Note

Sums of non-integral powers

Harold R. Parks

Department of Mathematics, Oregon State University, Corvallis, OR 97331, USA

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Abstract

We give an approximation to the sum of non-integral powers of positive integers which is a natural generalization of Faulhaber’s formula for the sum of integral powers of positive integers.

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Let \( k \) and \( N \) be positive integers. Faulhaber’s formula for the sum of the \( k \)th powers of the first \( N \) positive integers is

\[
\sum_{n=1}^{N} n^k = (k + 1)^{-1} \sum_{j=1}^{k+1} (-1)^{\delta_{jk}} \binom{k+1}{j} B_{k+1-j} N^j.
\]

In (1) the numbers \( B_m \) are Bernoulli numbers and the \( \binom{h}{j} \) are binomial coefficients. References are [1] and [3]. For the history of the formula we refer the reader to [2]. While we have not seen them ourselves, the significant early references are Johann Faulhaber’s \( Academia algebrae \) of 1631 and Jakob Bernoulli’s \( Ars conjectandi \) of 1713.

When the powers being summed are not integral, there is no perfect analogue of Faulhaber’s formula. One does have the following formula in terms of the Hurwitz zeta function\(^1\) \( \zeta(z, a) \) (see [5]):

\[ \zeta(z, a) = \sum_{n=0}^{\infty} (a + n)^{-z}, \]

for \( \text{Re}(z) > 1 \), and by analytic continuation at other values of \( z \) (except the pole at \( z = 1 \)). A reference is [6, Chapter 13].

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\[ \sum_{n=1}^{N} n^\alpha = \zeta(-\alpha, 1) - \zeta(-\alpha, N + 1). \]  

Unfortunately, the sum of powers on the left-hand side of (2) is more transparent than the difference of zeta functions on the right-hand side.

Motivated by the proof of Faulhaber’s formula for \( k = 1 \) in which one writes

\[(n + 1/2)^2 - (n - 1/2)^2 = 2n\]

and sums the telescoping series

\[ \sum_{n=1}^{N} [(n + 1/2)^2 - (n - 1/2)^2] \]

to obtain

\[ 2 \sum_{n=1}^{N} n = (N + 1/2)^2 - 1/4, \]

we have found the following result.

**Theorem 1.** Suppose \( \alpha > -1 \) is not an integer. Define \( J \) and \( \beta \) by requiring

\[ J \text{ to be even,} \quad 1 > \beta > -1, \quad \text{and} \quad \alpha = J + \beta. \]

Then there is a polynomial \( P_\alpha[X] \) of degree \( J \) and a finite real constant \( C_\alpha \) such that

\[ \sum_{n=1}^{N} n^\alpha - (N + 1/2)^\beta + 1 P_\alpha[N + 1/2] \to C_\alpha \]

as \( N \to \infty \); moreover, the coefficients of odd degree powers of \( X \) in \( P_\alpha[X] \) all vanish.

**Remark 2.**

1. The preceding theorem allows us to approximate the sum of non-integral powers of the first \( N \) positive integers using a formula similar to the Faulhaber formula.
2. The theorem is still true if \( \alpha \) is a non-negative integer, but, in that case, it is just a consequence of Faulhaber’s formula.
3. Since the preceding theorem seems to answer a natural question that could have been posed by Bernoulli, it is remarkable that we have not been able to find the result in the literature. We thank Kevin McGown for asking us the question.

Our starting point is the binomial theorem. Among many possible references, one is [6, §5.4].

**Newton’s binomial formula.** If \(-1 < t < 1\), then

\[ (1 + t)^Y = 1 + \sum_{r=1}^{\infty} \binom{Y}{r} t^r, \]
where
\[
\left( \frac{\gamma}{r} \right)_r = \frac{\gamma(\gamma-1) \cdots (\gamma-r+1)}{r!}.
\]

Lemma 3. For any positive integer \( N \) and any real \( \gamma \), it holds that
\[
(N + 1/2)^{\gamma+1} = 2^{-(\gamma+1)} + (\gamma + 1) \sum_{n=1}^{N} n^{\gamma} + \sum_{n=1}^{N} \sum_{k=1}^{\infty} \left( \frac{\gamma + 1}{2k + 1} \right) 2^{-2k} n^{\gamma-2k}.
\]

