IUTAM Symposium on 50 Years of Chaos: Applied and Theoretical

Nonlinear phenomena in hysteretic systems

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Abstract

This paper discusses the different transitions from periodicity to quasi-periodicity, intermittency and chaos in systems with hysteresis. Three types of hysteresis are considered: the Bouc-Wen law of hysteresis, the Masing law and the pseudoelastic constitutive law typical of shape memory alloys. The first two rate-independent models do not account for heat transfers, while in the third case the thermodynamic transformations are taken into account. It is shown that these systems share similar trends for the loss of stability of the fundamental response to highly nonlinear responses of various kinds.

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1. Introduction

Hysteresis is a pervasive phenomenon in mechanics and nanomechanics (e.g., elastoplasticity, pseudoelasticity, hysteretic friction in materials and systems, stick-slip motions, carbon nanotube-matrix interaction, are only a few examples) as well as in magnetism or ferroelectricity. The constitutive modeling of materials, including smart materials such as shape-memory alloys, electroactive foams, etc., must account for the description of hysteresis as a substantial feature of their physical nonlinear behaviors. The constitutive laws, often based on the introduction of internal variables devoted to the representation of the memory effects, may feature softening, hardening, pinching, thermomechanical coupling. Hysteresis thus delivers a highly nonlinear signature to the associated dynamic responses. Besides the theoretical interest towards nonlinear dynamics of such systems, there is a parallel technological interest in exploiting advantageously hysteresis in engineering applications such as in vibration absorbers or isolators.

In the 70s the analytical study of the dynamics of one-dof hysteretic systems led to important observations such as that of Iwan [1] according to which in a system with double bilinear hysteresis, jumps occur in the softening-type responses giving rise to multi-stability. This work initiated a wealth of
studies addressing nonlinear dynamics of hysteretic systems which highlighted new phenomena besides multi-stability, such as bifurcations, quasi-periodicity, and chaos. In the context of these studies, Vestroni and co-workers showed [2,3] that stiffness-degrading and stiffness-strength degrading systems exhibiting pinching do possess jumps at the fold bifurcations, thus proving the existence of multi-stable ranges with multiple coexisting solutions.

The use of continuation techniques [4] highlighted the fact that systems with softening Bouc-Wen hysteresis can exhibit multiple coexisting periodic attractors, such as quasi-periodic solutions through Neimark-Sacker bifurcations, phase-locked solutions, and chaotic attractors through torus breakdown. Shape-memory alloy (SMA) materials [5,6] exhibit hysteresis due to the phase transformations with recovery of the original shape. SMAs have been attracting interest in various applications in which continuous SMA elements are used in the form of wires, bars, rods often subject to dynamic excitations. The remarkable properties of SMAs motivate different applications in fields such as bioengineering, aerospace and civil engineering; examples are actuators for vibration and buckling control of flexible structures, fasteners, thermally-actuated switches, self-deployable structures, etc. SMA wires embedded in composite materials are also being used to modify appropriately their mechanical characteristics.

In a one-dof thermomechanical oscillator with a restoring force typical of a pseudoelastic material [7], the authors found that chaos emerges through period-doubling cascades both in the primary-resonance and superharmonic-resonance regions. The temperature effects were taken into account in a thermomechanical framework where the fast loading rates associated with nearly adiabatic conditions led to quasiperiodic responses via Neimark-Sacher bifurcations.

Other studies highlighted the occurrence of chaos in a von Mises (two-bar) SMA truss [8] or the transition from chaos to hyperchaos in a system of two coupled SMA oscillators [9]. These transitions were studied mostly by calculating the Lyapunov exponents. On the other hand, in [10] the method of Wandering Trajectories was employed to detect the sensitivity to initial conditions of the orbits of a thermomechanical SMA oscillator. In continuous SMA systems, nonlinear multi-mode responses are affected by highly nonlinear transfer of energy between the modes due to internal resonances which can give rise to a rich variety of nonlinear dynamical behaviors and transitions to chaos.

This work recasts in a unified framework the mentioned phenomenological models of hysteresis and discusses the common features underlined by the transitions to quasiperiodicity and chaos. Recent findings obtained in the context of a hysteretic vibration absorber [12,13] are shown to confirm experimentally these scenarios.

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<th>Nomenclature</th>
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2. Phenomenological Models of Hysteresis

There are various phenomenological models of hysteresis such as the Bouc-Wen model, the Masing model, and the model for pseudoelastic constitutive behavior, all characterized by the introduction of a differential equation regulating a suitable internal variable by which the evolution of the memory effects is described.

