (h_j(d_j(x)) + 1)}{q_n(x)^2 \cdot d_{n+1}(x)} \quad \text{for } x_{n+1} \leq \frac{1}{d_{n+1}(x)}. \quad (2.6)$$

On the other hand, by (2.3), $q_n \geq d_n q_{n-1} \geq \cdots \geq \prod_{k=1}^{n} d_k$. Moreover, $(d_1, \ldots, d_n)$ is admissible, then

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| \leq \frac{1}{q_n(x)} \prod_{j=1}^{n} \frac{h_j(d_j(x)) + 1}{d_{j+1}(x)} \leq \frac{1}{q_n(x)} \to 0.$$

The last assertion in the above formula is followed by the fact that $q_n \geq q_{n-1} + q_{n-2}$. □

The next proposition shows that the Oppenheim continued fraction expansion is unique.

**Proposition 2.6.** Let $(d_1, \ldots, d_n, \ldots)$ be admissible, and $\{p_n\}_{n \geq 1}$ and $\{q_n\}_{n \geq 1}$ be given by (2.2) and (2.3) with $b_n = d_n$ and $a_n = h_n(d_n) + 1$. Then $\frac{p_n}{q_n}$ converges to some $x \in (0, 1)$ and $d_n(x) = d_n$ for all $n \geq 1$.

**Proof.** The existence of the limit follows from Proposition 2.2(iii) and $|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}| \to 0$.

Since $0 < p_2/q_2 < x < p_1/q_1 \leq 1$, we have $x \in (0, 1)$. Notice that for any $n \geq 1$, $\frac{p_{2n}}{q_{2n}} < x < \frac{p_{2n+1}}{q_{2n+1}}$, thus $x$ can be expanded with the form

$$x = \frac{1}{d_1 + \frac{h_1(d_1) + 1}{d_2 + \cdots + \frac{h_{2n-1}(d_{2n-1}) + 1}{d_{2n} + x'}}}.$$
for some $0 < x' < 1$. Using similar arguments in the proof of Proposition 2.2, we get $d_j(x) = d_j$ for all $1 \leq j \leq 2n$.  

2.2. Some preliminary results

Here and in what follows, we use $\lambda$ to denote the Lebesgue measure.

**Definition 2.7.** Let $(d_1, d_2, \ldots, d_n)$ be an admissible sequence. Define

$$B(d_1, d_2, \ldots, d_n) := \{x \in (0, 1]: d_1(x) = d_1, \ d_2(x) = d_2, \ldots, d_n(x) = d_n\}$$

which is called an $n$th order cylinder.

For integers $1 \leq a \leq b$ and real number $0 \leq y < 1$, define

$$\Upsilon(a, b, y) = \frac{a(1 + y)(b + ay)(b + 1 + ay)}{a^2(b + ay)}.$$  

(2.7)

Then about the measure of a cylinder, we have

**Proposition 2.8.** Let $(d_1, d_2, \ldots, d_n)$ be an admissible vector. Then

$$\lambda(B(d_1, \ldots, d_n)) = \frac{\prod_{j=1}^{n-1}(h_j(d_j) + 1)}{q_n(q_n + q_{n-1})},$$

$$\frac{\lambda(B(d_1, \ldots, d_n, d_{n+1}))}{\lambda(B(d_1, \ldots, d_n))} = \Upsilon(h_n(d_n) + 1, d_{n+1}, y_n) \text{ with } y_n = \frac{q_{n-1}}{q_n}. \quad (2.8)$$

**Proof.** Look at (1.6). Since $0 \leq (h_n(d_n) + 1)x_{n+1} < 1$, $B(d_1, d_2, \ldots, d_n)$ is the interval with two endpoints $Q_n(d_1, \ldots, d_n; 0)$ and $Q_n(d_1, \ldots, d_n + 1)$. By Proposition 2.4, these two endpoints are $\frac{p_n}{q_n}$ and $\frac{p_n + p_{n-1}}{q_n + q_{n-1}}$. As a result,

$$\lambda(B(d_1, d_2, \ldots, d_n)) = \left| \frac{p_n}{q_n} - \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \right| = \frac{\prod_{j=1}^{n-1}(h_j(d_j) + 1)}{q_n(q_n + q_{n-1})}. \quad \square$$

Our further investigations are partially based on (2.8). We are led to study the function $\Upsilon(a, b, y)$.

**Lemma 2.9.** For any integers $1 \leq a \leq b$ and any real number $0 \leq y < 1$, we have

(i) $\Upsilon(a, b, y) = \frac{a(1 + y)}{b + ay} - \frac{a(1 + y)}{b + 1 + ay} = \int_{\frac{b+1+ay}{a(1+y)}}^{\frac{b+ay}{a(1+y)}} \frac{1}{x^2} \, dx$;

(ii) $|\Upsilon(a, b, y) - \Upsilon(a, b, 0)| \leq \frac{4ay}{b(b + 1)}$;

(iii) $\Upsilon(a, b, y) \leq \frac{2a}{b^2}$. 


