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Existence, Uniqueness, and Nonexistence of Limit Cycles for a Class of Quadratic Systems in the Plane

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In this paper we study the existence, uniqueness, and nonexistence of limit cycles for the class of quadratic systems

$$\begin{aligned}\dot{x} &= a_{11}x + a_{12}y + y^2,\\ \dot{y} &= a_{21}x + a_{22}y - xy + cy^2.\end{aligned} \tag{1}$$

This system arises naturally from Markus' classification of homogenous quadratic forms in [1]. All possible phase portraits for the system (1) were determined in [2, 3]. And examples of the system (1) with one and two limit cycles were obtained in [4] using the theory of rotated vector fields developed in [4, 5]. However, the uniqueness of the limit cycles was not established in [4].

In this paper we establish the existence *and* the uniqueness of limit cycles of (1) when $a_{11} = 0$. The uniqueness, which in general is very difficult to establish, follows in this case because the system (1) with $a_{11} = 0$, i.e.,

$$\begin{aligned}\dot{x} &= a_{12}y + y^2,\\ \dot{y} &= a_{21}x + a_{22}y - xy + cy^2,\end{aligned} \tag{1$'$}$$

can be put into the form of Lienard's equation and the results of Chang [6]

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apply. The existence of limit cycles of the system (1′) follows either from a straightforward application of the Poincaré–Bendixson theorem or from the theory of rotated vector fields as in [4]; we also obtain specific information concerning the global behavior of the limit cycles using the theory of rotated vector fields. The Bendixson–Dulac criterion or its generalization by Cheng [7] are used to establish that, under certain conditions on the coefficients, (1) has no limit cycles. Thus, the main purpose of this paper is to show that the system (1) either has no limit cycle or exactly one limit cycle around any one of its critical points. We also give concrete examples of quadratic systems with exactly one and with exactly two limit cycles in this paper.

There are very few results in the literature which determine the *exact* number of limit cycles for a given quadratic system or class of quadratic systems. Most of the major results concerning the number of limit cycles of a quadratic system, i.e., results related to Hilbert's 16th Problem for quadratic systems, determine that there are *at least* a certain number of limit cycles possessed by a given quadratic system or class of quadratic systems. For example, Bautin [8, p. 18] showed that there is a class of quadratic systems with three limit cycles disappearing into the origin; cf. [9] for a concrete example of Bautin's system with three limit cycles in a small neighborhood of the origin. It has not, however, been shown that the quadratic systems in [8, 9] do not have other limit cycles outside of a small neighborhood of the origin. Bautin, of course, also proved the important result that *at most* three limit cycles can disappear into a focus or center of a quadratic system. Shi [10] has recently shown that there is a quadratic system with *at least four* limit cycles. This proves that the assertion of Petrovskii and Landis [11] that the maximum number of limit cycles of a quadratic system is three is false. Tung [12] gave an example of a family of quadratic systems with *at least* three limit cycles around two different critical points. He also claimed to have given an example of a quadratic system with *exactly two* limit cycles in [12]; however, his proof that his system on p. 162 of [12] has at most two limit cycles is based on the invalid assertion of Petrovskii and Landis, cf. p. 167 in [12]; thus, all that can be said with certainty is that his system on p. 162 has *at least* two limit cycles. Yeh [13] also gave an example of a quadratic system with *at least* two limit cycles around two different critical points. He uses Duff's theory of rotated vector fields to establish the existence of these limit cycles, but he does not establish their uniqueness in [13]. Yeh does establish both the existence *and* the uniqueness of a single limit cycle for a quadratic system in [14]. And Frommer [15] also gave an example of a quadratic system with *exactly one* limit cycle.

In order to illustrate the precise type of results that follow from the theorems in this paper, we state two theorems which give concrete examples of quadratic systems with *exactly one* and with *exactly two* limit cycles, respectively.

THEOREM. *For $-2 < a_0 < 0$ and $0 < a < 2 + a_0$, the quadratic system*

$$\dot{x} = y + y^2,$$
$$\dot{y} = -x + ay - xy + (a - a_0)\, y^2,$$

has a unique limit cycle around the origin; this limit cycle is generated at the origin at $a = 0$ and it expands monotonically to infinity as a increases; there are no limit cycles for $a \leqslant 0$; and the separatrix configuration for $|a - a_0| < 2$ is given in Fig. 1. It is conjectured that the limit cycle for the system in the above theorem expands to infinity as a increases to $2 + a_0$. If this is the case, the separatrix configuration for $a = 2 + a_0$ is shown in Fig. 1(c).

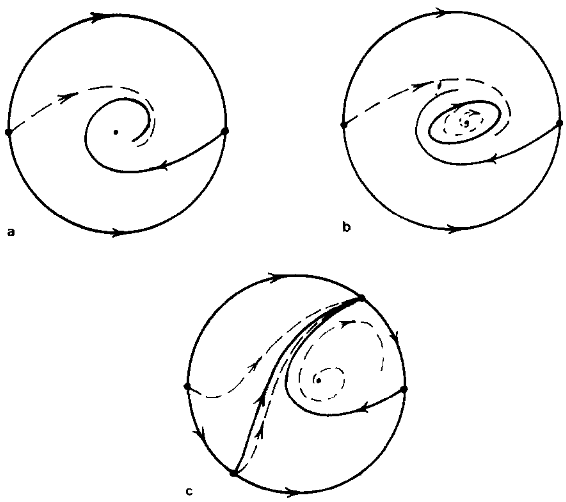

FIG. 1. (a) $-2 + a_0 < a \leqslant 0$; (b) $0 < a < 2 + a_0$; (c) $a = 2 + a_0$.

Numerical examples showing the phase portrait for this system with $a_0 = -1$ and with $a = \pm 0.2$ are given in Fig. 2.

THEOREM. *For $-1 < a_0 < 0$ and $0 < a < -a_0$, the quadratic system*

$$x = y + y^2,$$
$$y = -\frac{x}{2} + ay - xy + (a - a_0)\, y^2$$

has exactly one limit cycle around the origin and exactly one limit cycle around the critical point at $(2a_0, -1)$; these two limit cycles are generated at their respective critical points at $a = 0$, they expand monotonically as a

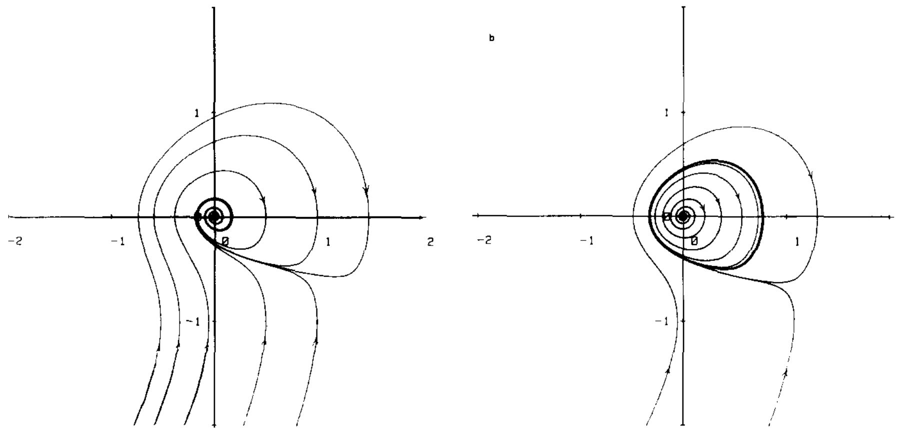

FIG. 2. (a) $\alpha = -0.2$; (b) $\alpha = 0.2$.

increases, and they intersect in the line $y = -\frac{1}{2}$ *at* $\alpha = -\alpha_0$; *there are no limit cycles for* $\alpha \leqslant 0$ *or for* $\alpha \geqslant -\alpha_0$; *and the separatrix configuration undergoes the continuous deformation depicted in Fig.* 3 *as* α *varies in* $(-2 + \alpha_0, 2 + \alpha_0)$.

Numerical examples showing the size and shape of the limit cycles of this system with $\alpha_0 = -0.8$ and with $\alpha = 0.2$ and 0.4 are given in Fig. 4.