2.1. The modified Bouc-Wen model of hysteresis

A modified nondimensional version of the classical Bouc-Wen law is proposed to account for the hardening behavior of the constitutive function expressed by

\[ N_{BW}^u = \delta u + \beta u^3 + (1-\delta)z, \quad \text{with} \quad \dot{z} = [1-(\gamma+\beta\text{sgn}(\dot{u}))^n]z^\| \dot{u} \]

where the overdot indicates differentiation with respect to time, the real parameter \( \delta \) has the meaning of ratio between the post-yielding stiffness (defined in the limit case for \( n \to \infty \)) and the total stiffness while \( \beta \) is the coefficient of the cubic term. The constitutive parameters \( (\beta, \gamma, n) \) govern the shape of the hysteretic part of the restoring force. It is possible to obtain three different constitutive behaviors during loading: (a) softening hysteresis if \( \gamma + \beta > 0 \) (the corresponding loops are shown in Fig. 1a), (b) quasilinear hysteresis if \( \gamma + \beta = 0 \), (c) hardening hysteresis if \( \gamma + \beta < 0 \). This model is governed by five parameters.

2.2. The Masing model of hysteresis

The Masing model is based on the experimental observation that the loading and unloading branches of the steady-state hysteretic response, for cyclically stabilized materials such as metals in a uniaxial stress state, are geometrically similar to the virgin loading curve with a two-fold magnification. If \( f(u,z) = 0 \) is the virgin loading curve, unloading and reloading take place on response curves described by

\[ f((u-u_l)/2,(z-z_l)/2) = 0 \]

where \((u_l,z_l)\) is the latest point of velocity reversal. There is an additional set of rules that defines the evolution of the constitutive function: (a) the equation of any hysteretic response curve can be obtained by applying the original Masing rule to the virgin loading curve using the latest point of velocity reversal; (b) the unloading and reloading branches follow the virgin loading curve whenever the latest achieved maximum displacement (or force) is overcome; and (c) when an interior curve under continued loading or unloading crosses a curve described in a previous load cycle, the force-deformation curve follows that of the previous cycle. The constitutive function is expressed by the following equations where the virgin loading curve \( g \) is given by the Ramberg-Osgood constitutive law:

\[ N^M = (1-\alpha)g(u) + \alpha z, \quad \dot{z} = g'_e((z-z_l)/2)\dot{u}, \quad g = \delta u + (1-\delta)u/[(1+|u|^n)^{1/n}] \]

where \( g'_e \) indicates the derivative of \( g \) with respect to \( u \). This model depends on three parameters, \( (\delta, \alpha, n) \). The parameter \( \alpha \) allows to tune the amount of hysteretic dissipation, \( \alpha = 1 \) corresponds to maximum hysteretic dissipation whereas \( \alpha = 0 \) corresponds to purely nonlinearly elastic behavior. The modified Masing model has the significant capability of describing the pinching phenomenon (see Fig. 1b) exhibited by materials and systems due to (a) encounter of additional stiffness or drop of stiffness, (b) strength in one direction of loading being different from the strength in the other direction.

2.3. The pseudoelastic constitutive model

Since the peculiar response of pseudoelastic (shape-memory) materials is determined by the austenite-martensite phase transformations [6], the fraction \( \xi \in [0,1] \) of martensite is introduced as the internal
variable that describes the evolution of the phase transformations [7]. In a fully martensitic state, denoted by $M$, $\xi = 1$, while in a fully austenitic state denoted by $A$ it is $\xi = 0$. The outstanding feature of the phase transformation is that a length increase is observed when $\xi$ increases ($A \rightarrow M$ forward transformation) and vice versa. This effect is inherent in the material microstructure and is taken into account in the model by the material parameter $\delta$ defined as $\delta := u(\xi = 1) - u(\xi = 0)$. Hence, $\delta$ represents the maximum transformation displacement. The pseudoelastic constitutive equation when thermal effects are neglected is given by

$$N^{SMA} = k(u - \text{sgn}(u)\delta\xi), \quad \dot{\xi} = G\dot{\Pi}, \quad \Pi = k\delta(|u| - \delta\xi) \quad (3)$$

where the differential law for $\dot{\xi}$ is the transformation kinetic, $\Pi$ is known as the thermodynamic driving force, and $G(|u|, \xi, \text{sgn}(\xi))$ is a hysteresis operator defined as

$$k_1(1-\xi)[1 + \tanh(k_1\Pi + k_2)] \text{ if } \dot{\xi} > 0 \text{ or } k_3\xi[1 - \tanh(k_3\Pi + k_4)] \text{ if } \dot{\xi} < 0 \quad (4)$$

