Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 237 (2007) 259–277

Journal of Differential Equations

www.elsevier.com/locate/jde

Traveling waves for non-local delayed diffusion equations via auxiliary equations ☆

Shiwang Ma *

School of Mathematical Sciences, Nankai University, Tianjin 300071, PR China

Received 28 April 2004; revised 10 May 2006

Available online 31 March 2007

To my parents

Abstract

In this paper, we study the existence of traveling wave solutions for a class of delayed non-local reaction–diffusion equations without quasi-monotonicity. The approach is based on the construction of two associated auxiliary reaction–diffusion equations with quasi-monotonicity and a profile set in a suitable Banach space by using the traveling wavefronts of the auxiliary equations. Under monostable assumption, by using the Schauder's fixed point theorem, we then show that there exists a constant $c_* > 0$ such that for each $c > c_*$, the equation under consideration admits a traveling wavefront solution with speed c, which is not necessary to be monotonic.

MSC: 34K30; 35B40; 35R10; 58D25

Keywords: Delayed non-local reaction–diffusion equation; Non-quasi-monotonicity; Monostable; Traveling wavefront; Existence; Schauder's fixed point theorem

1. Introduction and main result

Traveling wave solutions of reaction–diffusion equations have been extensively investigated due to their important role in describing the long term behavior of solutions to the associated initial value problems (see [2,11]). Such solutions also have their own practical background, such

☆ Research supported by the National Natural Science Foundation of China.
* On leave from the Department of Mathematics, Shanghai Jiaotong University, Shanghai 200030, China.
E-mail address: shiwangm@163.net.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.03.014

as transition between different states of a physical system, propagation of patterns, and domain invasion of species in population biology. For the reaction–diffusion equation of *monostable type*

$$w_t(t,x) = Dw_{xx}(t,x) + f\big(w(t,x)\big), \quad x \in \mathbb{R},\ t \geqslant 0, \tag{1.1}$$

with $f(w)$ satisfying $f(0) = f(k) = 0$ for some $k > 0$, and $f(w) > 0$ for all $w \in (0,k)$, it is well known from long time ago that in the case $f(w) \leqslant f'(0)w$ for all $w \in (0,k)$, $c_{\min} = 2\sqrt{Df'(0)} > 0$ is the minimal wave speed in the sense that (i) for every $c > c_{\min}$ there exists a traveling wavefront of the form $w(x,t) = u(x+ct)$ with $u(s)$ increasing and $u(-\infty) = 0$, $u(\infty) = k$; (ii) the wavefront is unique up to translation; (iii) for $c < c_{\min}$, there is no such monotone wavefront with speed c. It is also well known that the existence of traveling waves of Eq. (1.1) is independent of the monotonicity of the reaction function f.

Another area of intensive study in ecological modeling has been the incorporation of time lags and the role of time-delays in the dynamics of the solutions of the resulting equations. Not surprisingly, traveling wave solution of parabolic equations with delay have become the subject of considerable interest in recent years. We refer to the book by Wu [14] for theoretical discussion on delay equations with diffusion arising in biological and ecological problems.

Recently, traveling waves for non-local reaction–diffusion equations have also attracted much attention and up to now many significant results have been published [3,4,6,8–10]. We also refer the readers to [5] for a survey of the short history and the current status of the study of reaction–diffusion equations with non-local delayed interactions.

Mathematically, both time delay and non-local term present significant additional technical difficulties in the study of the existence of traveling wave solutions. In order to obtain some existence results, many researchers have proposed the so-called quasi-monotonicity assumptions on the reaction functions [1,5,7–10,13]. These conditions are very restrictive and most of reaction–diffusion equations with delay or non-local response *do not* satisfy such conditions. The existence of traveling waves for delayed non-local diffusion equations with non-quasi-monotonic reaction terms seem to be very interesting and challenging problem.