Proof. For \( n = 1, 2, \ldots \), we apply the binomial formula to \((1 + 2^{-1}n^{-1})^{\gamma+1}\) and to \((1 - 2^{-1}n^{-1})^{\gamma+1}\) to obtain
\[
(n + 1/2)^{\gamma+1} = n^{\gamma+1}(1 + 2^{-1}n^{-1})^{\gamma+1} = n^{\gamma+1} + \sum_{r=1}^{\infty} \left( \frac{\gamma + 1}{r} \right) 2^{-r} n^{\gamma-r+1}
\]
and
\[
(n - 1/2)^{\gamma+1} = n^{\gamma+1}(1 - 2^{-1}n^{-1})^{\gamma+1} = n^{\gamma+1} + \sum_{r=1}^{\infty} \left( \frac{\gamma + 1}{r} \right) (-1)^r 2^{-r} n^{\gamma-r+1}.
\]
Subtracting these two equations, we obtain
\[
(n + 1/2)^{\gamma+1} - (n - 1/2)^{\gamma+1} = \sum_{k=0}^{\infty} \left( \frac{\gamma + 1}{2k + 1} \right) 2^{-2k} n^{\gamma-2k}.
\]
(3)
Summing from \( n = 1 \) to \( n = N \), we obtain
\[
(N + 1/2)^{\gamma+1} = \sum_{n=1}^{N} \sum_{k=0}^{\infty} \left( \frac{\gamma + 1}{2k + 1} \right) 2^{-2k} n^{\gamma-2k}
\]
\[
= (\gamma + 1) \sum_{n=1}^{N} n^{\gamma} + \sum_{n=1}^{N} \sum_{k=1}^{\infty} \left( \frac{\gamma + 1}{2k + 1} \right) 2^{-2k} n^{\gamma-2k},
\]
and the result follows. \( \square \)

Theorem 4. For any real \( \beta < 1 \), it holds that
\[
\lim_{N \to \infty} \left[ (N + 1/2)^{\beta+1} - (\beta + 1) \sum_{n=1}^{N} n^{\beta} \right]
\]
\[
= 2^{-(\beta+1)} + \sum_{k=1}^{\infty} \left( \frac{\beta + 1}{2k + 1} \right) 2^{-2k} \zeta(2k - \beta),
\]
(4)
where \( \zeta(z) \) is Riemann’s zeta function given by
\[
\zeta(z) = \sum_{n=1}^{\infty} n^{-z},
\]
for \( \text{Re}(z) > 1 \) (see, for instance, [4]).
Proof. By Lemma 3, it will suffice to show that the right-hand side of (4) is convergent. Since, for \(1 \leq k\), it holds that \(\zeta(2k - \beta) \leq \zeta(2 - \beta) < \infty\) and since all the binomial coefficients \(\binom{\beta + 1}{2k + 1}\) have the same sign, it suffices to show that

\[
\sum_{k=1}^{\infty} \binom{\beta + 1}{2k + 1} 2^{-2k}
\]

is convergent. But (3) with \(n = 1\), tells us that

\[
\left(\frac{3}{2}\right)^{\beta + 1} - \left(\frac{1}{2}\right)^{\beta + 1} = \sum_{k=0}^{\infty} \binom{\beta + 1}{2k + 1} 2^{-2k}
\]

and the result follows. 

Corollary 5. Suppose \(1 > \beta > -1\) and \(\beta \neq 0\). Then Theorem 1 holds with \(P_\beta[X]\) equal to the constant polynomial \(1/(\beta + 1)\) and with \(C_\beta\) defined by requiring \(-(\beta + 1)C_\beta\) to equal the right-hand side of (4).

Proof of Theorem 1. By Corollary 5, we may suppose \(\alpha > 1\). Proceeding inductively, we may suppose the result has been proved for \(\alpha - 2, \alpha - 4, \ldots, \alpha - J = \beta\). Recall that \(J\) is even and set \(K = J/2\).

By Lemma 3, we have

\[
(N + 1/2)^{\alpha - 1} - (\alpha + 1) \sum_{n=1}^{N} n^{\alpha - 1} = 2^{-\alpha - 1} + \sum_{k=1}^{K} \left(\frac{\alpha + 1}{2k + 1}\right) 2^{-2k} \sum_{n=1}^{N} n^{\alpha - 2k} + \sum_{k=K+1}^{\infty} \left(\frac{\alpha + 1}{2k + 1}\right) 2^{-2k} \sum_{n=1}^{N} n^{\alpha - 2k}.
\]

By our induction hypothesis, we have

\[
\sum_{n=1}^{N} n^{\alpha - 2k} - (N + 1/2)^{\beta + 1} P_{\alpha - 2k}[N + 1/2] \to C_{\alpha - 2k} \quad \text{as} \quad N \to \infty.
\]

By the same argument as was used in the proof of Theorem 4, we have

\[
\sum_{k=K+1}^{\infty} \left(\frac{\alpha + 1}{2k + 1}\right) 2^{-2k} \sum_{n=1}^{N} n^{\alpha - 2k} \to \sum_{k=K+1}^{\infty} \left(\frac{\alpha + 1}{2k + 1}\right) 2^{-2k} \zeta(\alpha - 2k) \quad \text{as} \quad N \to \infty.
\]

