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Optimality conditions in a vector continuous-time optimization problem

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ABSTRACT

We consider nonlinear vector continuous-time optimization problems with inequality type of constraints. We derive new Fritz John and Karush–Kuhn–Tucker optimality conditions that are valid when all functions are preinvex.

1. Introduction

Let D be a nonempty subset of the Banach space $L_\infty^n[0, T]$ and let $f(t, x(t)) = \zeta(x)(t)$, $\zeta : D \to L_1^s[0, T]$ and $g(t, x(t)) = \xi(x)(t)$, $\xi : D \to L_1^m[0, T]$ be given functions. Here, by $L_\infty^n[0, T]$ we denote the space of all n-dimensional Lebesgue measurable essentially bounded functions defined on the interval $[0, T]$ and $L_1^n[0, T]$ denotes the space of all n-dimensional essentially bounded and Lebesgue integrable functions defined on the interval $[0, T]$ with the corresponding norms

$$\|x\|_\infty = \max_{1\le i\le n} \text{ ess sup}\{|x_i(t)|,\ 0 \le t \le T\}$$

and

$$\|x\|_1 = \max_{1\le i\le n} \int_0^T |x_i(t)|\, dt.$$

Note that, for each $t \in [0, T]$, $x_i(t)$ is the ith component of $x(t) \in R^n$.

Consider the following nonlinear continuous-time optimization problem:

$$J(x) = \int_0^T f(t, x(t))\, dt \to \inf; \tag{1}$$

s.t. $x \in \Omega$, where Ω is the set of all $x \in D$ for which the following vector inequality

$$g(t, x(t)) \le 0, \quad \text{a.e. in } [0, T], \tag{2}$$

is satisfied.

We assume that functions $t \to f(t, x(t))$ and $t \to g_i(t, x(t)),\ i = 1, \dots, m$ are Lebesgue measurable and integrable for all $x \in D$. Also, we assume that the admissible set Ω is nonempty.

Note that $g(t, x(t)) \le 0$ means that $g_i(t, x(t)) \le 0,\ i = 1, \dots, m$ and $g(t, x(t)) < 0$ means that $g_i(t, x(t)) < 0,\ i = 1, \dots, m$. Also, all vectors in our paper are column vectors and we use $'$ to denote transposition.

The minimization in the initial problem is in the sense of an efficient point.

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Definition 1.1. The point $\hat{x}$ is an efficient point of the problem (1)–(2) if there is no other $x \in \Omega$ such that

$$J(x) \leq J(\hat{x})$$

holds.

Scalar and vector continuous-time optimization problems have been intensively investigated in the past few years. For scalar optimization problems, we refer to [1,2] where all functions are locally Lipschitz and where the necessary and sufficient optimality conditions are obtained. Also, some related results can be found in [3–6].

The initial problem was considered in [7] with the minimization in the sense of a proper efficient solution. Among other interesting results, there are obtained Karush–Kuhn–Tucker optimality conditions with the assumption that the Slater condition holds. In this paper we derive new Fritz John and Karush–Kuhn–Tucker optimality conditions with the main characteristic that they depend on the set of constraints that are linear in x.

Now we will give definitions of an invex set and a preinvex function. For more information about generalized convexity, we refer to [8].

Definition 1.2. A subset S of a Banach space E is invex with respect to $\eta : S \times S \to E$ if for all $x_1, x_2 \in S$ and for each $\theta \in (0, 1)$,

$$x_2 + \theta\eta(x_1, x_2) \in S.$$

Definition 1.3. If S is invex with respect to $\eta : S \times S \to E$, a given function $p : S \to R^k$ is called preinvex with respect to η if for all $x_1, x_2 \in S$ and for each $\theta \in (0, 1)$,

$$p(x_2 + \theta\eta(x_1, x_2)) \leq \theta p(x_1) + (1 - \theta)p(x_2).$$

We will formulate the generalized Gordan's theorem that we shall use in our proof.