In this paper, we shall develop a new approach to obtain the existence of traveling wave solutions for delayed non-local reaction–diffusion equations without quasi-monotonicity. Our approach is based on the well-known Schauder's fixed point theorem. To apply this fixed point theorem, our main objective is to construct two associated auxiliary reaction–diffusion equations with quasi-monotonicity and a profile set in a suitable Banach space by using the traveling wavefronts of the auxiliary equations.

In the present paper, we consider the non-local delayed reaction–diffusion equation

$$w_t(t,x) = Dw_{xx}(t,x) - g\big(w(t,x)\big) + h\big(w(t,x)\big)\int_{\mathbb{R}} f\big(w(t-r,y)\big)J(x-y)\,dy, \tag{1.2}$$

where $D > 0$, $r \geqslant 0$ are constants, f, h and g are Lipschitz continuous functions on any compact interval, $f(0) = g(0) = 0$ and $g(K) = h(K)f(K)$ for some $K > 0$. We always assume that $J(y) = J(-y) \geqslant 0$ for all $y \in R$ and

$$\int_{\mathbb{R}} J(y)\,dy = 1, \qquad \int_{\mathbb{R}} e^{-\lambda y} J(y)\,dy < +\infty,$$

for any $\lambda > 0$.

When $J(x) = \delta(x)$, where $\delta(\cdot)$ is the Dirac delta function, then (1.2) reduces to the local equation

$$w_t(t,x) = Dw_{xx}(t,x) - g\big(w(t,x)\big) + h\big(w(t,x)\big) f\big(w(t-r,x)\big). \tag{1.3}$$

A prototype of such equations which has been widely investigated in the literature (see [8,9] and references therein) reads as

$$w_t(t,x) = Dw_{xx}(t,x) - dw(t,x) + \epsilon \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{(x-y)^2}{4\alpha}} b\big(w(t-r,y)\big)\,dy, \tag{1.4}$$

which was derived by So, Wu and Zou [9] as a model to describe the evolution of the adult population of a single species population with two age classes and moving around in a unbounded 1-dimensional spatial domain. In this context, $D > 0$ and $d > 0$ denote the diffusion rate and death rate of the adult population, respectively, $r \geqslant 0$ is the maturation time for the species, $b(\cdot)$ is the birth function, and $\epsilon > 0$ and $\alpha \geqslant 0$ reflect the impact of the death rate and the dispersal rate of the immature on the matured population, respectively. When $\alpha \to 0$, that is, as the immature become immobile, (1.4) reduces to

$$w_t(t,x) = Dw_{xx}(t,x) - dw(t,x) + \epsilon b\big(w(t-r,y)\big), \tag{1.5}$$

and the non-local effect disappears. We refer to So, Wu and Zou [9] for more details.

As another special case of Eq. (1.2), the following reaction–diffusion equation

$$w_t(t,x) = Dw_{xx}(t,x) - \beta w^2(t,x) + \alpha e^{-\gamma r} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi dr}} e^{-\frac{(x-y)^2}{4dr}} w(t-r,y)\,dy \tag{1.6}$$

with $\beta > 0$, $d \geqslant 0$, $\alpha > 0$ and $r \geqslant 0$, was also proposed by Gourley and Kuang [4] to describe the evolution of the mature population of a single species population with age structure.

We assume that there exists $K^* \geqslant K$ such that $g(K^*)/h(K^*) \geqslant \max\{f(u) \mid 0 \leqslant u \leqslant K^*\}$ and $g(u)/h(u) < g(K^*)/h(K^*)$ for all $u \in [0, K^*)$.

If $f'(0) > g'(0)/h(0)$ and $f(u) > 0$ for all $u \in (0, K^*]$, then

$$K_* := \inf\Big\{u \,\Big|\, g(u)/h(u) = \inf_{s\in(0,K^*]}\big\{f(s) \,\big|\, f(s) \leqslant g(s)/h(s)\big\}\Big\} > 0$$

is well defined and $f(u) > g(u)/h(u)$ for all $u \in (0, K_*)$.