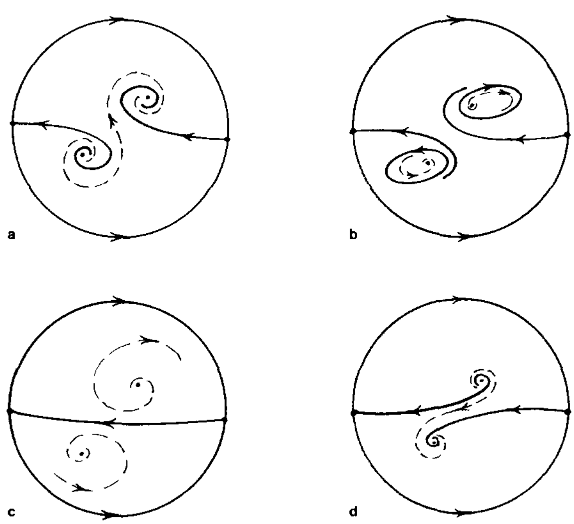

FIG. 3. (a) $-2 + \alpha_0 < \alpha \leqslant 0$; (b) $0 < \alpha < -\alpha_0$; (c) $\alpha = -\alpha_0$; (d) $-\alpha_0 < \alpha < 2 + \alpha_0$.

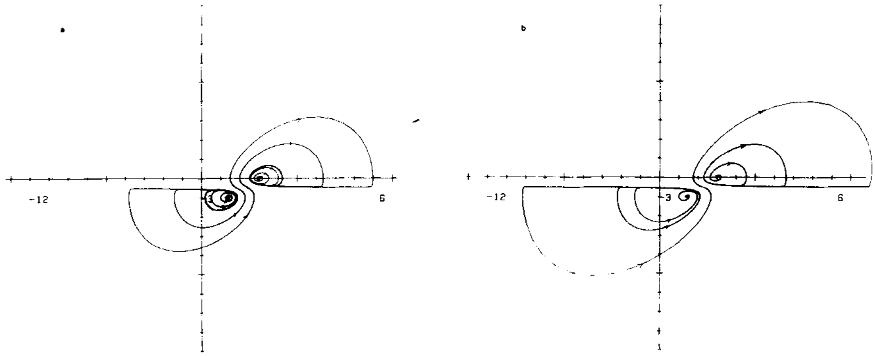

FIG. 4. (a) $\alpha = 0.2$; (b) $\alpha = 0.4$.

1. EXISTENCE OF LIMIT CYCLES

In this section we establish the existence of limit cycles of the system (1′). Under certain conditions on the coefficients the existence of a limit cycle of (1′) follows as an immediate consequence of the Poincaré–Bendixson theorem. The global behavior of the limit cycle can then be determined by applying the theory of rotated vector fields as was done in [4]. In certain other cases when the Poincaré–Bendixson theorem does not apply to establish the existence of a limit cycle for (1′), the theory of rotated vector fields can still be used to establish the existence and global behavior of limit cycles for (1′).

In order to establish the existence of limit cycles for the system (1′), it is first necessary to determine the nature of the finite critical points and the critical points at infinity for the system (1′). Using the basic theory for systems of ordinary differential equations in the plane (cf., e.g., [16], and the theorems in the appendix of [2]), it is easy to establish the following lemmas.

LEMMA 1. *The system (1′) has a critical point at the origin with determinant* $-a_{12}a_{21}$ *and trace* a_{22}; *for* $a_{12} + a_{21} \neq 0$ *there is a second critical point at*

$$\left(\frac{a_{22}a_{12} - a_{12}^2 c}{a_{12} + a_{21}}, -a_{12}\right)$$

with determinant $a_{12}(a_{12} + a_{21})$ *and trace*

$$\frac{a_{21}(a_{22} + ca_{21}) - c(a_{12} + a_{21})^2}{(a_{12} + a_{21})^2}$$

and these are the only critical points; for $a_{12}+a_{21}=0$ *and* $a_{21}(a_{22}+ca_{21})\neq 0$, *the origin is the only critical point; and for* $a_{12}+a_{21}=0$ *and* $a_{21}(a_{22}+ca_{21})=0$, *the line* $y=a_{21}$ *is a line of critical points.*

LEMMA 2. *For* $|c|<2$, *the system* (1′) *has a node at* $(\pm 1,0,0)$ *if* $a_{21}(a_{12}+a_{21})>0$, *a saddle at* $(\pm 1,0,0)$ *if* $a_{21}(a_{12}+a_{21})<0$, *and a saddle-node at* $(\pm 1,0,0)$ *for* $a_{12}+a_{21}=0$ *and* $a_{21}(a_{22}+ca_{21})\neq 0$. *There are no other critical points at infinity for* $|c|<2$ *and the behavior near the equator of the Poincaré sphere is shown in Fig.* 5 *for these three cases, respectively.*

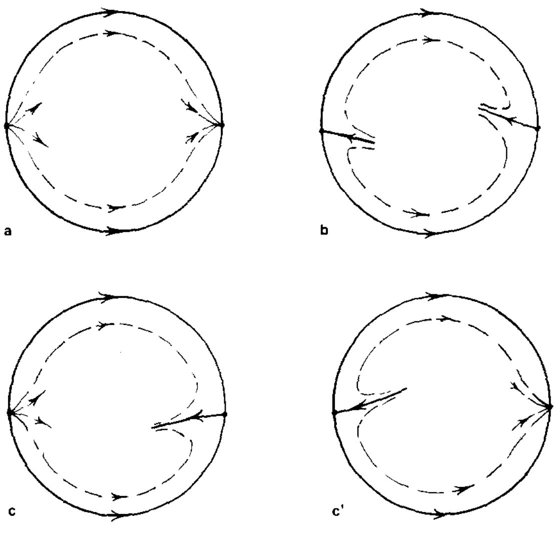

FIG. 5. (c) $a_{22}+ca_{21}<0$; (c′) $a_{22}+ca_{21}>0$.

We first consider the system (1′) with $a_{12}+a_{21}=0$, $|c|<2$, and $a_{21}(a_{22}+ca_{21})\neq 0$. The following theorem which establishes the existence of a limit cycle around the origin for the system (1′) is an immediate consequence of the Poincaré–Bendixson theorem. This follows, for example, if $a_{22}>0$ and $a_{22}+ca_{21}<0$, since by Lemma 1 the origin is the only (finite) critical point of (1′) and under the above conditions it is an *unstable* focus; and by Lemma 2 the behavior near infinity for $a_{22}+ca_{21}<0$ is shown in Fig. 5(c). It therefore follows from the Poincaré–Bendixson theorem that there must be at least one externally stable limit cycle around the origin of (1′) that is the ω-limit set of the trajectories shown in Fig. 5(c). Note that it follows from the recent work of Chicone and Shafer [17] that there can be at most a finite number of limit cycles on the interior of this externally stable limit cycle.

THEOREM 1. *For* $|c| < 2$, $a_{21} \neq 0$, $a_{12} + a_{21} = 0$, *and* $a_{22}(a_{22} + ca_{21}) < 0$, *the system* (1′) *has a limit cycle around the origin. Assuming that this limit cycle is unique, the separatrix configuration for* (1′) *with* $a_{22} + ca_{21} < 0$ *is determined by Fig.* 6, *and for* $a_{22} + ca_{21} > 0$ *it is determined by one of the configurations in Fig.* 6 *rotated about the* y *axis with the direction of the arrows reversed.*

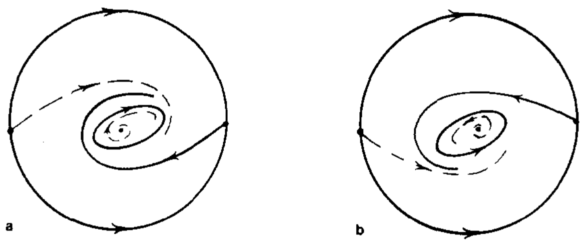

FIG. 6. (a) $a_{21} < 0$; (b) $a_{21} > 0$.

Remark 1. It is shown in the next section that the system (1′) in Theorem 1 has at most one limit cycle around the origin; i.e., the limit cycle whose existence is established in Theorem 1 is indeed unique. This fact allows us to uniquely determine the phase portrait for this system as in Fig. 6.

Remark 2. The transformation $x \to -x$, $t \to -t$ transforms the system (1′) into

$$\dot{x} = a_{12} y + y^2,$$
$$\dot{y} = a_{21} x - a_{22} y - xy - cy^2$$

and we see that changing the sign of a_{22} and c, i.e., changing the sign of $a_{22} + ca_{21}$, causes a rotation of the phase portrait of (1′) about the y axis with the direction of the arrows being reversed.