Proof. The proof is elementary. \( \square \)

**Lemma 2.10.** For any integer \( a, \tau \in \mathbb{N} \) and any real number \( 0 \leq y < 1 \), we have

\[
\frac{1}{\tau + 1} \leq \sum_{b \geq \tau a} \Upsilon(a, b, y) \leq \frac{2}{\tau + 1}.
\] (2.9)

Proof. It suffices to remark that, by (i) in Lemma 2.9, we have

\[
\sum_{b \geq \tau a} \Upsilon(a, b, y) = \sum_{b \geq \tau a} \left( \frac{a(1 + y)}{b + ay} - \frac{a(1 + y)}{b + 1 + ay} \right) = \frac{1 + y}{\tau + y}. \square
\]

**Lemma 2.11.** For any integer \( a \in \mathbb{N} \) and any real number \( 0 \leq y < \frac{1}{a} \), we have

\[
\sum_{b \geq a} \frac{a}{b} \cdot \Upsilon(a, b, y) \leq \frac{11}{12}.
\]

Proof. For the first term, we have

\[
\frac{a}{a} \cdot \Upsilon(a, a, y) = \frac{a}{a(a + ay)(a + 1 + ay)} = \frac{1}{a + 1 + ay} \leq \frac{1}{a + 1}.
\]

For other terms, notice that \( a(1 + y) < a + 1 \), we have

\[
\frac{a}{b} \cdot \Upsilon(a, b, y) \leq \frac{a}{b} \cdot \frac{a + 1}{(b + ay)(b + 1 + ay)} \leq \frac{a(a + 1)}{b^2(b + 1)}.
\]

However,

\[
\sum_{b \geq a+2} \frac{1}{b^2(b + 1)} \leq \int_{a+1}^{+\infty} \frac{1}{z^2(z + 1)} \, dz \leq \frac{1}{2(a + 1)^2}.
\]

Therefore, the sum to be estimated is bounded by

\[
h(a) := \frac{1}{a + 1} + \frac{a}{(a + 1)(a + 2)} + \frac{a}{2(a + 1)}.
\]

Since \( \frac{1}{a+1} + \frac{a}{(a+1)(a+2)} \) is decreasing and \( \frac{a}{2(a+1)} \) is bounded by \( \frac{1}{2} \), we have

\[
h(a) \leq \max \left\{ h(1), h(2), h(3), \frac{4}{4 + 1} + \frac{4}{(4+1)(4+2)} + \frac{1}{2} \right\} = h(1) = \frac{11}{12}. \square
\]
Lemma 2.12. For any integer \( a \in \mathbb{N} \) and any real number \( 0 \leq y < 1 \), we have
\[
\sum_{b \geq a} \log \frac{b}{a} \cdot \Upsilon(a, b, y) \leq 2 \sum_{k=1}^{+\infty} \frac{\log(k+1)}{k^2} < +\infty.
\]

Proof. It follows just by \( \Upsilon(a, b, y) \leq 2a/b^2 \) (see Lemma 2.9(iii)). \( \square \)

Lemma 2.13. For any fixed integer \( a \in \mathbb{N} \) and real number \( 0 \leq y < 1 \), we have, for \( \tau \) sufficient large,
\[
\sum_{a \leq b \leq \tau a} b \Upsilon(a, b, y) = (1 + o(1) + O(1) y) \log \tau.
\]

Proof. By Lemma 2.9(ii), we know
\[
\left| \frac{b}{a} \Upsilon(a, b, y) - \frac{1}{b+1} \right| \leq \frac{4y}{b+1}.
\]

Lemma 2.14. For any integer \( a \in \mathbb{N} \) and any real number \( 0 \leq y < \frac{1}{a} \), we have
\[
\left| \sum_{b \geq a} \Upsilon(a, b, y) \cdot e^{it \log \frac{b}{a}} - \frac{1}{1-it} \right| \leq \frac{|t|}{a}.
\]

Proof. We denote by \( \alpha(t, a, y) \) the quantity defined by the sum. Lemma 2.9 gives
\[
\left| \alpha(t, a, y) - \frac{1}{1-it} \right| = \left| \sum_{b \geq a} \int_{\frac{b+ay}{a(1+y)}}^{\frac{b+1+ay}{a(1+y)}} \frac{1}{x^2} e^{ixt \log \frac{b}{a}} - e^{ixt \log x} \, dx \right|.
\]

We can get the desired result since by the mean value theorem and when \( \frac{b+ay}{a(1+y)} \leq x \leq \frac{b+1+ay}{a(1+y)} \), we have
\[
\left| e^{ixt \log \frac{b}{a}} - e^{ixt \log x} \right| \leq |t| \left| \log \frac{b}{a} - \log x \right| = \frac{|t|}{a}.
\]

3. Metric theory of OCF expansions

In this section, we will investigate the metric properties of the digits \( d_j \). In most cases, we follow the ideas due to J. Galambos [3–5] who studied the metric properties of Oppenheim series expansions.