Theorem 1.1. *Let $D \subset L^n_\infty[0, T]$ and let $g : [0, T] \times R^n \to R^m$, where $g(t, x(t)) = \xi(x)(t)$, a.e. in $[0, T]$ and $\xi : D \to L^m_1[0, T]$. Let D be the invex set with respect to $\eta : D \times D \to L^n_\infty[0, T]$ and let g be preinvex in x on $[0, T]$ with respect to the same η. Then, exactly one of the following systems is consistent:*

I. *there exists $x \in D$ such that*

$$g(t, x(t)) < 0 \quad \text{a.e. in } [0, T];$$

II. *there exist $u \in L^m_\infty[0, T]$, $u(t) \geq 0$, $u(t) \neq 0$ a.e. in $[0, T]$ such that*

$$\int_0^T u'(t)g(t, x(t))\, dt \geq 0, \quad \forall x \in D. \tag{3}$$

The proof of the preceding theorem is given in [7], which is essentially based on the general idea from [9].

2. Fritz John optimality conditions

Let A be the subset of indices from the set $\{1, \ldots, m\}$ for which the functions g_i are nonlinear in x and let $\overline{A}$ be the set $\{1, \ldots, m\} \backslash A$, i.e., the subset of indices for which the functions g_i are linear in x. Similarly, by B we denote the set of indices for which the coordinate cost functions f_i are nonlinear in x and by $\overline{B}$ we denote the set $\{1, \ldots, s\} \backslash B$. Also, if all functions f_i and g_i are nonlinear in x we assume that $A = \{1, \ldots, m\}$ and $B = \{1, \ldots, s\}$. By f_A we denote all components of f with indices from the set A. According to the preceding notations we have that $u(t) = (u_A(t), u_{\overline{A}}(t))$ and $g(t, x(t)) = (g_A(t, x(t)), g_{\overline{A}}(t, x(t)))$.

Theorem 2.1. *Let the set D be invex with respect to $\eta : D \times D \to L^n_\infty$, let the functions $f(t, x(t)) = \zeta(x)(t)$ and $g(t, x(t)) = \xi(x)(t)$ be preinvex in x with respect to the same η. Let $\hat{x}$ be an efficient point of the problem (1)–(2). Assume that the following control qualifications are satisfied:*

CQ1: *there exists $x_1 \in \operatorname{int} D$ such that*

$$g_{\overline{A}}(t, x_1(t)) \leq 0, \quad \text{a.e. in } [0, T];$$

CQ2: *there exists $x_2 \in D$ such that*

$$g_{\overline{A}}(t, x_2(t)) < 0, \quad \text{a.e. in } [0, T].$$

Then there exist the multipliers $\hat{\lambda} \in R^s$, $\hat{u} \in L^m_\infty[0, T]$ such that the following conditions are satisfied:

0.

$$(\hat{\lambda}, \hat{u}_A(t)) \neq 0, \quad \text{a.e. in } [0, T],$$

1.

$$\hat{\lambda} \geq 0, \hat{u}(t) \geq 0 \quad \text{a.e. in } [0, T],$$

2.

$$\int_0^T \hat{u}'(t)g(t,\hat{x}(t))\,dt = 0,$$

3.

$$\int_0^T \hat{\lambda}'f(t,x(t))\,dt + \int_0^T \hat{u}'(t)g(t,x(t))\,dt \geq \int_0^T \hat{\lambda}'f(t,\hat{x}(t))\,dt + \int_0^T \hat{u}'(t)g(t,\hat{x}(t))\,dt, \quad \forall x \in D.$$

Proof. Let $i \in \{1,\dots,s\}$ be fixed. Put

$$J_i(x) = \int_0^T f_i(t,x(t))\,dt.$$

It is known (see [10]) that if $\hat{x}$ is an efficient point of the problem (1)–(2), then we have that there is no $x \in D$ such that the following system is consistent

$$J_i(x) < J_i(\hat{x}),$$
$$g(t,x(t)) < 0, \quad \text{a.e. in } [0,T],$$
$$\int_0^T f_j(t,x(t))\,dt < \int_0^T f_j(t,\hat{x}(t))\,dt, \quad j=1,\dots,s,\ j \neq i,\ \text{a.e. in } [0,T].$$