More specific information about the global behavior of the limit cycle in Theorem 1 can be obtained by using the theory of rotated vector fields; cf. [4, 5]. In order to illustrate this point, we consider the following special case of system (1′):

$$\begin{aligned} \dot{x} &= a_{12} y + y^2, \\ \dot{y} &= a_{21} x + \alpha a_{12} y - xy + (\alpha - \alpha_0) y^2. \end{aligned} \tag{2}$$

The critical points of (2) as given by Lemma 1 are independent of α, and the angle of the field vector (P, Q) defined by the right-hand side of (2), i.e., $\theta = \tan^{-1} Q/P$, satisfies

$$\frac{\partial \theta}{\partial \alpha} = \frac{P^2}{P^2 + Q^2} > 0$$

and

$$\tan\theta = \alpha + \frac{a_{21}x - xy - \alpha_0 y^2}{P} \to \pm\infty \qquad \text{as} \quad \alpha \to \pm\infty$$

for $P \neq 0$. The system (2) therefore forms a *semi-complete family* (mod $P = 0$) in the terminology of [4]. The following theorem is an immediate consequence of Theorem D and the corollaries to Theorems G and H in [4]. (Note that it follows from the fact that any limit cycle of a quadratic system encloses a convex region [12] that any limit cycle of (2) crosses the x axis at most twice.)

THEOREM 1′. *For $a_{21} < 0$ and $a_{12} + a_{21} = 0$, the system* (2) *forms a semi-complete family* (mod $P = 0$) *with parameter $\alpha \in (-\infty, \infty)$. The only critical point is at the origin and a unique limit cycle is generated at the origin at $\alpha = 0$. Under the assumption that there is at most one limit cycle around the origin, this limit cycle is stable and it expands monotonically to infinity as α increases to some positive number $\alpha^* \geqslant 2 + \alpha_0$ when $-2 < \alpha_0 < 0$; furthermore the separatrix configuration is determined by Fig.* 1(a) *for $-2 + \alpha_0 < \alpha \leqslant 0$ and by Fig.* 1(b) *for $0 < \alpha < 2 + \alpha_0$.*

Remark 3. For $-2 < \alpha_0 < 0$, it is conjectured that $\alpha^* = 2 + \alpha_0$. If this is the case, then the separatrix configuration for the system (2) with $\alpha = \alpha^*$ is shown in Fig. 1(c) and the separatrix configuration for $\alpha > \alpha^*$ is shown in Fig. 7. However, if $\alpha^* > 2 + \alpha_0$, then for $2 + \alpha_0 < \alpha < \alpha^*$ we would obtain some intermediate configurations with the separatrices from the saddles at infinity possibly being connected (but this seems unlikely) and then we would once again obtain the configuration in Fig. 7 for $\alpha > \alpha^*$. Note that the system (2) has a saddle-node at $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}, 0)$ for $\alpha - \alpha_0 = 2$ which splits into a saddle and a node for $\alpha - \alpha_0 > 2$ (cf. [2]). Also, the case when $0 < \alpha_0 < 2$ follows from the above theorem by employing the coordinate transformation in Remark 2.

Remark 4. It is shown in the next section that there is at most one limit cycle around the origin of the system (2) under the conditions of Theorem 1′

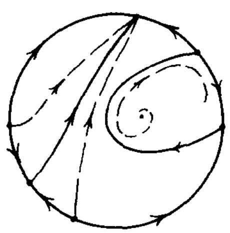

FIGURE 7

when $\alpha > 0$. And it is shown in Section 3 that there are no limit cycles around the origin for $\alpha \leqslant 0$. This information is critical in determining the global phase portrait for the system (2) and the global behavior of the limit cycle generated at the origin at $\alpha = 0$; cf. Remark 7 in Section 3. For example if we did not know that there was at most one limit cycle around the origin of the system (2), then any number of semi-stable limit cycles could appear as α increases in the interval $(0, \alpha^*)$ and we could even end up with the separatrix cycle at infinity shown in Fig. 1(c) or in Fig. 7 containing an infinite number of limit cycles on its interior. This is still an open possibility for quadratic systems as is pointed out in [17].

We next consider the system (1′) with $a_{12} + a_{21} > 0$, $|c| < 2$, $a_{21} < 0$, and $a_{22} + ca_{21} < 0$. It follows from Lemma 1 that there are two (finite) critical points in this case, one at the origin and one in the lower half plane. The behavior near the equator of the Poincaré sphere is described by Fig. 5(b) in this case. The following theorem is an immediate consequence of the Poincaré–Bendixson theorem.

THEOREM 2. *For* $a_{21} < 0$, $a_{12} + a_{21} > 0$, $a_{22} > 0$, $a_{22}/|a_{21}| < c < 2$, *and* $a_{21}(a_{22} + ca_{21}) < c(a_{12} + a_{21})^2$, *the system* (1′) *has a limit cycle around each of its two critical points and assuming that these limit cycles are unique, the separatrix configuration is determined by Fig.* 3(b).

Proof. For $a_{21} < 0$ and $a_{22} + ca_{21} < 0$, we have $\dot{y} = a_{21}(a_{22} + ca_{21}) > 0$ on the line $y = a_{21}$. The separatrix from the saddle at $(1, 0, 0)$, shown in Fig. 5(b), is therefore into the region $y > a_{21}$. The origin is the only critical point in this region since $-a_{12} < a_{21}$ and for $a_{22} > 0$ and $a_{22} + ca_{21} < 0$ the origin is an unstable focus. It therefore follows from the Poincaré–Bendixson theorem that there is a limit cycle around the origin, stable on its exterior, that is the ω-limit set of the separatrix from the saddle at $(1, 0, 0)$. Similarly, the separatrix going into the saddle at $(-1, 0, 0)$ in Fig. 5(b) lies in the region $y < a_{21}$. There is only one critical point in this region and for $a_{21}(a_{22} + ca_{21}) < c(a_{12} + a_{21})^2$ it is stable. (The conditions of this theorem imply that it is in fact a stable focus.) It therefore follows from the Poincaré–Bendixson theorem that there is a limit cycle, unstable on its exterior, around the lower critical point. Under the assumption that the limit cycle around the origin and around the lower critical point are unique, it follows from the fact that the flow across the transversal $y = a_{21}$ is upward and from the Poincaré–Bendixson theorem that the separatrix configuration for the system (1′) under the conditions of this theorem is given by Fig. 3(b). This completes the proof of Theorem 2.

It is shown in the next section that under certain additional conditions on the coefficients, the limit cycles established in Theorem 2 are indeed unique.

We next obtain more specific information concerning the global behavior of the limit cycles in Theorem 2 by employing the theory of rotated vector fields. We consider the special case $a_{12} + 2a_{21} = 0$ since it is easier to describe what happens to the two limit cycles in this case.

THEOREM 2′. *For* $a_{21} < 0$ *and* $a_{12} + 2a_{21} = 0$, *the system* (2) *forms a semi-complete family* (mod $P = 0$) *with parameter* $\alpha \in (-\infty, \infty)$. *There are two critical points, one at the origin and one at* $(2a_{12}\alpha_0, -a_{12})$. *For* $-1 < \alpha_0 < 0$, *a unique limit cycle is generated at each of these critical points at* $\alpha = 0$ *and, under the assumption that there is at most one limit cycle around each of these critical points, these limit cycles expand monotonically as* α *increases and intersect in the line* $y = a_{21}$ *at* $\alpha = -\alpha_0$; *futhermore, the separatrix configurations for* $-2 + \alpha_0 < \alpha < 2 + \alpha_0$ *are given by Figs.* 3(a)–(d).

Proof. By Lemma 1, the determinant at both the upper and the lower critical points is equal to $-a_{12}a_{21}$ which is positive. By Lemma 7 in [2] neither critical point is a center for $\alpha = 0$ provided $\alpha_0 \neq 0$. By Lemma 1 the trace at the origin is equal to αa_{12} and the trace at the lower critical point is equal to α/a_{21}. It then follows from the corollary to Theorem H in [4] that a unique limit cycle is generated at each critical point of (2) at $\alpha = 0$. Under the assumption that there is at most one limit cycle around each of the critical points of (2), it follows from Theorem D in [4] (using Lemma 2 and the fact that the flow is upward across $y = a_{21}$ to determine the stability of the limit cycles) that for $-1 < \alpha_0 < 0$ the limit cycles expand monotonically as α increases; and from the corollary to Theorem G in [4], it follows that the two limit cycles expand until they meet the critical points at infinity at $(\pm 1, 0, 0)$. This is only possible if they intersect in a trajejctory joining $(1, 0, 0)$ to $(-1, 0, 0)$ and this can only occur at $\alpha = -\alpha_0$ when the line $y = a_{21}$ is such a trajectory. The separatrix configuration for $\alpha = -\alpha_0$ is given by Fig. 3(c). The other separatrix configurations in Fig. 3 are easy consequences of Lemma 2, the fact that the flow is upward across $y = a_{21}$, and the Poincaré–Bendixson theorem. This completes the proof of Theorem 2′.