In the sequel, we always assume that the following hypothesis holds:
\[
h_j(n) \geq n - 1 \quad \text{for all } j \geq 1 \text{ and } n \geq 1.
\]
3.1. About \( d_j(x) \) and \( \frac{1}{x_j} - d_j \)

**Proposition 3.1.** Assume \((H)\). We have

\[
\sum_{d_1, d_2, \ldots, d_n} \frac{\lambda(B(d_1, d_2, \ldots, d_n))}{d_n} \leq \left(\frac{11}{12}\right)^n,
\]

where the summation takes over all admissible vectors \((d_1, d_2, \ldots, d_n)\).

**Proof.** Let \( S_n \) be the sum in (3.1). We are going to show that

\[ S_{n+1} \leq \frac{11}{12} S_n. \]

By (2.8), we can write

\[ S_{n+1} = \sum_{d_1, \ldots, d_n} \frac{\lambda(B(d_1, \ldots, d_n))}{d_n} \cdot \Theta(d_n, y_n) \]

where

\[ \Theta(d_n, y_n) := \sum_{d_n+1 \leq h_n(d_n)+1} \frac{d_n}{d_{n+1}} \gamma(h_n(d_n)+1, d_{n+1}, y_n), \]

and the summation in \( S_{n+1} \) takes over all admissible vectors \((d_1, \ldots, d_n)\), i.e. \( d_{j+1} \geq h_j(d_j) + 1 \).

According to Lemma 2.11, \( \Theta(d_n, y_n) \) is bounded by \( \frac{11}{12} \). So, we get \( S_{n+1} \leq \frac{11}{12} \cdot S_n \). \( \square \)

**Theorem 3.2.** Assume \((H)\). Then for almost all \( x \in (0, 1) \), \( d_j(x) \) is strictly increasing when \( j \) is sufficiently large.

**Proof.** We can show: \( \lim \inf_{j \to \infty} d_{j+1}(x) - d_j(x) \geq 1 \) for almost all \( x \). By Proposition 2.2 and \((H)\), we have \( d_{j+1} \geq h_j(d_j) + 1 \geq d_j \) for all \( x \). Consider \( A_n = \{d_{n+1} = h_n(d_n) + 1\} \). Then

\[ \lambda(A_n) = \sum_{d_1, \ldots, d_n} \lambda(B(d_1, \ldots, d_n, h_n(d_n) + 1)). \]

Notice that \( \gamma(a, a, y) = \frac{1}{a+1+ay} \leq \frac{1}{a+1} \) and \( h_n(d_n)+1 \geq d_n \). We get

\[ \frac{\lambda(B(d_1, \ldots, d_n, h_n(d_n) + 1))}{\lambda(B(d_1, \ldots, d_n))} = \gamma(h_n(d_n)+1, d_n(h_n(d_n)+1), y_n) \leq \frac{1}{d_n+1}. \]

Hence, by Proposition 3.1, we get \( \lambda(A_n) \leq \left(\frac{11}{12}\right)^n \). We conclude by Borel–Cantelli lemma. \( \square \)

For every \( x \in (0, 1) \) and \( j \geq 1 \), define

\[ z_j(x) = (h_j(d_j(x)) + 1)x_{j+1} = \frac{1}{x_j} - d_j = \left\{ \frac{1}{x_j} \right\}. \]

Next, we show that \( z_j \) converges in distribution to the uniform distribution on \([0, 1]\).
Theorem 3.3. Assume (H). Then for any $0 < c \leq 1$,
\begin{equation}
|\lambda\{x \in (0, 1): z_n(x) < c\} - c| \leq c \left(\frac{11}{12}\right)^n.
\end{equation}

Proof. Write
\begin{align*}
\lambda(z_{n+1}(x) < c) &= \lambda\left\{x \in (0, 1): x_{n+1}(h_n(d_n(x)) + 1) < c\right\} \\
&= \sum_{d_1, \ldots, d_n} \lambda\left\{x: d_1(x) = d_1, \ldots, d_n(x) = d_n, x_{n+1}(h_n(d_n(x)) + 1) < c\right\} \\
&= \sum_{d_1, \ldots, d_n} \lambda(B(d_1, \ldots, d_n; c)).
\end{align*}