We also need the following assumptions:

(H1) $f'(0) > g'(0)/h(0)$ and there is a $\nu \in (0,1]$ such that

$$\limsup_{u\to 0^+}\big[f'(0) - f(u)/u\big]u^{-\nu} < +\infty, \qquad \limsup_{u\to 0^+}\big[g(u)/u - g'(0)\big]u^{-\nu} < +\infty$$

and

$$\limsup_{u\to 0^+}\big[h(0) - h(u)\big]u^{-(1+\nu)} < +\infty;$$

(H2) $f'(0)u \geqslant f(u) > 0$, $g(u) \geqslant g'(0)u$, $h(0) \geqslant h(u) > 0$ and $f'(0)u > g(u)/h(u)$ for all $u \in (0, K^*]$;
(H3) $g(u)/h(u)$ is strictly increasing on $[K_*, K^*]$ and

$$g(u)/h(u) < f(u) < 2g(K)/h(K) - g(u)/h(u) \quad \text{for } u \in [K_*, K),$$
$$g(u)/h(u) > f(u) > 2g(K)/h(K) - g(u)/h(u) \quad \text{for } u \in (K, K^*].$$

In the above assumptions, by $f'(0) > g'(0)/h(0)$, we mean that $f(u)$ and $g(u)$ are differentiable at $u = 0$ and $f'(0) > g'(0)/h(0)$, and the others can be understood similarly. It is easily seen that if f, g, h are of class C^2, $f'(0) > g'(0)/h(0)$ and $h'(0) = 0$, then (H1) holds spontaneously.

In the present paper, we are interested in finding traveling waves $w(t, x) = U(x + ct)$ of Eq. (1.2), with $\lim_{\xi\to-\infty} U(\xi) = 0$ and $\lim_{\xi\to+\infty} U(\xi) = K$. To this end, we need to find a solution $U(\xi)$, where $\xi = x + ct$, for the following associated wave equation

$$cU'(\xi) - DU''(\xi) + g\big(U(\xi)\big) - h\big(U(\xi)\big) \int_{\mathbb{R}} f\big(U(\xi - y - cr)\big) J(y)\, dy = 0, \tag{1.7}$$

subject to the boundary conditions

$$U(-\infty) := \lim_{\xi\to-\infty} U(\xi) = 0, \qquad U(+\infty) := \lim_{\xi\to+\infty} U(\xi) = K. \tag{1.8}$$

If the function $f(u)$ is nondecreasing on the interval $[0, K]$, then the problem becomes easier since the whole interaction term is quasi-monotone. One can apply the upper–lower solutions and monotone iteration technique developed in Wu and Zou [12] to establish the existence of monotone traveling wavefronts. This trick has been used in several papers (see [7,9,12,13]). When K is such that $f(u)$ is not increasing on $[0, K]$, the monotone iteration method developed in Wu and Zou [12] cannot be used as the involved iteration scheme is no longer monotone. Due to lack of quasi-monotonicity, the problem becomes much harder and a little has been done for the existence of traveling waves for reaction–diffusion equations without quasi-monotonicity. For such delayed equations without quasi-monotonicity, some existence results for traveling waves have been obtained in Huang and Zou [6] and Wu and Zou [12,13] by using the idea of the so-called exponential ordering for delayed differential equations. But these results are valid only for small values of the delay r, and their application to particular model equations is not trivial as it requires construction of very demanding upper–lower solutions.

In this paper, we *do not* assume that the function $f(u)$ is nondecreasing on the interval $[0, K]$. Thus, we cannot apply the upper–lower solutions and monotone iteration technique developed in Wu and Zou [12] to establish the existence of monotone traveling wavefronts. Fortunately, we can use an argument as used in Ma [7] and the Schauder's fixed point theorem to prove the existence of traveling waves that is not necessary to be monotonic.