Remark 5. If $\alpha_0 = 0$ in Theorem 2′, it follows from Lemma 7 in [2] that the critical points are centers for $\alpha = 0$ and we have the separatrix configuration shown in Fig. 8 for $a_{21} < 0$, $a_{12} + 2a_{21} = 0$, and $\alpha = \alpha_0 = 0$ in (2). In this case, the configurations for $-2 < \alpha < 0$ and $0 < \alpha < 2$ are given by Figs. 3(a) and (d), respectively. The case when $0 < \alpha_0 < 1$ follows from Theorem 2′ by employing the coordinate transformation in Remark 2.

Finally, we consider the existence of limit cycles of (1′) for the case when $a_{12} + a_{21} < 0$. In this case, the results differ from Theorem 2 only when $a_{21} \leqslant 0$. If $a_{21} = 0$ the x axis is either a trajejctory or a line of critical points

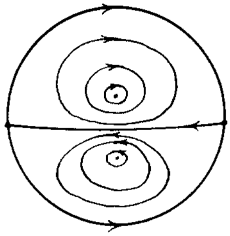

FIGURE 8

and there is no limit cycle around the origin. For $a_{21} < 0$ we take $a_{12} > 0$ in order to get a positive determinant at the origin. This implies, by Lemma 1, that there is a saddle in the lower half plane. For $|c| < 2$, the behavior near the equator of the Poincaré sphere is then given by Fig. 5(a) in Lemma 2. However, in this case we can *not* obtain the existence of a limit cycle around the origin of (1′) using the Poincaré–Bendixson theorem. On the other hand, the existence and global behavior of a limit cycle for the system (2) does follow from the theory of rotated vector fields. The following theorem follows from Theorem D and the corollaries to Theorems G and H in [4] as in the proof of Theorem 2′.

THEOREM 3. *For $a_{12} > 0$ and $a_{12} + a_{21} < 0$, the system* (2) *forms a semi-complete family* (mod $P = 0$) *with parameter $\alpha \in (-\infty, \infty)$. There are two critical points, one at the origin and a saddle in the lower half plane. For $\alpha_0 \neq 0$ a unique limit cycle is generated at the origin at $\alpha = 0$ and, under the assumption that there is at most one limit cycle around the origin, this limit cycle expands monotonically with monotonically varying α and intersects the saddle in the lower half plane, forming a separatrix cycle, at some value of $\alpha = \alpha^*$ where $\alpha^* \neq 0$; for $\alpha_0 < 0$ and $\alpha^* > 0$, the separatrix configuration undergoes the continuous deformation depicted in Fig.* 9.

Figure 10 shows the size and shape of the limit cycle of system (2) with $a_{12} = 1$, $a_{21} = -2$, $\alpha_0 = -1$, and $\alpha = 0.2$. Numerical evidence indicates that $\alpha^* \approx 0.52$ in this case.

Remark 6. Numerical evidence makes it clear that the sign of α^* is determined by the sign of α_0, i.e., that $\alpha_0 \alpha^* < 0$. Also, we note that if $\alpha_0 = 0$, then it follows from Lemma 7 in [2] that the critical point at the origin is a center for $\alpha = 0$ and we have the separatrix configuration shown in Fig. 11, the configurations for $-2 < \alpha < 0$ and $0 < \alpha < 2$ being given by Figs. 9(a) and (d), respectively. The case when $\alpha_0 > 0$ follows from Theorem 3 by employing the coordinate transformation in Remark 2. Finally, we note the

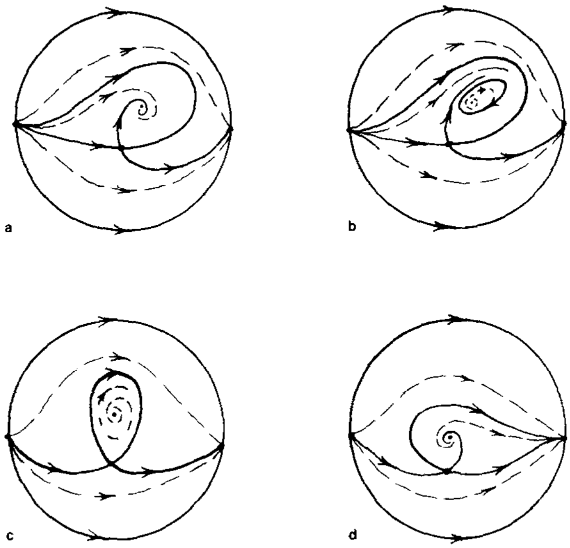

FIG. 9. (a) $-2+\alpha_0<\alpha\leqslant 0$; (b) $0<\alpha<\alpha^*$; (c) $\alpha=\alpha^*$; (d) $\alpha^*<\alpha<2+\alpha_0$.

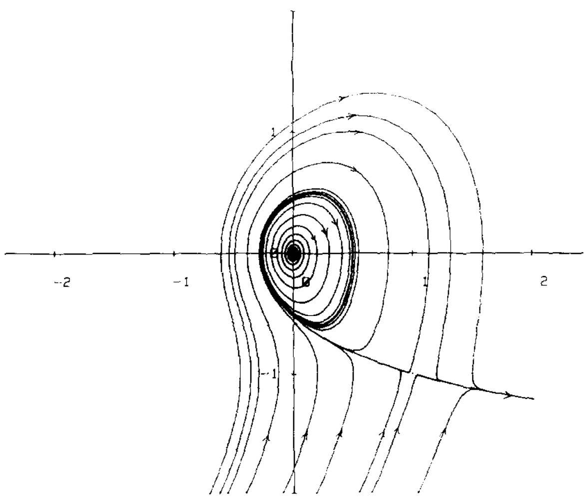

FIGURE 10

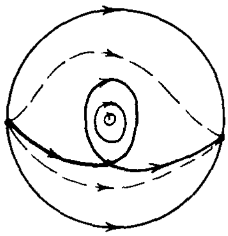

FIGURE 11

similarity of the behavior of the system (2) in Theorem 3 as depicted in Fig. 9 with the behavior of the system (6) in Theorem I of [4, Fig. 3].

Figure 12 shows a numerical example of the configuration in Fig. 11 for the system (2) with $a_{12} = 1$, $a_{21} = -2$, and $\alpha = \alpha_0 = 0$.

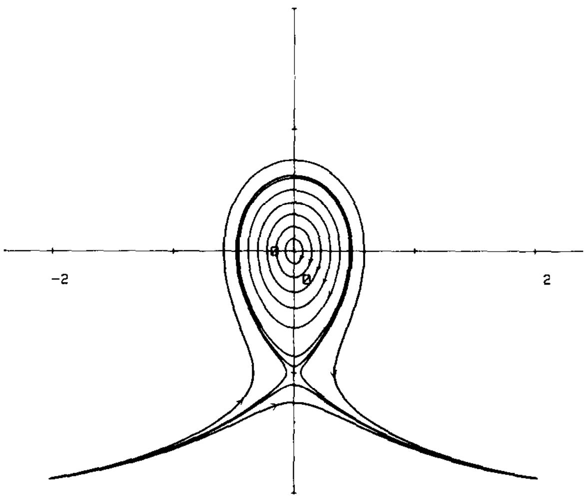

FIGURE 12

2. UNIQUENESS OF LIMIT CYCLES

In this section, we establish the uniqueness of the limit cycles whose existence was established in Section 1. Our uniqueness proofs are based on the results of Chang [6]. For convenience, a translation of Chang's results in [6] is given in the Appendix at the end of this paper. As was noted in the introduction, the system (1′) can be put into the form of Lienard's equation for which we have a uniqueness result. However, the hypotheses of Lienard's theorem are much too stringent and do not apply to our system (1′). On the other hand, the results of Chang [6] do apply and yield some interesting uniqueness results.