From (1.6), it is evident that $B(d_1, \ldots, d_n; c)$ is the interval with the two endpoints $Q_n(d_1, \ldots, d_n; 0)$ and $Q_n(d_1, \ldots, d_n; c)$. Then, by Proposition 2.4, we have
\begin{equation}
\lambda(B(d_1, \ldots, d_n; c)) = \left|\frac{p_n}{q_n} - \frac{p_n + cp_{n-1}}{q_n + cq_{n-1}}\right| = \frac{c \prod_{j=1}^{n-1}(h_j(d_j) + 1)}{q_n(q_n + cq_{n-1})}.
\end{equation}
Taking summation over $d_1, \ldots, d_n$ and using Proposition 3.1, we get the desired result. \qed

3.2. Growth rate of digits $d_n(x)$

The condition $d_{j+1} \geq h_j(d_j) + 1$ can be viewed as a growth rate of the sequence $\{d_n(x)\}_{n \geq 1}$. While, for a rational number $x$, the digits $d_n(x)$ terminate at finite steps. So, a general result for all numbers with faster growth rate is hardly to seek. For each $x \in (0, 1)$, define
\begin{equation}
R_n := R_n(x) = \frac{d_{n+1}(x)}{h_n(d_n(x)) + 1}.
\end{equation}

We can consider the behavior of $R_n$ at the infinity as a growth rate of $d_n$.

Theorem 3.4. Let $\varphi: \mathbb{N} \to \mathbb{N}$. Then

(i) If $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < +\infty$, then $\lambda\{x \in (0, 1): R_n(x) > \varphi(n) i.o.\} = 0$.
(ii) If $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} = +\infty$, then $\lambda\{x \in (0, 1): R_n(x) > \varphi(n) i.o.\} = 1$.

Before giving the proof, we establish a general estimation of conditional probability that will be used several times.

Lemma 3.5. Let $(A_1, \ldots, A_n, \ldots)$ be a sequence of nonempty subsets of $\mathbb{N}$, and $\varphi: \mathbb{N} \to \mathbb{N}$. Then we have
\begin{equation}
\frac{1}{\varphi(n) + 1} \leq \frac{\lambda\{x: R_1(x) \in A_1, \ldots, R_{n-1}(x) \in A_{n-1}, R_n > \varphi(n)\}}{\lambda\{x: R_1(x) \in A_1, \ldots, R_{n-1}(x) \in A_{n-1}\}} \leq \frac{2}{\varphi(n) + 1}.
\end{equation}
Proof. The numerator is equal to

\[ \lambda \{ x : R_1(x) \in A_1, \ldots, R_{n-1}(x) \in A_{n-1}, R_n > \varphi(n) \} \]

\[ = \sum_{d_1, \ldots, d_n} \sum_{d_{n+1} > \varphi(n)(d_n + 1)} \lambda(B(d_1, \ldots, d_n, d_{n+1})) \]

\[ = \sum_{d_1, \ldots, d_n} \lambda(B(d_1, \ldots, d_n)) \sum_{d_{n+1} > \varphi(n)(h_n(d_n) + 1)} \gamma(h_n(d_n) + 1, d_{n+1}, y_n) \]

where the first sum takes over all \((d_1, \ldots, d_n)\) satisfying \(R_j \in A_j\). An application of Lemma 2.10 yields that the inner sum is bounded by \(1/\varphi(n)+1\) and \(2/\varphi(n)+1\). This finishes the proof. \(\square\)

Proof of Theorem 3.4. Notice that Lemma 3.5 gives

\[ \frac{1}{\varphi(n)+1} \leq \lambda \{ x \in (0, 1) : R_n(x) > \varphi(n) \} \leq \frac{2}{\varphi(n)+1}. \] (3.4)

Therefore, Borel–Cantelli lemma guarantees the first part of the theorem.

For the second part, consider the sets

\[ A_N(M) = \{ x \in (0, 1) : R_n(x) \leq \varphi(n), N \leq n \leq N + M \} \quad (N \geq 1, M \geq 1) \]

and \(A = \bigcup_{N=1}^{\infty} \bigcap_{M=1}^{\infty} A_N(M)\). Then \(A\) is the set of the points \(x\) for which \(R_n(x) \leq \varphi(n)\) holds for all but a finite number of \(n\). Therefore we only need to show that \(\lambda(A) = 0\), or equivalently, \(\lim_{M \to \infty} \lambda(A_N(M)) = 0\) for all \(N\).