From the generalized Gordan's theorem we have that there exist the multipliers $\hat{\lambda}^i \in R$, $\hat{\beta}^i \in R^{s-1}$ and $\hat{u}^i \in L_\infty^m[0,T]$ such that

$$\hat{\lambda}^i \geq 0, \qquad \hat{\beta}^i \geq 0, \qquad \hat{u}^i(t) \geq 0, \quad (\hat{\lambda}^i, \hat{\beta}^i, \hat{u}^i(t)) \neq 0,\ \text{a.e. in } [0,T],$$

such that

$$\int_0^T \hat{\lambda}^i f_i(t,x(t))\,dt + \sum_{\substack{j=1\\ j\neq i}}^{s} \int_0^T \hat{\beta}_j^i f_j(t,x(t))\,dt + \int_0^T (\hat{u}^i)'(t)g(t,x(t))\,dt$$
$$\geq \int_0^T \hat{\lambda}^i f_i(t,\hat{x}(t))\,dt + \sum_{\substack{j=1\\ j\neq i}}^{s} \int_0^T \hat{\beta}_j^i f_j(t,\hat{x}(t))\,dt$$

holds for all $x \in D$.

Putting $x = \hat{x}$ we obtain

$$\int_0^T (\hat{u}^i)'(t)g(t,\hat{x}(t))\,dt \geq 0.$$

Since the point $\hat{x}$ is admissible, we have that the opposite inequality is also satisfied.

From the fact that

$$\int_0^T (\hat{u}^i)'(t)g(t,\hat{x}(t))\,dt = 0$$

we obtain that

$$\int_0^T \hat{\lambda}^i f_i(t,x(t))\,dt + \sum_{\substack{j=1\\ j\neq i}}^{s} \int_0^T \hat{\beta}_j^i f_i(t,x(t))\,dt + \int_0^T (\hat{u}^i)'(t)g(t,x(t))\,dt$$
$$\geq \int_0^T \hat{\lambda}^i f_i(t,\hat{x}(t))\,dt + \sum_{\substack{j=1\\ j\neq i}}^{s} \int_0^T \hat{\beta}_j^i f_i(t,\hat{x}(t))\,dt + \int_0^T (\hat{u}^i)'(t)g(t,\hat{x}(t))\,dt \tag{4}$$

holds for all $x \in D$.

Put $\hat{\lambda} = (\hat{\lambda}_1,\dots,\hat{\lambda}_s)$ and $\hat{u}(t) = (\hat{u}_1(t),\dots,\hat{u}_m(t))$, where

$$\hat{\lambda}_i = \hat{\lambda}^i + \sum_{\substack{j=1\\ j\neq i}}^{s} \hat{\beta}_i^j, \quad i=1,\dots,s,$$

and

$$\hat{u}_l(t) = \sum_{i=1}^{s} \hat{u}_l^i(t), \quad l = 1, \dots, m.$$

Summing inequalities (4), for $i = 1, \dots, s$, we obtain that for such $\hat{\lambda}$ and $\hat{u}(t)$ the conditions 1–3 are satisfied. □

Now, we shall prove that the condition 0 holds. We will suppose that is not true. Let

$$(\hat{\lambda}, \hat{u}_A(t)) = 0, \quad \text{a.e. in } [0, T].$$

From the condition 3 we obtain

$$\int_0^T \sum_{i \in \overline{A}} \hat{u}_i(t) g_i(t, x(t))\, dt \geq 0, \quad \forall x \in D.$$

For the point $x = x_1$, from CQ1, we have that

$$\int_0^T \sum_{i \in \overline{A}} \hat{u}_i(t) g_i(t, x_1(t))\, dt \leq 0$$

holds. It follows that

$$\int_0^T \sum_{i \in \overline{A}} \hat{u}_i(t) g_i(t, x_1(t))\, dt = 0.$$

Since the function

$$G(x) = \int_0^T \sum_{i \in \overline{A}} \hat{u}_i(t) \xi_i(x)(t)\, dt$$

is linear in x, nonnegative on the set D, and since x_1 belongs to the interior of the set D, we can conclude that the function $G(x)$ vanish on the whole set D. Specifically, for $x = x_2$, we have that

$$\int_0^T \sum_{i \in \overline{A}} \hat{u}_i(t) g_i(t, x_2(t))\, dt = 0$$

holds, which is the contradiction with CQ2. Indeed, from CQ2 and from the fact that $\hat{u}_{\overline{A}}(t) \neq 0$, a.e. in $[0, T]$, we obtain

$$\int_0^T \sum_{i \in \overline{A}} \hat{u}_i(t) g_i(t, x_2(t))\, dt < 0.$$

As an illustration, we will consider the following simple example from [10]:

$$\left(\int_0^1 |x_1(t) - t|\, dt, \int_0^1 |x_2(t) - t|\, dt \right) \to \inf;$$
$$-x_1(t) \leq 0, \quad \text{a.e. in } [0, 1],$$
$$-x_2(t) \leq 0, \quad \text{a.e. in } [0, 1],$$
$$x_1(t) + x_2(t) - 2t \leq 0, \quad \text{a.e. in } [0, 1].$$