Now we formulate our main result as follows:

Theorem 1.1. *Assume that* (H1) *and* (H2) *hold. Then there exists $c_* > 0$ such that for every $c > c_*$, Eq.* (1.2) *admits a traveling wave solution $w(t, x) = U(x + ct)$ satisfying $U(-\infty) = 0$ and $K^* \geqslant \limsup_{\xi\to+\infty} U(\xi) \geqslant \liminf_{\xi\to+\infty} U(\xi) \geqslant K_* > 0$. If, in addition,* (H3) *also holds, then $U(+\infty) = K$.*

Remark 1.1. The result obtained in Theorem 1.1 is still valid for the local equation (1.3). In our main Theorem 1.1, instead of (H3), we may assume that the following weaker condition holds: $g(u)/h(u)$ is strictly increasing in $[K_*, K^*]$, $g(u)/h(u) < f(u) \leqslant 2g(K)/h(K) - g(u)/h(u)$ for $u \in [K_*, K)$, $g(u)/h(u) > f(u) \geqslant 2g(K)/h(K) - g(u)/h(u)$ for $u \in (K, K^*]$, and there is no pair $\alpha, \beta \in [K_*, K^*]$ with $\alpha < K < \beta$, such that $g(\alpha)/h(\alpha) \geqslant f(\beta)$ and $g(\beta)/h(\beta) \leqslant f(\alpha)$. In particular, if $g(K)/h(K) \geqslant f(u) > g(u)/h(u)$ for all $u \in (0, K)$, then $K^* = K = K_*$, and hence these conditions hold spontaneously.

Remark 1.2. In [9], the authors have taken a particular birth function $b(u) = pue^{-au}$, where $p > 0$ and $a > 0$ are parameters, and it has been shown that if $1 < \frac{\epsilon p}{d} \leqslant e$, then there exists $c_* > 0$ such that for every $c > c_*$, Eq. (1.4) has a nondecreasing wave solution which connects the trivial equilibrium $w_1 = 0$ and $w_2 = \frac{1}{a}\ln\frac{\epsilon p}{d}$. Recently, it has been shown in [3] that if $e < \frac{\epsilon p}{d} \leqslant e^2$, then there exist $r^* > 0$ and a *sufficiently large* $c^* > 0$ such that for every $c > c^*$, Eq. (1.4) with $r < r^*$ admits a traveling wave solution connecting w_1 and w_2. As a direct consequence of Theorem 1.1, we can easily show that if $1 < \frac{\epsilon p}{d} \leqslant e^2$, then for every $c > c_*$, Eq. (1.4) admits a traveling wave solution connecting w_1 and w_2 for all values of the delay $r \geqslant 0$.

Remark 1.3. In contrast to the results in Huang and Zou [6] and Wu and Zou [12,13], our main Theorem 1.1 is valid for all values of the delay $r \geqslant 0$.

Remark 1.4. The technique used in the present paper can been used to obtain analogous results for lattice differential equations

$$u_n'(t) = D\sum_{i\in\mathbb{Z}\setminus\{0\}} J(i)\big[u_{n-i}(t) - u_n(t)\big] - du_n(t) + \sum_{i\in} Q(i)b\big(u_{n-i}(t-r)\big), \tag{1.9}$$

where $u_n(t) \in \mathbb{R}$, $n \in \mathbb{Z}$, $t > 0$, $D, d > 0$, $r \geqslant 0$, $b(\cdot)$ is a Lipschitz continuous function on any compact interval and $b(0) = dK - b(K) = 0$ for some $K > 0$. Moreover, the kernel functions J and Q are non-negative and satisfy

$$\sum_{i\in\mathbb{Z}\setminus\{0\}} J(i) = 1, \qquad \sum_{i\in\mathbb{Z}\setminus\{0\}} J(i)e^{-\lambda i} < +\infty,$$
$$\sum_{i\in\mathbb{Z}} Q(i) = 1, \qquad \sum_{i\in\mathbb{Z}} Q(i)e^{-\lambda i} < +\infty,$$

for any $\lambda > 0$. Under the assumptions

(H4) $b'(0) > d$ and $\limsup_{u\to 0^+}[b'(0) - b(u)/u]u^{-\nu} < +\infty$ for some $\nu \in (0, 1]$;
(H5) $\min\{b'(0)u, dK^*\} \geqslant b(u) > 0$ for some $K^* \geqslant K$ and for all $u \in (0, K^*]$;
(H6) $du < b(u) < d(2K - u)$ for $u \in [K_*, K)$, and $du > b(u) > d(2K - u)$ for $u \in (K, K^*]$, where $K_* := \frac{1}{d}\inf_{u\in(0,K^*]}\{b(u) \mid b(u) \leqslant du\}$

the following result can been established.