Since \(\lambda(A_N(M)) = \lambda(A_N(M) \cap A_N(M+1)) + \lambda(A_N(M+1))\), by Lemma 3.5, we get

\[ \left(1 - \frac{2}{\varphi(N+M+1)+1}\right) \leq \frac{\lambda(A_N(M+1))}{\lambda(A_N(M))} \leq \left(1 - \frac{1}{\varphi(N+M+1)+1}\right). \]

Therefore the divergence of \(\sum_{n=1}^{\infty} \frac{1}{\varphi(n)}\) implies \(\lim_{M \to \infty} \lambda(A_N(M)) = 0\) for each \(N \geq 1\). Thus \(\lambda(A) = 0\). \(\square\)

As a corollary, we have

Theorem 3.6. For almost all \(x \in (0, 1)\),

\[ \limsup_{n \to \infty} \frac{\log R_n(x) - \log n}{\log \log n} = 1, \quad \liminf_{n \to \infty} \frac{\log R_n(x) - \log n}{\log \log n} = -\infty. \] (3.5)

Proof. Applying the preceding theorem to the two cases \(\varphi(n) = n \log n\) and \(n(\log n)^{1+\epsilon}\) allows us to get immediately the first equality. To show the second one, we define \(A_n(\alpha) = \{ R_n(x) \leq n(\log n)^{-\alpha} \}\), for any \(\alpha > 0\) and \(n \geq 1\). Then by Fatou Lemma and Lemma 3.5, we get \(\lambda(\limsup_{n \to +\infty} A_n(\alpha)) \geq \limsup_{n \to +\infty} \lambda(A_n(\alpha)) \geq 1\). This implies that for almost all \(x \in (0, 1)\)
\[
\lim \inf_{n \to \infty} \frac{\log R_n(x) - \log n}{\log \log n} \leq -\alpha.
\]
This proves the second equality, for \(\alpha > 0\) is arbitrary. \(\Box\)

For a further investigation on the ratio \(R_n\), we consider

\[
L_n = \max_{1 \leq j \leq n} R_n.
\]

**Theorem 3.7.** For almost all \(x \in (0, 1)\),

\[
\lim \sup_{n \to \infty} \frac{\log L_n(x) - \log n}{\log \log n} = 1, \quad \lim \inf_{n \to \infty} \frac{\log L_n(x) - \log n}{\log \log n} = 0.
\]

**Proof.** We only establish that \(\lim \inf_{n \to \infty} \frac{\log L_n(x) - \log n}{\log \log n} \geq 0\), since the other assertions can be done similar to Theorem 3.6 by replacing \(A_n(\alpha)\) by \(\tilde{A}(\beta) = \{L_n(x) \leq n(\log n)^{\beta}\}\), for \(\beta > 0\). To get the desired inequality, it suffices to show that for any \(0 < \alpha < 1\), for almost all \(x \in (0, 1)\), \(L_n(x) \leq n(\log n)^{-\alpha} := \varphi(n)\) holds for only finite times. But by Theorem 3.4, for almost all \(x \in (0, 1)\), \(L_n(x) > \varphi(n)\) for infinitely many \(n\). Thus it follows that \(L_n(x) \leq \varphi(n)\) holds for only finite times is equivalent to

\[
L_n(x) \leq \varphi(n), \quad \text{but} \quad L_{n+1}(x) > \varphi(n+1) \quad \text{holds for only finite times.}
\]

Let \(A_n = \{x \in (0, 1): L_n(x) \leq \varphi(n), \ L_{n+1}(x) > \varphi(n+1)\}\). Then Lemma 3.5 is enough to make sure that \(\sum_{n=1}^{\infty} \lambda(A_n) < \infty\). Following the Borel–Cantelli lemma, we complete the proof. \(\Box\)

### 3.3. Laws of large numbers on \(\log R_n\) and \(R_n\)

In this subsection, we will study the law of large numbers (L-N) of the digits occurring in the Oppenheim continued fraction expansions. We shall make use of the following more general law of large numbers (see Galambos [3]).

**Lemma 3.8.** Let \(X_1, X_2, \ldots, X_n, \ldots\) be a sequence of random variables defined on a given probability space and assume that

(i) \(\lim_{n \to \infty} \mathbb{E}X_n = m\), for some constant \(m \in \mathbb{R}\).

(ii) \(\text{Var}(\sum_{i=1}^{n} X_i) = O(n^t)\) for some \(0 < t < 2\).

(iii) \(X_i \geq M (i = 1, 2, \ldots)\), for some \(M \in \mathbb{R}\).

Then \(\frac{1}{n} \sum_{i=1}^{n} X_i\) converges almost surely to \(m\).

Using this lemma, we will prove the following law of large numbers for \(\log R_n\).
Theorem 3.9. Assume (H). Then for almost all \( x \in (0, 1) \),
\[
\lim_{n \to +\infty} \frac{1}{n} \left( \log R_1(x) + \log R_2(x) + \cdots + \log R_n(x) \right) = 1. \tag{3.7}
\]

Proof. We apply Lemma 3.8 to \( X_j = \log R_j \). It is easy to see that \( X_j \geq 0 \) because \( R_j \geq 1 \).