It is obvious that $\hat{x}(t) = (\hat{x}_1(t), \hat{x}_2(t)) = (t, t)$ is an efficient point of the preceding problem. Since $D = L^2_\infty[0, 1]$ we have that the condition CQ1 is satisfied for any admissible point. Also, we can take that $x_1(t) = \frac{t}{2}$, $x_2(t) = \frac{t}{2}$ and we have that CQ2 holds. From Theorem 2.1 immediately follows that the multiplier that correspond to the cost functions must be nonzero. We have that necessary optimality conditions are satisfied for

$$\hat{\lambda} = (1, 1), \qquad \hat{u}_1(t) = \hat{u}_2(t) = 0, \qquad \hat{u}_3(t) = 1.$$

Remark 2.1. If the set D is open, or if $D = L^n_\infty[0, T]$, then CQ2 implies CQ1 and Theorem 2.1 can be proved only with the assumption that CQ2 holds.

Remark 2.2. It is obvious that the set $\overline{A}$ in CQ1 and CQ2 can be replaced with a set $\overline{A_1} \subset \overline{A}$ and in that case the condition 0 becomes

$$(\hat{\lambda}, \hat{u}_{A_1}(t)) \neq 0, \quad \text{a.e. in } [0, T],$$

where $A_1 = \{1, \dots, m\} \setminus \overline{A_1}$.

3. Karush–Kuhn–Tucker optimality conditions

With the additional assumption we can obtain Karush–Kuhn–Tucker optimality conditions for the initial problem.

Theorem 3.1. *Let the set D be invex with respect to $\eta : D \times D \to L^n_\infty$, let the functions $f(t, x(t)) = \zeta(x)(t)$ and $g(t, x(t)) = \xi(x)(t)$ be preinvex in x with respect to the same η. Let $\hat{x}$ be a global minimizer of the problem (1)–(2). Assume that the following conditions hold:*

CQ1: *there exists $x_1 \in \operatorname{int} D$ such that*

$$g_{\overline{A}}(t, x_1(t)) \le 0, \quad \text{a.e. in } [0, T];$$

CQ2: *there exists $x_2 \in D$ such that*

$$g_{\overline{A}}(t, x_2(t)) < 0, \quad \text{a.e. in } [0, T];$$

CQ3: *there exists an admissible point $x_3 \in D$ such that*

$$g_A(t, x_3(t)) < 0, \quad \text{a.e. in } [0, T].$$

Then there exist the multipliers $\hat{\lambda} \in R^s$, $\hat{u} \in L^m_\infty[0, T]$ such that the following conditions are satisfied:

0.

$$\hat{\lambda} \ne 0,$$

1.

$$\hat{u}(t) \ge 0 \quad \text{a.e. in } [0, T],$$

2.

$$\int_0^T \hat{u}'(t) g(t, \hat{x}(t))\, dt = 0,$$

3.

$$\int_0^T \hat{\lambda}' f(t, x(t))\, dt + \int_0^T \hat{u}'(t) g(t, x(t))\, dt \ge \int_0^T \hat{\lambda}' f(t, \hat{x}(t))\, dt + \int_0^T \hat{u}'(t) g(t, \hat{x}(t))\, dt, \quad \forall x \in D.$$