Theorem 1.2. *Assume that* (H4) *and* (H5) *hold. Then there exists* $c_* > 0$ *such that for every* $c > c_*$, *Eq.* (1.9) *admits a traveling wave solution* $u_n(t) = U(n+ct)$ *satisfying* $U(-\infty) = 0$

and $K^* \geqslant \limsup_{\xi\to+\infty} U(\xi) \geqslant \liminf_{\xi\to+\infty} U(\xi) \geqslant K_* > 0$. *If, in addition,* (H6) *also hold, then* $U(+\infty) = K$.

The rest of this paper is organized as follows. In Section 2, by using the super- and subsolution technique and the Schauder's fixed point theorem, we study the existence and the behavior of traveling wavefronts of (1.2) when the function f is nondecreasing in $[0, K]$. In Section 3, we firstly construct two auxiliary reaction–diffusion equations with quasi-monotonicity and then prove our main result by using the results obtained in Section 2 and the Schauder's fixed point theorem.

2. Preliminary lemmas

Our approach for the existence of traveling wavefronts of Eq. (1.2) is based on the Schauder's fixed point theorem. To this end, we need to construct a closed subset in a suitable Banach space. In the present paper, this closed subset is constructed by using the nondecreasing traveling wavefronts of two auxiliary reaction–diffusion equations with quasi-monotonicity. Therefore, we must study the existence and the behavior of traveling wavefronts for delayed reaction–diffusion equations with quasi-monotonicity.

In this section, we always assume that (H1), (H2) hold, $K^* = K$ and $f(u)$ is nondecreasing on $[0, K]$.

At first, we establish the existence of traveling waves of (1.2) under these conditions by using the sub–super-solution technique and the Schauder's fixed point theorem.

We set

$$\Delta(c,\lambda) := c\lambda - D\lambda^2 + g'(0) - h(0)f'(0)\int_{\mathbb{R}} e^{-\lambda(y+cr)} J(y)\,dy. \tag{2.1}$$

If $f'(0) > g'(0)/h(0)$, we have $\Delta(c,0) = g'(0) - h(0)f'(0) < 0$ for all $c \geqslant 0$ and $\lim_{\lambda\to+\infty} \Delta(c,\lambda) = -\infty$. For fixed $c \geqslant 0$ and any $\lambda_1, \lambda_2 \geqslant 0$ with $\lambda_1 \neq \lambda_2$, we have

$$\begin{aligned}
&\frac{1}{2}\big[\Delta(c,\lambda_1) + \Delta(c,\lambda_2)\big] \\
&\quad = c\frac{\lambda_1+\lambda_2}{2} - D\frac{\lambda_1^2+\lambda_2^2}{2} + g'(0) - h(0)f'(0)\int_{\mathbb{R}} \frac{e^{-\lambda_1(y+cr)} + e^{-\lambda_2(y+cr)}}{2} J(y)\,dy \\
&\quad < c\frac{\lambda_1+\lambda_2}{2} - D\left(\frac{\lambda_1+\lambda_2}{2}\right)^2 + g'(0) - h(0)f'(0)\int_{\mathbb{R}} e^{-(\lambda_1+\lambda_2)(y+cr)/2} J(y)\,dy \\
&\quad = \Delta\left(c, \frac{\lambda_1+\lambda_2}{2}\right).
\end{aligned}$$