The identification of the parameters $k_i$ from the force–displacement experimental curves at constant temperature is obtained considering that the forces $N_{Ms}$ and $N_{Mt}$ at which the forward transformation $A \rightarrow M$ starts and finishes, respectively, are experimentally measurable quantities. The same holds for the inverse transformation. The following nondimensional parameters are introduced: $q_1 = N_{Ms}/N_{Ms}$, $q_3 = N_{Ms}/N_{Mt}$, $\lambda = \delta/um_s$ where $um_s$ is the displacement at the start of $A \rightarrow M$. To preserve the symmetry of the loops, the condition $q_3 = N_{Ms}/N_{Ms} = (1 + q_3)/q_3$ is enforced. An additional parameter, the residual Martensite at the conventional start of the $A \rightarrow M$ transformation denoted by $\xi_r$ regulates the smoothness of the transition around $N_{Ms}$ and $N_{Mt}$.

3. Transitions to quasi-periodicity, phase-locking, and chaos

In previous studies by the authors [4,7], a pseudo-arclength continuation strategy was devised by employing the Poincaré map for detecting through its fixed points the periodic solutions. The map computed via finite differences in state space perturbing in all state and control parameter directions. The continuation method together with the Floquet theory enables the unfolding of the bifurcations of harmonically forced single-dof hysteretic systems governed by at least three state equations

$$\dot{u} = v, \quad \dot{v} = -2\xi v - N(u, \dot{u}, z) + f \cos(\Omega t), \quad \dot{z} = Z(|u|, z, \text{sgn}(z)) \quad (5)$$

where $z$ denotes the internal variable (hysteretic part of the restoring force, or phase fraction), $Z$ is a general hysteresis operator, $\xi$ is the excitation frequency $\Omega$ is elected as control parameter. For the thermomechanical pseudoelastic oscillator, the temperature is the fourth state variable. The continuation analysis in $\Omega$ yields the ranges where the fundamental periodic response loses the stability through various bifurcations. Subsequently, the numerical construction of
bifurcation diagrams unfolds very rich scenarios of transition to quasiperiodicity, phase locking and chaos highlighting the prominent role of the different nonlinear hysteretic material behaviors.

The bifurcation diagram in Fig. 2c of the undamped Bouc-Wen softening oscillator in the superharmonic resonance region shows the quasi-periodicity that arises out of the Neimark-Sacher bifurcation and the repeated occurrence of phase-locking/synchronization in the zoomed-in diagram in Fig. 2d. A thorough exploration of the dynamics near phase-locking has revealed the occurrence of the intermittency phenomenon [4]. For frequencies past the frequency-locking phenomenon, the ratio between the modulation frequency and the carrier frequency is very close to a rational number and, as a result, an orbit on this attractor spends long stretches of time near the ghost or phantom of the frequency-locked orbit, from which it unlocks and then relocks nearly periodically. It was found that chaos emerged in this oscillator out of a torus breakdown scenario. The type of hysteresis here explored (see Fig. 2b) resembles very closely that exhibited by a magnetic nanocomposite whose experimental loop is reported in Fig. 2a [11]. Moreover, interesting features were found in the microhysteresis exhibited by carbon nanotube-nanocomposites due to the stick-slip nanomotions of the weakly bonded carbon nanotubes [12].

Fig. 2. (a) Bifurcation diagrams of the Bouc-Wen hysteretic oscillator for the following set of parameters: \( n=1, \delta_1=0, \delta = 1/21, \beta = 0.005, \gamma = 1.0 \). Part (b) is an enlargement of (a) in a narrow frequency band. The hysteretic loop shown in part (a) is the experimental room-temperature loop of Fe2O3/SiO2 nanocomposite [12].

On the other hand, for the Masing and the isothermal pseudelastic oscillators which have in common a pinching-type behavior, most of the rich nonlinear dynamics are triggered by superharmonic resonances of order one-third and one-fifth. Under these resonances, the response loses its stability by a symmetry-breaking bifurcation, followed by complete sequences of period-doubling bifurcations leading to chaos. One such example is the bifurcation diagram shown in Fig. 3a together with the phase portrait and restoring force of a typical chaotic attractor. Within the full bubble structure culminating into chaos, a boundary crisis was found, as well as fold bifurcations leading to period-five solutions and subsequent period-doubling cascades. The dynamics are profoundly modified if the constitutive model of pseudelastic materials is enriched by the description of the heat exchanges and temperature variations which affect the phase transformation forces. In general, the temperature increases the apparent stiffness of the oscillator and the responses do not undergo jumps at all, are periodic and are stable in a relatively large frequency range. The most interesting scenario occurs when the system is in nearly adiabatic conditions in which the heat exchanges take place very slowly. Under these conditions, the fundamental response of the 4-dimensional oscillator (i.e., endowed with four state variables) loses its stability via a Neimark-Sacher bifurcation out of which a
surprisingly interesting response is born. The amplitude-modulated responses exhibit a very large modulation period which turns out to be of the order of the characteristic time regulating the mechanism of heat convection described by Newton’s law of cooling. Moreover, the distinguished character of the response frequency content is that, around the excitation frequency and its odd harmonics, a dense distribution of side bands is developed which makes the shape of the amplitude modulation quite complex.