THEOREM 4. *For* $c > 0$, $a_{22} > 0$, $a_{21} < 0$, $a_{12} + a_{21} = 0$, *and* $a_{22} + ca_{21} < 0$, *the system* (1′) *has at most one limit cycle around the origin.*

Proof. For $a_{12} + a_{21} = 0$, the system (1′) can be written in the form

$$\dot{x} = y(y - a_{21})$$
$$\dot{y} = -x(y - a_{21}) + a_{22} y + cy^2.$$

On the line $y = a_{21}$ we have $\dot{x} = 0$ and $\dot{y} = a_{21}(a_{22} + ca_{21})$. The line $y = a_{21}$ is therefore either a transversal or a line of rest points; thus no limit cycle intersects $y = a_{21}$. We define a new independent variable by $d\tau = (y - a_{21})\, dt$. The above system (1′) then becomes

$$\frac{dx}{d\tau} = y,$$
$$\frac{dy}{d\tau} = -x - \frac{a_{22} y + cy^2}{a_{21} - y}.$$

This has the form of Chang's system (1) in [6] with $\phi(x) = x$, $g(y) = y$, and

$$F(y) = \frac{a_{22} y + cy^2}{a_{21} - y}.$$

Cf. the Appendix at the end of this paper. It follows that

$$f(y) = F'(y) = \frac{-cy^2 + 2ca_{21} y + a_{21} a_{22}}{(y - a_{21})^2}.$$

Clearly $yg(y) = y^2 > 0$ for $y \neq 0$, $G(y) = \int_0^y \eta \, d\eta = y^2/2 \to \infty$ as $y \to \pm\infty$, and $f(y)$ is continuous for $y > a_{21}$. Since any limit cycle around the origin lies in the region $y > a_{21}$, it suffices to show that $f(y)/y$ is nondecreasing for $y > a_{21}$ and $y \neq 0$ in order to apply Chang's Theorem 1 in [6]. To do this, we compute

$$\left[\frac{f(y)}{y}\right]' = \frac{H(y)}{y^2 (y - a_{21})^3},$$

where

$$H(y) = cy^3 - 3ca_{21} y^2 - 3a_{21} a_{22} y + a_{21}^2 a_{22}.$$

Then under the hypotheses $c > 0$, $a_{22} > 0$, and $a_{21} < 0$, we clearly have $H(y) > 0$ for $y > 0$. To show that $H(y) \geqslant 0$ for $a_{21} < y < 0$, we compute

$$H'(y) = 3(cy^2 - 2ca_{21}y - a_{21}a_{22})$$

and

$$H''(y) = 6c(y - a_{21}).$$

For $a_{21} < 0$ and $ca_{21} + a_{22} < 0$ the quadratic equation $H'(y) = 0$ has exactly one root in the interval $(a_{21}, 0)$, namely,

$$y_1 = a_{21} + \sqrt{a_{21}^2 + (a_{21}a_{22}/c)}.$$

And under the hypotheses of this theorem, it follows that

$$H''(y_1) = 6c\sqrt{a_{21}^2 + (a_{21}a_{22}/c)} > 0$$

and

$$H(y_1) = -2a_{21}(a_{22} + ca_{21})y_1 > 0.$$

Thus, $H(y)$ has a positive minimum at $y = y_1$; i.e., $H(y) \geqslant H(y_1) > 0$ for all $y \in (a_{21}, 0)$. It therefore follows from Chang's Theorem 1 that under the hypotheses of this theorem, the system (1′) has at most one limit cycle around the origin. This completes the proof of Theorem 4.

The following corollaries are an immediate consequence of Theorems 1,. 1′, and 4 in this paper.

COROLLARY 1. *For* $0 < c < 2$, $a_{22} > 0$, $a_{21} < 0$, $a_{12} + a_{21} = 0$, *and* $a_{22} + ca_{21} < 0$, *the system* (1′) *has exactly one limit cycle around the origin and the separatrix configuration for this system is given by Fig.* 6(a).

COROLLARY 1′. *For* $a_{21} < 0$, $a_{12} + a_{21} = 0$, $-2 < \alpha_0 < 0$, *and* $0 < \alpha < 2 + \alpha_0$, *the system* (2) *has exactly one limit cycle around the origin and the separatrix configuration is given by Fig.* 6(a).

The following theorem also follows from Chang's Theorem 1 and it can be used to establish the uniqueness of the limit cycles in Theorems 2 and 2′.

THEOREM 5. *For* $c > 0$, $a_{22} \geqslant 0$, $a_{21} < 0$, $-a_{21} \leqslant a_{12} \leqslant -2a_{21}$, $a_{22} + ca_{21} \leqslant 0$, *and* $c(4a_{21} + a_{12}) + 3a_{22} \geqslant 0$, *the system* (1′) *has at most one limit cycle around the origin.*

Proof. Since on $y = a_{21}$ we have $\dot{y} = a_{21}(a_{22} + ca_{21}) \geqslant 0$, the line $y = a_{21}$ is a transversal (or a line of rest points) and no limit cycle intersects $y = a_{21}$. We again show that the conditions of Chang's Theorem 1 in [6] are satisfied in the region $y > a_{21}$. If we define the new independent variable τ by $d\tau = (y - a_{21})\,dt$, the system (1′) becomes

$$\frac{dx}{d\tau} = y\left(\frac{y + a_{12}}{y - a_{21}}\right),$$

$$\frac{dy}{d\tau} = -x - \frac{a_{22}y + cy^2}{a_{21} - y}.$$

This has the form of Chang's system (1) with $\phi(x) = x$,

$$g(y) = y\left(\frac{y + a_{12}}{y - a_{21}}\right),$$

and

$$F(y) = \frac{a_{22}y + cy^2}{a_{21} - y}.$$

It follows that

$$G(y) = \int_0^y g(\eta)\,d\eta = \frac{y^2}{2} + (a_{12} + a_{21})\,y + a_{21}(a_{12} + a_{21})\log\left(\frac{y - a_{21}}{|a_{21}|}\right)$$

and that

$$f(y) = F'(y) = \frac{-cy^2 + 2ca_{21}y + a_{21}a_{22}}{(y - a_{21})^2}.$$

Clearly $yg(y) > 0$ for $y \neq 0$ and $y > a_{21}$; $G(y) \to \infty$ as $y \to \pm\infty$; and $f(y)$ is continuous for $y > a_{21}$. It remains to show that $f(y)/g(y)$ is nondecreasing for $y > a_{21}$ and $y \neq 0$. From above

$$\frac{f(y)}{g(y)} = \frac{-cy^2 + 2ca_{21}y + a_{21}a_{22}}{y(y - a_{21})(y + a_{12})}$$

and it follows that

$$\left[\frac{f(y)}{g(y)}\right]' = \frac{H(y)}{y^2(y - a_{21})^2(y + a_{12})^2}$$

where

$$H(y) = cy^4 + 4cky^3 + k(2ck + a_{12}c + 3a_{22})\,y^2 + 2ka_{22}(a_{12} + k)\,y + a_{12}a_{22}k^2$$

and $k = -a_{21} > 0$. Under the hypotheses $c > 0$, $a_{12} > 0$, and $a_{22} > 0$, it follows that $[f(y)/g(y)]' > 0$ for $y > 0$. Also, since by hypothesis

$-a_{12} \leqslant a_{21}$, it follows that the above denominator is positive for $a_{21} < y < 0$. It remains to show that $H(y) \geqslant 0$ for $a_{21} < y < 0$. We do this by showing that under the hypotheses of this theorem $H(a_{21}) \geqslant 0$ and $H'(y) \geqslant 0$ for $a_{21} < y < 0$. First

$$H(a_{21}) = -a_{21}^2(ca_{21} + a_{22})(a_{21} + a_{12}) \geqslant 0$$

since by hypothesis $ca_{21} + a_{22} \leqslant 0$ and $a_{21} + a_{12} \geqslant 0$. We next show that $H'(y) \geqslant 0$ for $a_{21} < y < 0$ by showing that $H'(a_{21}) \geqslant 0$ and that $H''(y) \geqslant 0$ for $a_{21} < y < 0$. But

$$H'(a_{21}) = -2a_{21}(ca_{21} + a_{22})(2a_{21} + a_{12}) \geqslant 0$$

since by hypothesis $a_{21} < 0$, $ca_{21} + a_{22} \leqslant 0$, and $2a_{21} + a_{12} \leqslant 0$. And finally

$$H''(y) = 12c\left[y^2 + 2ky + \frac{k(2ck + a_{12}c + 3a_{22})}{6c}\right] \geqslant 0$$

since under the hypotheses of this theorem $H''(0) > 0$ and the discriminant of the above quadratic

$$4\left[k^2 - \frac{k(2ck + a_{12}c + 3a_{22})}{6c}\right] = \frac{2a_{21}}{3c}\left[(4a_{21} + a_{12})c + 3a_{22}\right] \leqslant 0.$$

It therefore follows from Chang's Theorem 1 that the system (1′) has at most one limit cycle around the origin. This completes the proof of Theorem 5.