Comparing \( z_j = (h_j(d_j) + 1)x_{j+1} \) with the definition of \( R_j \), we get
\[
X_j = -\log z_j + O \left( \frac{1}{d_{j+1}} \right)
\]
with a uniform constant involved in \( O(\cdot) \). Then by Theorem 3.3 and Proposition 3.1, we obtain
\[
\mathbb{E} X_j = 1 + O \left( \left( \frac{11}{12} \right)^j \right), \quad \mathbb{E} (X_j^2) = 2 + O \left( \left( \frac{11}{12} \right)^{j/2} \right). \tag{3.8}
\]

Now we estimate \( E(X_i X_j), i \neq j \). We assume \( i < j \).

\[
\mathbb{E}(X_i X_j) = \int_0^1 \left( \log \frac{d_{i+1}}{h_i(d_i) + 1} \right) \left( \log \frac{d_{j+1}}{h_j(d_j) + 1} \right) d\lambda
\]
\[
= \sum_{d_1, \ldots, d_{j+1}} \left( \log \frac{d_{i+1}}{h_i(d_i) + 1} \right) \left( \log \frac{d_{j+1}}{h_j(d_j) + 1} \right) \lambda(B(d_1, \ldots, d_{j+1}))
\]
\[
= \sum_{d_1, \ldots, d_j} \left( \log \frac{d_{i+1}}{h_i(d_i) + 1} \right) \lambda(B(d_1, \ldots, d_j))
\]
\[
\times \left( \sum_{d_{j+1} \geq h_j(d_j) + 1} \left( \log \frac{d_{j+1}}{h_j(d_j) + 1} \right) \Psi(h_j(d_j) + 1, d_{j+1}, y_j) \right),
\]

where the summation takes over all admissible vectors \( (d_1, d_2, \ldots, d_{j+1}) \). By applying Lemma 2.12, the inner sum is bounded by a constant \( M \). So,
\[
E(X_i X_j) \leq M \cdot E(X_i). \tag{3.9}
\]

Combining (3.8) and (3.9), we get
\[
\mathbb{E} \left( \left( \sum_{i=1}^n X_i \right)^2 \right) = O(n). \tag{3.10}
\]

Thus \( \{X_j\} \) satisfies all conditions in Lemma 3.8. This completes the proof. \( \square \)

For \( R_n \), we also have the following weak law of large numbers.

Theorem 3.10. Assume (H). Then the sequence \( \frac{1}{n \log n} \sum_{j=1}^n R_j(x) \) converges in law to 1.
Proof. Fix $n$. For any $1 \leq k \leq n$, define

$$U_k(x) = R_k(x) \cdot 1_{\{R_k(x) \leq n \log n\}}(x) \quad \text{and} \quad V_k(x) = R_k(x) \cdot 1_{\{R_k(x) > n \log n\}}(x).$$

Set $A_n = \{x \in (0, 1): R_k = U_k, 1 \leq k \leq n\}$. Then

$$\lambda \left\{ x \in (0, 1): \left| \frac{1}{n \log n} \sum_{k=1}^{n} R_k(x) - 1 \right| > \epsilon \right\} \leq \lambda \left\{ x \in A_n: \left| \frac{1}{n \log n} \sum_{k=1}^{n} U_k(x) - 1 \right| > \epsilon \right\} + \lambda \left( A_n^c \right).$$

By (3.4), we have

$$\lambda \{ x \in (0, 1): V_k(x) \neq 0 \} = \lambda \{ x \in (0, 1): R_k(x) > n \log n \} \leq \frac{2}{n \log n + 1},$$

thus $\lambda \left( A_n^c \right) \leq \sum_{k=1}^{n} \frac{2}{n \log n + 1} \to 0$ as $n \to \infty$.

The following is devoted to showing that

$$\lambda \left\{ x \in (0, 1): \left| \frac{1}{n \log n} \sum_{k=1}^{n} U_k(x) - 1 \right| > \epsilon \right\} \to 0.$$

Using similar arguments in estimating $E(X_i X_j)$ in Theorem 3.9, and by Lemma 2.13 prepared before and Proposition 3.1, we have, for all $1 \leq k \leq n$,

$$E(U_k(x)) = \log n + o(\log n), \quad \text{Var} \left( \sum_{k=1}^{n} U_k(x) \right) = o(n^2(\log n)^2). \quad (3.11)$$

Therefore, by Chebyshev inequality, we get

$$\lambda \left\{ x \in (0, 1): \left| \sum_{k=1}^{n} U_k(x) - \sum_{k=1}^{n} E(U_k(x)) \right| > \epsilon \sum_{k=1}^{n} E(U_k(x)) \right\} = o(1).$$

That is to say, $\sum_{k=1}^{n} \frac{1}{E(U_k(x))} \sum_{k=1}^{n} U_k(x)$ converges in probability to 1. Since (3.11) gives $\lim_{n \to \infty} \frac{E(U_k(x))}{\log n} = 1$, then we get $\lim_{n \to \infty} \frac{1}{n \log n} \sum_{k=1}^{n} E(U_k(x)) = 1$. Thus $P(A_n) \to 0$ as $n \to \infty$. □

3.4. Central limit theorem for $\log R_n(x)$

In this subsection, we consider some central limit theorems related to $R_n(x) = \frac{d_{n+1}(x)}{h_n(d_n(x)) + 1}$.