![Fig. 3. (a) force Bifurcation diagram of the shape memory oscillator subject to a superharmonic resonance of order one-third; (b) phase portrait; (c) hysteresis loops of the pseudoelastic](image)

4. **A physical system: the hysteretic vibration absorber**

In a recent series of works [13,14], a nonlinear hysteretic tuned mass damper was devised and experimentally investigated. The restoring force of the secondary mass of the device is provided by short wire ropes under flexure (see Figs. 4 a,b,c). The hysteresis loops of the steel wire ropes, acquired by using an MTS Universal testing machine and a load transducer, exhibit softening or hardening hysteresis depending on whether the clamps of the wire ropes are laterally unconstrained (Fig. 4e) or constrained (Fig. 4f) when they are displaced in the vertical direction. These force-displacement loops are identified by the Bouc-Wen law given by Eq. (1) and represented by the solid lines.

The continuation analysis performed on the two-dof system represented by the main structure and the absorber hysteretic hardening mass reveal the occurrence of Neimark-Sacher bifurcations in the region of the out-of-phase mode (denoted by points A,B,C,D along the frequency-response curves in Fig. 5, left). The bifurcation diagrams in the frequency bands between the Neimark-Sacher bifurcations shown in Fig. 5 (right) exhibit the same scenarios of quasiperiodicity, phase-locking, chaos found in the simpler context of the one-dof Bouc-Wen oscillator. The practical relevance of this finding is that the emergence of these attractors is highly beneficial to the absorber performance since the amplitude-modulated, high-periodicity, and chaotic responses exhibit lower amplitudes than the corresponding softening mode of operation of the absorber where the resonance of the out-of-phase mode is stable gives rise to large-amplitude responses.
Concluding remarks

The bifurcation scenarios leading to chaos have been extensively studied in the literature in the context of deterministic nonlinear systems with emphasis on geometric nonlinearities. Far less attention has been placed on the role of material nonlinearities which become prominent for new classes of smart materials employed as dynamic actuators, vibration absorbers, energy harvesting systems. In such systems, nonlinear couplings arising from internal resonances due to the material nonlinearities are responsible for bifurcations leading to attractors of various kinds, including quasi-periodic attractors, phase-locked. The bifurcation scenarios leading to chaos have been extensively studied in the literature in the context of deterministic nonlinear systems with emphasis on geometric nonlinearities. Far less attention has been placed on the role of material nonlinearities which become prominent for new classes of smart materials employed as dynamic actuators, vibration absorbers, energy harvesting systems. In such systems, nonlinear couplings arising from internal resonances due to the material nonlinearities are responsible for
bifurcations leading to attractors of various kinds, including quasi-periodic attractors, phase-locked solutions, and chaotic attractors. The diverse transitions to highly nonlinear responses and chaos documented in this study lead to the following main conclusions.

Primary resonances exhibit, for most hysteretic and pseudoelastic materials, softening or softening behaviors at low amplitude and hardening at higher amplitudes. Superharmonic resonances (of order one third and one-fifth depending on the excitation amplitude) are strong triggers of nonlinear complex motions.

In purely hysteretic systems (a la Bouc-Wen), Neimark-Sacher bifurcations, synchronization, intermittency, are the precursors of chaos which emerge out of a torus breakdown. The same phenomenology was found in a vibration absorber where the restoring force is provided by wires ropes having hardening behavior. The phenomenology is beneficial to the absorber performance.

In systems with pseudoelasticity (shape-memory materials) and hysteretic systems with pinching, the route to chaos is through full cascades of period doublings. When the physics are enriched to include thermodynamics with heat exchanges, the overall dynamical behaviors are strongly modified. The heat exchanges can drive the phase transformations and hence the mechanical variables in such a way that the ensuing dynamical responses are very slowly and densely modulated in time. This is a chapter of material behavior under dynamic excitations yet to be explored and opens many challenging and interesting opportunities.

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References