The following corollary is an immediate consequence of Theorems 2 and 5.

COROLLARY 2. *For* $a_{22} > 0$, $a_{21} < 0$, $a_{22}/|a_{21}| < c < 2$, $-a_{21} < a_{12} \leqslant -2a_{21}$, $a_{21}(a_{22} + ca_{21}) < c(a_{12} + a_{21})^2$, *and* $c(4a_{21} + a_{12}) + 3a_{22} \geqslant 0$, *the system* (1′) *has exactly one limit cycle around the origin and it is stable.*

And the following corollary is a consequence of Theorems 2′ and 5.

COROLLARY 2′. *For* $a_{21} < 0$, $a_{12} + 2a_{21} = 0$, $-1 < \alpha_0 < 0$, *and* $-\alpha_0/2 \leqslant \alpha < -\alpha_0$, *the system* (2) *has exactly one limit cycle around the origin and exactly one limit cycle around the lower critical point at* $(2\alpha_0 a_{12}, -a_{12})$. *The separatrix configuration is given by Fig.* 3(b).

Proof. Under the above hypotheses, it follows from Theorem 2′ that there is at least one limit cycle around each of the critical points of system (2) for $0 < \alpha < -\alpha_0$. The uniqueness of the limit cycle around the origin for $-\alpha_0/2 \leqslant \alpha < -\alpha_0$ follows from Theorem 5 since we have $c = \alpha - \alpha_0 > -\alpha_0 > 0$, $a_{12} = -2a_{21} > 0$, $a_{22} = \alpha a_{12} > 0$, $a_{22} + ca_{21} = a_{12}(\alpha + \alpha_0)/2 < 0$, and $c(4a_{21} + a_{12}) + 3a_{22} = a_{12}(2\alpha + \alpha_0) \geqslant 0$ for the system (2). The uniqueness of the limit cycle around the lower critical point at $(2\alpha_0 a_{12}, -a_{12})$ follows by translating the origin to that critical point. This yields

$$\dot{x} = -a_{12}y + y^2,$$
$$\dot{y} = -a_{21}x - \alpha a_{12}y - xy + (\alpha - \alpha_0)y^2.$$

But under the coordinate transformation $x \to -x$, $y \to -y$, and $t \to -t$, this system is transformed into the system (2). It therefore follows from the uniqueness of the limit cycle around the origin of (2) established in the first part of this proof that, under the hypotheses of this corollary, there is exactly one limit cycle around the origin of the above system; i.e., around the lower critical point of (2). This completes the proof of Corollary 2′.

For $a_{12} + 2a_{21} = 0$, it follows from Theorem 5 that the system (1′) has at most one limit cycle around the origin if $2ca_{21} + 3a_{22} \geqslant 0$. We now show that this result also holds if $2ca_{21} + 3a_{22} < 0$, and this allows us to establish the uniqueness of the limit cycles in Theorem 2′ over their entire interval of existence.

THEOREM 6. *For* $c > 0$, $a_{22} > 0$, $a_{21} < 0$, $a_{12} + 2a_{21} = 0$, *and* $2ca_{21} + 3a_{22} < 0$, *the system* (1′) *has at most one limit cycle around the origin.*

Proof. As in the proof of Theorem 5, we use Chang's Theorem 1. The proof of this theorem is exactly the same as the proof of Theorem 5 except in showing that the function $H(y)$, defined in the proof of Theorem 5, is positive for $a_{21} < y < 0$. In this case with $a_{12} = -2a_{21} = 2k < 0$, we have

$$H(y) = cy^4 + 4cky^3 + k(4ck + 3a_{22})y^2 + 6k^2a_{22}y + 2k^3a_{22},$$

and it follows that

$$H'(y) = 2(y + k)(2cy^2 + 4cky + 3a_{22}k).$$

We show that $H(y) > 0$ for $a_{21} < y < 0$ by showing that $H(y)$ has only one minimum in $(a_{21}, 0)$ and that it is positive. For $a_{21} < 0$ and $2ca_{21} + 3a_{22} < 0$, the equation $H'(y) = 0$ has exactly one root in $(a_{21}, 0)$, namely,

$$y_1 = a_{21} + \sqrt{a_{21}^2 + (3a_{21}a_{22}/2c)}.$$

And under the hypotheses of this theorem,

$$H''(y_1) = 4a_{21}(2ca_{21} + 3a_{22}) > 0$$

and

$$H(y_1) = -a_{21}^2 a_{22}(8ca_{21} + 9a_{22})/4c > 0,$$

since $8ca_{21} + 9a_{22} < 4(2ca_{21} + 3a_{22}) < 0$. It follows that $H(y)$ has a positive minimum in $(a_{21}, 0)$; i.e., $H(y) \geqslant H(y_1) > 0$ for all $y \in (a_{21}, 0)$. It therefore follows from Chang's Theorem 1 in [6] that under the hypotheses of this theorem, the system (1′) has at most one limit cycle around the origin. This completes the proof of Theorem 6.

The following corollary is an immediate consequence of Theorems 2′ and 6. We simply note that for the system (2), $2ca_{21} + 3a_{22} = a_{12}(2\alpha + \alpha_0) < 0$ is equivalent to $\alpha < -\alpha_0/2$ for $a_{12} > 0$, that $a_{22} = \alpha a_{12} > 0$ is equivalent to $\alpha > 0$ for $a_{12} > 0$, and that the uniqueness of the limit cycle around the lower critical point of (2) follows exactly as in the proof of Corollary 2′.

COROLLARY 3. *For* $a_{21} < 0$, $a_{12} + 2a_{21} = 0$, $-1 < \alpha_0 < 0$, *and* $0 < \alpha < -\alpha_0/2$, *the system* (2) *has exactly one limit cycle around each of its critical points and the separatrix configuration is given by Fig.* 3(b).

The uniqueness of the limit cycle around the origin of (2), whose existence was established in Theorem 3 for $a_{12} + a_{21} < 0$, can probably also be established using Chang's Theorem 1, at least for α in some subinterval of $(0, \alpha^*)$; however, the details of the proof appear to be much more complicated in this case and the results much less interesting than the results in Corollaries 2′ and 3.

3. Nonexistence of Limit Cycles

In this section, we employ the Bendixson–Dulac criterion or its generalization given by Cheng [7] and various other results in order to show that under certain conditions on the coefficients, the system (1) has no limit cycles around the origin.

We first note that if $a_{21} = 0$ then the x axis consists of trajectories and there can be no limit cycles around the origin. Also if $a_{11} = a_{12} = 0$, then the flow across the y axis is everywhere to the right except at the origin and, once again, there can be no limit cycles around the origin. The next theorem follows from the Bendixson–Dulac criterion.

THEOREM 7. *If* $ca_{21}(ca_{21} + a_{22}) \leqslant -a_{11}^2$, *the system* (1) *has no limit cycles around the origin.*

Proof. If $a_{21} = 0$ there are no limit cycles around the origin and if $a_{21} \neq 0$ then, as in the proof of Theorem 4, the line $y = a_{21}$ is either a transversal or a line of rest points and therefore no limit cycle around the origin can intersect this line. In order to prove this theorem, we define the Dulac function

$$B(x, y) = \frac{1}{a_{21} - y}$$

and compute div(BP, BQ) with P and Q defined by the right-hand side of (1). We find

$$\frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} = \frac{-cy^2 + (2ca_{21} - a_{11})y + a_{21}(a_{11} + a_{22})}{(a_{21} - y)^2}.$$

If $ca_{21}(ca_{21} + a_{22}) \leqslant -a_{11}^2$, div$(BP, BQ)$ does not change sign for $y \neq a_{21}$ since the discriminant of the quadratic in the numerator of div(BP, BQ) is equal to $ca_{21}(ca_{21} + a_{22}) + a_{11}^2$. It therefore follows from the Bendixson–Dulac criterion, cf., e.g., [18, p. 176, or 19, p. 530], that under the hypotheses of this theorem there are no limit cycles in either of the regions $y < a_{21}$ or $y > a_{21}$. Since for $a_{21} \neq 0$ the origin lies in one of these regions, it follows that there are no limit cycles around the origin of the system (1) under the hypotheses of this theorem. This completes the proof of Theorem 7.