Thus we have

\[
\sum_{n=1}^{N} n^{\alpha} - (N + 1/2)^{\beta + 1}(\alpha + 1)^{-1}
\]
\[ \times \left[ (N + 1/2)^J - \sum_{k=1}^{K} \left( \frac{\alpha + 1}{2k + 1} \right) 2^{-2k} P_{\alpha - 2k}[N + 1/2] \right] \]

\[ \to - (\alpha + 1)^{-1} \left[ 2^{-x-1} + \sum_{k=1}^{K} \left( \frac{\alpha + 1}{2k + 1} \right) 2^{-2k} C_{\alpha - 2k} \right] + \sum_{k=K+1}^{\infty} \left( \frac{\alpha + 1}{2k + 1} \right) 2^{-2k} \xi(\alpha - 2k) \] as \( N \to \infty \),

so if we set

\[ P_{\alpha}[X] = (\alpha + 1)^{-1} \left[ X^{J/2} - \sum_{k=1}^{K} \left( \frac{\alpha + 1}{2k + 1} \right) 2^{-2k} P_{\alpha - 2k}[X] \right], \]

then the result follows. \( \Box \)

Some specific cases of interest are obtained using the following corollary.

**Corollary 6.** For \( 1 > \beta > -1, \beta \neq 0 \), it holds that

\[ P_{\beta} \equiv \frac{1}{\beta + 1}, \]

\[ P_{\beta+2}[X] = \frac{1}{\beta + 3} X^{2} - \frac{\beta + 2}{24}, \]

\[ P_{\beta+4}[X] = \frac{1}{\beta + 5} X^{4} - \frac{\beta + 4}{24} X^{2} + (\beta + 4)(\beta + 3)(\beta + 2) \frac{7}{2^7 \cdot 3^2 \cdot 5}. \]

**Example 7.** Taking \( \beta = -1/2 \) and \( \beta = 1/2 \) in the preceding corollary, we obtain

\[ \sum_{n=1}^{N} n^{-1/2} \approx 2(N + 1/2)^{1/2}, \]

\[ \sum_{n=1}^{N} n^{1/2} \approx (2/3)(N + 1/2)^{3/2}, \]

\[ \sum_{n=1}^{N} n^{3/2} \approx (2/5)(N + 1/2)^{5/2} - (1/16)(N + 1/2)^{1/2}, \]

\[ \sum_{n=1}^{N} n^{5/2} \approx (2/7)(N + 1/2)^{7/2} - (5/48)(N + 1/2)^{3/2}, \]

\[ \sum_{n=1}^{N} n^{7/2} \approx (2/9)(N + 1/2)^{9/2} - (7/48)(N + 1/2)^{5/2} - (49/3072)(N + 1/2)^{1/2}, \]

\[ \sum_{n=1}^{N} n^{9/2} \approx (2/11)(N + 1/2)^{11/2} - (3/16)(N + 1/2)^{7/2} - (49/1024)(N + 1/2)^{3/2}. \]
As a simple illustration, in Table 1, we give a comparison, for the case \( N = 348 \), of \( \sum_{n=1}^{N} n^\alpha \) and \( F_\alpha[100] \). This modest value of \( N = 100 \) was chosen to avoid any delicate issues of numerical analysis and so that the table would not be too large.

**Remark 8.** In the spirit of [3, Eq. (4)], still supposing \( 1 > \beta > -1, \beta \neq 0 \), by Lemma 3, we see that

\[
\begin{pmatrix}
(N + 1/2)^{\beta + 1} \\
(N + 1/2)^{\beta + 3} \\
(N + 1/2)^{\beta + 5} \\
\vdots
\end{pmatrix}
\]

differs from

\[
\begin{pmatrix}
\beta + 1 & 0 & 0 & \cdots \\
2^{-2}(\beta + 3) & \beta + 3 & 0 & \cdots \\
2^{-2}(\beta + 5) & 2^{-2}(\beta + 3) & \beta + 5 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
\sum_{n=1}^{N} n^\beta \\
\sum_{n=1}^{N} n^\beta+2 \\
\sum_{n=1}^{N} n^\beta+4 \\
\vdots
\end{pmatrix}
\]

by constants and sums of powers of \( n \) of degree \( \beta - 2 \) and smaller. Since the first \( \ell \) rows of the inverse of a lower-triangular matrix can be computed from the first \( \ell \) rows of the original matrix, this approach gives us an algorithm for generating the coefficients of the
polynomial $P_{\beta+2\ell}$. Thus we see that

$$\begin{pmatrix}
\sum_{n=1}^{N} n^\beta \\
\sum_{n=1}^{N} n^{\beta+2} \\
\sum_{n=1}^{N} n^{\beta+4} \\
\vdots
\end{pmatrix}$$

differs from

$$\begin{pmatrix}
\frac{1}{\beta+1} & 0 & 0 & \cdots & (N + 1/2)^{\beta+1} \\
\frac{1}{\beta+2} & \frac{1}{\beta+3} & 0 & \cdots & (N + 1/2)^{\beta+3} \\
\frac{1}{\beta+4} & \frac{1}{\beta+5} & \cdots & (N + 1/2)^{\beta+5} \\
\vdots & \vdots & \vdots & \ddots & \vdots
\end{pmatrix}$$

by constants and sums of powers of $n$ of degree $\beta - 2$ and smaller.

References