We will show that $R_n^{-1}$ converges in law to the uniform distribution on $[0, 1]$ and $\log R_n$ satisfies the central limit theorem (CLT).
Proposition 3.11. Assume (H). We have for any \( 0 < c < 1 \),
\[
\lim_{j \to \infty} \lambda \{ x \in (0, 1) : R_n^{-1} < c \} = c.
\] (3.12)

Proof. It is easy to know that \( z_j \leq R_n(x)^{-1} < z_j + \frac{1}{d_{j+1}} \). Then by Theorem 3.2 that \( d_j \to +\infty \) for almost all \( x \in (0, 1) \) and by Theorem 3.3 that \( Z_j \to U(0, 1) \), we get (3.12). □

Theorem 3.12. Assume (H). Then \( \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \log R_n \) converges in law to the normal distribution \( N(1, 1) \).

Proof. By the continuity theorem of characteristic functions [13, p. 191], we only need to show that the characteristic function of \( (\log R_1 + \log R_2 + \cdots + \log R_{n} - n) \cdot n^{-1/2} \), denoted by \( \psi_n(t) \), converges to \( e^{-\frac{1}{2} t^2} \).

Let
\[
\varphi_n(t) = E(e^{it(\log R_1 + \cdots + \log R_n)}) \quad \text{and} \quad \psi_n(t) = \varphi_n \left( \frac{t}{n^{1/2}} \right) e^{-itn^{1/2}}.
\]

Similar to estimate \( E(X_i X_j) \) in Theorem 3.9, in the light of Lemma 2.14, we have
\[
\left| \varphi_n(t) - \frac{1}{1-it} \varphi_{n-1}(t) \right| \leq |t| \cdot \left( \frac{11}{12} \right)^n.
\]

Thus
\[
\left| \varphi_n(t) - \left( \frac{1}{1-it} \right)^n \right| \leq \sum_{j=0}^{n-1} |t| \cdot \left( \frac{11}{12} \right)^{n-j} \leq M_1 \cdot |t|
\]
for some absolute constant \( M_1 \).

Set \( \varphi_n(t) = \beta^n(t) + D_n(t) \cdot t \) with \( |D_n(t)| \leq M_1 \), where \( \beta(t) = \frac{1}{1-it} \). We have
\[
\psi_n(t) = \left( \beta^n \left( \frac{t}{n^{1/2}} \right) + D_n \left( \frac{t}{n^{1/2}} \right) \cdot \frac{t}{n^{1/2}} \right) e^{-itn^{1/2}} \to e^{-\frac{1}{2} t^2} \quad \text{as} \quad n \to \infty. \] □

3.5. Some special cases

In this subsection, we give some applications of Theorems 3.9, 3.10 and 3.12.

Example 1. Engel continued fraction expansion case: \( h_j(n) = n - 1 \) for all \( j \geq 1 \) and \( n \geq 1 \). Then \( R_n(x) = \frac{d_{n+1}(x)}{d_n(x)} \) for each \( n \geq 1 \). The results in Theorems 3.9, 3.10 and 3.12 are stated as follows.

Corollary 3.13. (See [11].) For Engel continued fraction expansion, we have

(i) \( \frac{\log d_{n+1}(x)}{\sqrt{n}} \) converges in law to \( N(1, 1) \).
(ii) For almost all \(x \in (0, 1)\), \(\lim_{n \to \infty} \frac{1}{n} \log d_n(x) = 1\).

(iii) \(\frac{1}{n \log n} \sum_{j=1}^{n} \frac{d_{j+1}(x)}{d_j(x)}\) converges in law to 1.

**Example 2.** Sylvester continued fraction expansion case: \(h_j(n) = n(n - 1)\) for all \(j \geq 1\) and \(n \geq 1\). Then \(R_n(x) = \frac{d_{n+1}(x)}{d_n(x)(d_n(x) - 1) + 1}\) for each \(n \geq 1\).