Note that for $a_{11} = 0$, Theorem 7 complements Theorems 4 and 5 in the previous section wherein $ca_{21}(ca_{21} + a_{22}) \geqslant 0$. The following corollaries are an immediate consequence of Theorem 7; the fact that there are no limit cycles around the lower critical point of (2) in Corollary 5 follows by translating the origin to the lower critical point and making the coordinate transformation $x \to -x$, $y \to -y$, and $t \to -t$, as in the proof of Corollary 2′, and then employing Theorem 7.

COROLLARY 4. *If* $a_{21} < 0$, $a_{12} + a_{21} = 0$, $\alpha_0 < 0$, *and* $\alpha \leqslant \alpha_0$, *the system* (2) *has no limit cycles around the origin.*

COROLLARY 5. *If* $a_{21} < 0$, $a_{12} + a_{21} > 0$, $\alpha_0 < 0$, *and either* $\alpha \leqslant \alpha_0$ *or* $\alpha > \alpha_0 a_{21}/(a_{12} + a_{21})$, *the system* (2) *has no limit cycles around either one of its critical points.*

The next result follows from Cheng's generalization of the Bendixson–Dulac criterion given in [7]. For convenience, a translation of Cheng's result is given in the Appendix at the end of this paper. We first note that for $a_{11} \neq 0$ the system (1) is transformed into the system (1) with $a_{11} = -1$ under the coordinate transformation $x \to -a_{11}x$, $y \to -a_{11}y$, and $t \to -t/a_{11}$. We therefore assume that $a_{11} = -1$ in the statement of the next theorem. This theorem can be used to show that, under certain conditions on the coefficients, the system (1) in the form considered in Theorems I and II in [4] has no limit cycles.

THEOREM 8. *For* $a_{11} = -1$, $a_{21} < 0$, $2c - a_{12} < 0$, *and* $a_{12}^2 - 2ca_{12} + 2a_{22} - 2 < 0$, *the system* (1) *has no limit cycles around the origin.*

Proof. Considering the flow on the x axis, we see that any limit cycle around the origin of (1) must be negatively oriented. In order to apply Cheng's criterion in [7], we define

$$B(x, y) = \left(\frac{2}{2c - a_{12}}\right) e^{-x}, \qquad M(x, y) = e^{-x}, \qquad \text{and} \qquad N(x, y) = 0,$$

and compute

$$\begin{aligned} &\frac{\partial}{\partial x}(NQ) - \frac{\partial}{\partial y}(MP) + \operatorname{div}(BP, BQ) \\ &\qquad = e^{-x}\left[\frac{(a_{12}^2 - 2ca_{12} + 2a_{22} - 2)}{2c - a_{12}} + \frac{2}{a_{12} - 2c} y^2\right]. \end{aligned}$$

Under the conditions of this theorem, the above quantity is *positive*, and it follows from Cheng's criterion [7], given in the Appendix, that there are no limit cycles around the origin. This completes the proof of Theorem 8.

We conclude this section by citing a result of Cherkas [20] that allows us to complete the proofs of the two theorems in the introduction. In order to complete the proofs of those theorems, it is necessary to show that the systems in question have no limit cycles for $\alpha_0 < \alpha < 0$.

Remark 7. If we did not know that there are no limit cycles around the origin of the system (2) with $a_{12} = -a_{21} = 1$ and $\alpha_0 < \alpha < 0$, it would then be possible for a semi-stable limit cycle (or any number of semi-stable limit cycles) to appear around the origin. In this case an *unstable* limit cycle would necessarily be generated at the origin at $\alpha = 0$ and it would expand with *decreasing* α rather than with increasing α; and instead of the simple behavior depicted in Fig. 1, we would then have the configuration in Fig. 1(a) for $-2 + \alpha_0 < 2 < \alpha_1$, a semi-stable limit cycle appearing at $\alpha = \alpha_1$, two (or more) limit cycles around the origin for $\alpha_1 < \alpha < 0$ and the configuration in Fig. 1(b) for $0 \leqslant \alpha < 2 + \alpha_0$; viz. Fig. 13. Fortunately, this does not happen; the nonexistence of limit cycles for $\alpha_0 < \alpha < 0$ is a consequence of the following lemma which is Cherkas' Theorem 5(1) in [20].

LEMMA. *If* $b < 0$, $ac \geqslant 0$, *and* $ac \leqslant 0$, *then the system*

$$\begin{aligned} \dot{x} &= y + y^2, \\ \dot{y} &= -x + \alpha y + ax^2 + bxy + cy^2 \end{aligned}$$

has no limit cycles around the origin.

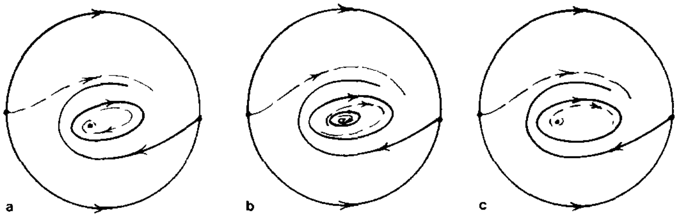

FIG. 13. (a) $\alpha = \alpha_1$; (b) $\alpha_1 < \alpha < 0$; (c) $0 \leqslant \alpha < 2 + \alpha_0$.

Cherkas proves this lemma by showing that if the above system has a limit cycle around the origin then, under the conditions of the above lemma, the limit cycle and the origin have the same stability; and this leads to a contradiction. The next corollary follows immediately from this lemma with $b = -1$, $a = 0$, $\alpha \leqslant 0$, and $c = \alpha - \alpha_0 \geqslant 0$.

COROLLARY 6. *If* $a_{12} = -a_{21} = 1$, $\alpha_0 < 0$, *and* $\alpha_0 \leqslant \alpha \leqslant 0$, *the system* (2) *has no limit cycles around the origin.*

The first theorem in the introduction then follows from Theorem 1′ and Corollaries 1′, 4, and 6 in this paper. In order to complete the proof of the second theorem in the introduction, we simply note that the system (2) with $a_{12} = 1$ and $a_{21} = -\frac{1}{2}$ is transformed, under the coordinate transformation $x \to x/\sqrt{2}$, $t \to t/\sqrt{2}$, into

$$\dot{x} = y + y^2,$$
$$\dot{y} = -x + \sqrt{2}\,\alpha y - 2xy + \sqrt{2}\,(\alpha - \alpha_0)\,y^2.$$

It then follows immediately from the above lemma that the system (2) with $a_{12} = 1$, $a_{21} = -\frac{1}{2}$, $\alpha_0 < 0$, and $\alpha_0 \leqslant \alpha \leqslant 0$ has no limit cycles around the origin. It follows by translating the origin to the lower critical point as in the proof of Corollary 2′, making the coordinate transformation $x \to -x/\sqrt{2}$, $y \to -y$, $t \to -t/\sqrt{2}$ to obtain the above system, and employing the above lemma, that there are no limit cycles around the lower critical point of (2) for $a_{12} = 1$, $a_{21} = -\frac{1}{2}$, $\alpha_0 < 0$, and $\alpha_0 \leqslant \alpha \leqslant 0$. This proves the following corollary.

COROLLARY 7. *If* $a_{12} = 1$, $a_{21} = -\frac{1}{2}$, $\alpha_0 < 0$, *and* $\alpha_0 \leqslant \alpha \leqslant 0$, *the system* (2) *has no limit cycles around either one of its critical points.*

The second theorem in the introduction then follows from Theorem 2′ and Corollaries 2′, 3, 5, and 7 in this paper.

This concludes this paper on existence, uniqueness, and nonexistence of limit cycles for the class of quadratic systems (1). We note that the methods used in this paper can also be applied to establish the existence, uniqueness, and nonexistence of limit cycles of more general classes of polynomial systems in the plane. Also note that in order to uniquely determine the global phase portrait of a planar system or to determine the global behavior of the limit cycles of a planar system, it is first necessary to determine the exact number of limit cycles around each critical point of the system. This has been accomplished in this paper by showing that there are either no limit cycles or exactly one limit cycle around each of the critical points of (1′). Finally, we note that we establish an example of a quadratic system with *exactly two* limit cycles in this paper. This is apparently the first time that such an example has been extablished in the literature. It would certainly be desirable to establish that Example 3 of Tung in [12] and the example of Bautin's system in [9] have *exactly three* limit cycles and that the example of Shi in [10] has *exactly four* limit cycles.