Proof. As in the proof of Theorem 2.1., we have that the conditions 1–3 are satisfied. Also, we have that

$$(\hat{\lambda}, \hat{u}_A(t)) \ne 0, \quad \text{a.e. in } [0, T]$$

holds. Let us assume that $\hat{\lambda} = 0$. In that case, the condition 3 becomes

$$\int_0^T \hat{u}'(t) g(t, x(t))\, dt \ge 0, \quad \forall x \in D.$$

According to the fact that CQ3 holds, we obtain that

$$\int_0^T \sum_{i \in A}^{m} \hat{u}_i(t) g_i(t, x_3(t))\, dt + \int_0^T \sum_{i \in \overline{A}}^{m} \hat{u}_i(t) g_i(t, x_3(t))\, dt < 0$$

which is the contradiction. □

Remark 3.1. Theorem 3.1 can be proved with the assumption that the classical Slater condition holds, i.e., without CQ1, CQ2 and CQ3. Note that by the fact that the Slater condition holds, we mean that there exists a point $\overline{x} \in D$ such that

$$g(t, \overline{x}(t)) < 0, \quad \text{a.e. in } [0, T].$$

However, if the set D is open or if $D = L^n_\infty[0, T]$, then we have that Theorem 3.1 was proved with the weaker assumptions, since the Slater condition implies CQ2 and CQ3.

Remark 3.2. With the assumptions of the preceding theorem the condition 3 can be formulated as the saddle point optimality criteria.

Define the function

$$L : D \times R^s \times L^m_\infty[0, T] \to R$$

by

$$L(x, \lambda, u) = \int_0^T \lambda' f(t, x(t))\, dt + \int_0^T u'(t) g(t, x(t))\, dt. \tag{5}$$

The condition 3 can be replaced with

$$L(\hat{x}, \lambda, u) \leq L(\hat{x}, \hat{\lambda}, \hat{u}) \leq L(x, \hat{\lambda}, \hat{u}), \tag{6}$$

for all $x \in D$, $u \in L_\infty^m[0, T]$, $u(t) \geq 0$ a.e. in $[0, T]$. Indeed, we have that

$$\int_0^T \lambda' f(t, \hat{x}(t))\, dt + \int_0^T \hat{u}'(t) g(t, \hat{x}(t))\, dt \geq \int_0^T \lambda' f(t, \hat{x}(t))\, dt + \int_0^T u'(t) g(t, \hat{x}(t))\, dt,$$

since $\lambda \geq 0$, $u(t) \geq 0$ a.e. in $[0, T]$ and $g(t, \hat{x}(t)) \leq 0$ hold.

Remark 3.3. The assertions of Theorem 3.1 are also sufficient for the optimality of the point $\hat{x}$. Indeed, if we put $u(t) \equiv 0$ in (6), we obtain

$$\int_0^T \hat{\lambda}' f(t, \hat{x}(t))\, dt \leq \int_0^T \hat{\lambda}' f(t, x(t))\, dt + \int_0^T \hat{u}'(t) g(t, x(t))\, dt.$$

For the admissible point x holds

$$\int_0^T \hat{\lambda}' f(t, \hat{x}(t))\, dt \leq \int_0^T \hat{\lambda}' f(t, x(t))\, dt.$$

According to the known fact (see [7]) that the solution of the so-called weighting scalar problem associated with the initial problem is the efficient solution for the initial problem, we have that

$$\int_0^T f(t, \hat{x}(t))\, dt \leq \int_0^T f(t, x(t))\, dt$$

holds.

Using the same approach, we can obtain Karush–Kuhn–Tucker optimality conditions where all multipliers that correspond to the nonlinear coordinate cost functions are nonzero. We will introduce constraint qualifications and regularity conditions in order to prove the assertions.

Theorem 3.2. *Let the set D be invex with respect to $\eta : D \times D \to L_\infty^n$, let the functions $f(t, x(t)) = \zeta(x)(t)$ and $g(t, x(t)) = \xi(x)(t)$ be preinvex in x with respect to the same η. Let $\hat{x}$ be a global minimizer of the problem (1)–(2). Assume that the following conditions hold:*

RCQ1: *there exists $x_1 \in \operatorname{int} D$ such that*

$$g_{\overline{A}}(t, x_1(t)) \leq 0, \quad \text{a.e. in } [0, T],$$
$$f_{\overline{B}}(t, x_1(t)) \leq f_{\overline{B}}(t, \hat{x}(t)), \quad \text{a.e. in } [0, T];$$

RCQ2: *there exists $x_2 \in D$ such that*

$$g_{\overline{A}}(t, x_2(t)) < 0, \quad \text{a.e. in } [0, T],$$
$$f_{\overline{B}}(t, x_2(t)) < f_{\overline{B}}(t, \hat{x}(t)), \quad \text{a.e. in } [0, T];$$