**Corollary 3.14.** For Sylvester continued fraction expansion, we have

(i) \(\log \left| x - \frac{p_n(x)}{q_n(x)} \right| = -\left(1 + o(1)\right) \sum_{j=1}^{n} \log d_j\) \hspace{1cm} (4.1)

**Proof.** Similar to (2.6), since \(\frac{1}{2d_{n+1}} < x_{n+1} \leq \frac{1}{d_{n+1}}\) and \(d_nq_{n-1} < q_n \leq 2d_nq_{n-1}\), we have

\[
2^{-n-2} \frac{\prod_{j=1}^{n} R_j}{\prod_{j=1}^{n} d_j} \leq \left| x - \frac{p_n}{q_n} \right| \leq \frac{\prod_{j=1}^{n} R_j}{\prod_{j=1}^{n} d_j}. \hspace{1cm} (4.2)
\]

By Theorems 3.2 and 3.9, respectively, we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \log d_j = \lim_{n \to \infty} \log d_n = +\infty, \quad \sum_{j=1}^{n} \log R_j = n + o(n) \quad \lambda\text{-a.e.}
\]

Thus \(\sum_{j=1}^{n} \log R_j = o(1) \sum_{j=1}^{n} \log d_j\). As a consequence, we get the claim. \(\square\)

Now we consider the special cases when \(h_j(n)\) are polynomials of \(n\) for all \(j \geq 1\).

**Corollary 4.2.** Suppose for all \(j \geq 1\), \(h_j(n) = An + B\) with integers \(A \geq 1\) and \(B\) satisfying \(An + B \geq n - 1\) for any \(n \geq 1\), then for almost all \(x \in (0, 1)\),

\[
\lim_{n \to \infty} \frac{1}{n^2} \log \left| x - \frac{p_n(x)}{q_n(x)} \right| = -\frac{1}{2} (1 + \log A). \hspace{1cm} (4.3)
\]
Proof. For this special case, an application of Theorem 3.9 yields
\[ \lim_{n \to \infty} \frac{1}{n} \log d_n(x) = 1 + \log A, \quad \lambda\text{-a.e.} \]  

Definition 4.3. Let \( t > 1 \). We say that the function \( h_j(n) \) is of the order \( t \) if there exist constants \( 0 < K_1 \leq K_2 \) (independent of \( j \)) such that
\[ K_1 \leq \frac{h_j(n)}{n^t} \leq K_2 \quad \text{for all large } n. \]

Theorem 4.4. Let \( h_j(n) \) be of order \( t > 1 \). Then for almost all \( x \in (0, 1) \),
\[ \lim_{n \to \infty} \frac{1}{t^n} \log d_n(x) = G(x) \]  
(4.4)

where
\[ G(x) = t^{-1} \left\{ \log d_1(x) + \sum_{n=1}^{+\infty} t^{-n} \log \frac{d_{n+1}(x)}{d_n(x)} \right\}. \]  
(4.5)

Proof. It suffices only to show that \( G \) is meaningful almost everywhere, since
\[ \sum_{n=1}^{N} \frac{1}{t^n} \log \frac{d_{n+1}(x)}{d_n(x)} = t^{-N} \log d_{N+1}(x) - \log d_1(x). \]

By the definition of \( z_n(x) = x_{n+1}(h_n(d_n(x)) + 1) \), and \( \frac{1}{d_{n+1}+1} < x_{n+1} \leq \frac{1}{d_n+1} \), we get, when \( n \) is large enough,
\[ \frac{K_1}{z_n(x)} \leq \frac{d_{n+1}(x)}{(d_n(x))^t} \leq \frac{d_{n+1}(x)}{h_n(d_n(x)) + 1} \frac{h_n(d_n(x)) + 1}{(d_n(x))^t} \leq \frac{2K_2}{z_n(x)}. \]

Recalling the distribution of \( z_j \) in Theorem 3.3, we get \( \mathbb{E}(|\log \frac{d_{n+1}(x)}{d_n(x)}|) \leq M_1 \). Moreover
\[ \mathbb{E}(\log d_1(x)) = \sum_{d_1=1}^{\infty} \log d_1 \cdot \frac{1}{d_1(d_1 + 1)} := M_2 < \infty. \]

These two integrations assert that \( \int_0^1 G(x) \, dx < +\infty \) which implies \( G(x) \) is meaningful for almost all \( x \in (0, 1) \).  

Corollary 4.5. If \( h_j(n) \) is of order \( t > 1 \), then for almost all \( x \in (0, 1) \),
\[ \lim_{n \to \infty} \frac{1}{t^n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right| = -\frac{t}{t-1} G(x), \]  
(4.6)

where \( G(x) \) is defined by (4.5).
From Corollaries 4.2 and 4.5, we have the following corollary immediately.

**Corollary 4.6.** The Lebesgue measure of those $x \in (0, 1)$ which are of same Engel and Sylvester continued fraction expansions is 0.

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**References**