APPENDIX

In this Appendix we give a translation of Chang's theorems in [6] and of Cheng's theorem in [7] for easy reference. Chang's Theorem 1 in [6] is essentially reproduced as Theorem 13 in [19]; however, the hypothesis $g(\pm\infty)=\infty$ in Theorem 13 in [19] should read $G(\pm\infty)=\infty$.

In [6], Chang considers the system

$$\begin{aligned} \dot{x} &= g(y) \\ \dot{y} &= -\phi(x) - F(y) \end{aligned} \tag{A}$$

and states the following theorem.

THEOREM 1. *If* $\phi(x)=x$, $f(y)=F'(y)$ *is continuous on* $-\infty<y<0$ *and on* $0<y<\infty$, *and if the following conditions are satisfied*:

(1) $yg(y)>0$ *for* $y\neq 0$, $G(\pm\infty)=\infty$, *where* $G(y)=\int_0^y g(\eta)\,d\eta$, $g(y)$ *is continuous and satisfies a Lipschitz condition on any interval*;

(2) $f(y)/g(y)$ *is nondecreasing on* $-\infty<y<0$ *and on* $0<y<\infty$, $f(y)/g(y)$ *is not constant in any neighborhood of* $y=0$ *and* $F(0)=0$; *it then follows that the system* (A) *has at most one limit cycle around the origin, and if that limit cycle exists, it is stable.*

The proof of this theorem follows by showing that if the origin is a stable critical point then under the hypotheses (1) and (2) of this theorem there can be no limit cycle around the origin. And if the origin is unstable and L_1 is the smallest limit cycle around the origin, it must be stable on its interior and then by the Poincaré criterion

$$\oint_{L_1} f[x_1(t)]\,dt \geqslant 0$$

where $x = x_1(t)$, $y = y_1(t)$ describes the limit cycle L_1. But if L_2 is any other limit cycle enclosing L_1, it can be shown that, under the hypotheses of this theorem,

$$\oint_{L_2} f[x_2(t)]\,dt > \oint_{L_1} f[x_1(t)]\,dt;$$

i.e.,

$$\oint_{L_2} f[x_2(t)]\,dt > 0$$

where $x = x_2(t)$, $y = y_2(t)$ describes the limit cycle L_2. Thus, by the criterion of Poincaré, L_2 is stable. It follows that if L_1 is stable, then no limit cycle can exist on the exterior of L_1. If L_1 is semistable, then any small rotation of the vector field (in the appropriate sense) would cause L_1 to split into a stable limit cycle L_{1s} and an unstable limit cycle L_{1u} with $L_{1s} \subset \mathrm{Int}(L_{1u})$. And for a sufficiently small rotation it can similarly be shown that any limit cycle on the exterior of L_{1s} is stable. Thus, L_1 cannot be semistable. Therefore, if the origin is unstable, the system (A) has at most one limit cycle around the origin, and if that limit cycle exists, it is stable. (The details of this proof can be found in Chang's doctoral dissertation, Moscow State University, 1958.)

Chang also states that Theorem 1 holds if in addition to the above conditions we require that $x\phi(x) > 0$ for $x \neq 0$, $\phi(\pm\infty) = \infty$, $\phi(x)$ is continuous, strictly monotonic, and satisfies a Lipschitz condition (on any interval); $\phi(x)$ has right- and left-hand derivatives at the origin, $\phi'_+(0)$ and $\phi'_-(0)$; and $\phi'_+(0) = \phi'_-(0) \neq 0$ in the case when $f(0) = 0$.

The following theorem, although not used in this paper, also gives a useful uniqueness criterion.

THEOREM 2. *If condition* (1) *of Theorem* 1 *holds and if the following condition is satisfied*:

(2′) *$f(y)$ is continuous and $F[y(u)]/u$ is nondecreasing for increasing $|u|$, where $y = y(u)$ is defined by the equation $u = u(y) = \mathrm{sgn}(y)\sqrt{2G(y)}$; it then follows that the system* (A) *has at most one limit cycle around the origin.*

THEOREM (Cheng's Criterion). *Let G be a simply connected region in the plane. If*

$$\dot{x} = P(x, y)$$
$$\dot{y} = Q(x, y)$$

has a positively (negatively) oriented limit cycle Γ in G, then for any functions B(x, y), M(x, y), N(x, y) having continuous first partial derivatives in G with M and N either identically zero or positive in G, but not both identically zero, it follows that

$$\iint_{\mathrm{Int}(\Gamma)} \left[\frac{\partial}{\partial x}(NQ) - \frac{\partial}{\partial y}(MP) + \frac{\partial}{\partial x}(BP) + \frac{\partial}{\partial y}(BQ)\right] dx\, dy > 0 \quad (<0).$$

REFERENCES

1. L. MARKUS, Quadratic differential equations and non-associative algebras, Ann. of Math. Stud., No. 45, pp. 185–213, Princeton Univ. Press, Princeton, N.J., 1960.
2. R. J. DICKSON AND L. M. PERKO, Bounded quadratic systems in the plane, *J. Differential Equations* **7** (1970), 251–273.
3. L. M. PERKO, "Phase Portraits for Quadratic Systems in the Plane I," Lockheed research report, LMSC 6-78-68-9, 1968.
4. L. M. PERKO, Rotated vector fields and the global behavior of limit cycles for a class of quadratic systems in the plane, *J. Differential Equations* **18** (1975), 63–86.
5. G. F. D. DUFF, Limit cycles and rotated vector fields, *Ann. of Math.* **67** (1953), 15–31.
6. C.-F. CHANG, On the uniqueness of limit cycles of certain equations of nonlinear oscillations, *Dokl. Akad. Nauk. USSR* **19** (1958), 659–662. [Russian]
7. G.-C. CHENG, A new criterion for the non-existence of limit cycles and applications, *Acta Math. Sinica* **20** (1977), 281–284. [Chinese]
8. N. N. BAUTIN, On the number of limit cycles which appear with variation of coefficients from an equilibrium position of focus or center type, *Mat. Sb.* **30** (1952), 181–196; *Amer. Math. Soc. Transl.* (1962), 396–413.
9. Y.-S. CHIN ET AL., Concrete examples of existence of three limit cycles for the system $dx/dt = X_2(x, y)$, $dy/dt = Y_2(x, y)$, *Acta Math. Sinica* **9** (1959), 213–226. [Chinese]
10. S.-L. SHI, A concrete example of the existence of four limit cycles for plane quadratic systems, *Sci. Sinica* **23** (1980), 153–158.
11. I. G. PETROVSKII AND E. M. LANDIS, On the number of limit cycles of the equation $dy/dx = P(x, y)/Q(x, y)$ where P and Q are polynomials of the second degree, *Mat. Sb.* **37** (1955), 209–250; *Amer. Math. Soc. Transl.* **10** (1958), 177–221.
12. C.-C. TUNG, Position of limit cycles of the system $dx/dt = \cdots$, $dy/dt = \cdots$, *Sci. Sinica* **8** (1959), 151–171.
13. Y.-C. YEH, Periodic solutions and limit cycles of certain nonlinear differential equations, *Sci. Record* **1** (1957), 391–394.
14. Y.-C. YEH, A qualitative study of the integral curves of the differential equation $dy/dx = \cdots$. II. Uniqueness of limit cycles, *Chinese Math.* **3** (1963), 62–70.
15. M. FROMMER, Über das Auftreten von Wirbeln und Strudeln in der Urngebung rationaler Unbestinmtheitsstellen, *Math. Ann.* **109** (1934), 395–424.

16. S. LEFSCHETZ, "Differential Equations: Geometric Theory," Interscience, New York, 1957.
17. C. CHICONE AND D. S. SHAFER, Separatrix and limit cycles of quadratic systems and Dulac's theorem, *Trans. Amer. Math. Soc.* **278** (1983), 585–612.
18. C. CHICONE AND J.-H. TIAN, On general properties of quadratic systems, *Amer. Math. Monthly* **89** (1982), 167–178.
19. L. A. CHERKAS, Estimation of the number of limit cycles of autonomous systems, *Differential Equations* **13** (1977), 529–547.
20. L. A. CHERKAS AND L. I. ZHILEVICH, Some criteria for the absence of limit cycles and for the existence of a single limit cycle, *Differential Equations* **6** (1970), 891–897.