RCQ3: *there exists an admissible point $x_3 \in D$ such that*

$$g_A(t, x_3(t)) < 0, \quad \text{a.e. in } [0, T],$$
$$f_{\overline{B}}(t, x_3(t)) < f_{\overline{B}}(t, \hat{x}(t)), \quad \text{a.e. in } [0, T].$$

Then there exist the multipliers $\hat{\lambda} \in R^s$, $\hat{u} \in L_\infty^m[0, T]$ such that the following conditions are satisfied:

0.

$$\hat{\lambda}_B \neq 0,$$

1.

$$\hat{u}(t) \geq 0 \quad \text{a.e. in } [0, T],$$

2.

$$\int_0^T \hat{u}'(t) g(t, \hat{x}(t))\, dt = 0,$$

3.

$$\int_0^T \hat{\lambda}' f(t, x(t))\, dt + \int_0^T \hat{u}'(t) g(t, x(t))\, dt \geq \int_0^T \hat{\lambda}' f(t, \hat{x}(t))\, dt + \int_0^T \hat{u}'(t) g(t, \hat{x}(t))\, dt, \quad \forall x \in D.$$

Note that by $\hat{\lambda}_B$ we denote the set of multipliers $\hat{\lambda}_i$ from $\hat{\lambda}$ that correspond to the nonlinear coordinate cost functions.

Proof. As in the proof of Theorem 2.1., we have that the conditions 1–3 are satisfied. Moreover, from RCQ1 and RCQ2 we have that

$$(\hat{\lambda}_B, \hat{u}_A(t)) \neq 0, \quad \text{a.e. in } [0, T]$$

holds. If we assume that $\hat{\lambda}_B = 0$, then the condition 3 becomes

$$\int_0^T \hat{\lambda}'_{\overline{B}} f_{\overline{B}}(t, x(t))\, dt + \int_0^T \hat{u}'(t) g(t, x(t))\, dt \geq \int_0^T \hat{\lambda}'_{\overline{B}} f_{\overline{B}}(t, \hat{x}(t))\, dt,$$

for all $x \in D$. According to the fact that RCQ3 holds, we obtain that

$$\int_0^T \hat{\lambda}'_{\overline{B}} f_{\overline{B}}(t, x_3(t))\, dt + \int_0^T \hat{u}'(t) g(t, x_3(t))\, dt < \int_0^T \hat{\lambda}'_{\overline{B}} f_{\overline{B}}(t, \hat{x}(t))\, dt$$

which is the contradiction. □

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References

[1] A.J.V. Brandao, M.A. Rojas-Medar, G.N. Silva, Nonsmooth continouous-time optimization problems: necessary conditions, Comput. Math. Appl. 41 (2001) 1447–1456.
[2] A.J.V. Brandao, M.A. Rojas-Medar, G.N. Silva, Nonsmooth continouous-time optimization problems: sufficient conditions, J. Math. Anal. Appl. 227 (1998) 305–318.
[3] R. Bellman, Bottleneck problems and dynamic programming, Proc. Natl. Acad. Sci. USA 39 (1953) 947–951.
[4] M.A. Hanson, B. Mond, A class of continuous convex programming problems, J. Math. Anal. Appl. 22 (1968) 427–437.
[5] G.J. Zalmai, The Fritz John and Kuhn–Tucker optimality conditions in continuous-time nonlinear programming, J. Math. Anal. Appl. 110 (1985) 503–518.
[6] G.J. Zalmai, Optimality conditions and Lagrangian duality in continuous-time nonlinear programming, J. Math. Anal. Appl. 109 (1985) 426–452.
[7] V.A. de Oliveira, Vector continouous-time programming without differentiabillity, J. Comput. Appl. Math. 234 (2010) 924–933.
[8] S.K. Mishra, S.Y. Wang, K.K. Lai, Generalized Convexity and Vector Optimization, Springer, Berlin, 2009.
[9] G.J. Zalmai, A continuous-time generalization of Gordan's transposition theorem, J. Math. Anal. Appl. 110 (1985) 130–140.
[10] S. Nobakhtian, M.R. Pouryayevali, Optimality criteria for nonsmooth continuous-time problems of multiobjective optimization, J. Optim. Theory Appl. 136 (2008) 69–76.