Differentiating $\Delta(c,\lambda)$ with respect to c, we get

$$\frac{\partial}{\partial c}\Delta(c,\lambda) = \lambda + \lambda r h(0)f'(0)\int_{\mathbb{R}} e^{-\lambda(y+cr)} J(y)\,dy > 0 \quad \text{for all } \lambda > 0.$$

Furthermore, for each fixed $\lambda > 0$, we have $\lim_{c\to+\infty} \Delta(c,\lambda) = +\infty$ and

$$\Delta(0,\lambda) = -D\lambda^2 + g'(0) - h(0)f'(0)\int_{\mathbb{R}} e^{-\lambda y}J(y)\,dy < 0.$$

Therefore, we have the following observations:

Lemma 2.1. *Assume that $f'(0) > g'(0)/h(0)$. Then there exists a unique $c_* > 0$ such that*

(i) *if $c \geqslant c_*$, then there exist two positive numbers $\Lambda_1(c)$ and $\Lambda_2(c)$ with $\Lambda_1(c) \leqslant \Lambda_2(c)$ such that*

$$\Delta\big(c, \Lambda_1(c)\big) = \Delta\big(c, \Lambda_2(c)\big) = 0;$$

(ii) *if $c < c_*$, then $\Delta(c,\lambda) < 0$ for all $\lambda \geqslant 0$;*
(iii) *if $c = c_*$, then $\Lambda_1(c) = \Lambda_2(c) := \Lambda_*$, and if $c > c_*$, then $\Lambda_1(c) < \Lambda_* < \Lambda_2(c)$ and*

$$\Delta(c,\cdot) > 0 \quad \text{in } \big(\Lambda_1(c), \Lambda_2(c)\big), \qquad \Delta(c,\cdot) < 0 \quad \text{in } \mathbb{R}\setminus\big[\Lambda_1(c), \Lambda_2(c)\big].$$

Definition 2.1. A continuous function $\phi : \mathbb{R} \to [0,K]$ is called a *supersolution* of (1.7) if there exists a finite subset $S \subset \mathbb{R}$ such that ϕ is twice continuously differentiable in $\mathbb{R}\setminus S$ and

$$\begin{aligned} N_c[\phi](\xi) :={}& c\phi'(\xi) - \phi''(\xi) + g\big(\phi(\xi)\big) \\ &- h\big(\phi(\xi)\big)\int_{\mathbb{R}} f\big(\phi(\xi - y - cr)\big)J(y)\,dy \geqslant 0, \quad \text{on } \mathbb{R}\setminus S. \end{aligned} \tag{2.2}$$

A *subsolution* of (1.7) is defined in a similar way by reversing the inequality in (2.2).

Lemma 2.2. *Assume that* (H1), (H2) *hold, $K^* = K$ and $f(u)$ is nondecreasing on* $[0,K]$. *Let $c > c_*$ and $\Lambda_1(c), \Lambda_2(c)$ be defined as in Lemma* 2.1. *Then for every $\gamma \in (1, \min\{1+\nu, \frac{\Lambda_2(c)}{\Lambda_1(c)}\})$, there exists $Q(c,\gamma) \geqslant 1$, such that for any $q \geqslant Q(c,\gamma)$ and any $\xi^\pm \in \mathbb{R}$, the functions $\phi^\pm$ defined by*

$$\phi^+(\xi) := \min\big\{K, e^{\Lambda_1(c)(\xi+\xi^+)} + qe^{\gamma\Lambda_1(c)(\xi+\xi^+)}\big\}, \quad \xi \in \mathbb{R}, \tag{2.3}$$

and

$$\phi^-(\xi) := \max\big\{0, e^{\Lambda_1(c)(\xi+\xi^-)} - qe^{\gamma\Lambda_1(c)(\xi+\xi^-)}\big\}, \quad \xi \in \mathbb{R}, \tag{2.4}$$

are a supersolution and a subsolution to (1.7), *respectively.*

Proof. It is easily seen that there exists $\xi^* \leqslant -\xi^+ - \frac{1}{\beta\Lambda_1(c)}\ln\frac{q}{K}$, such that $\phi^+(\xi) = K$ for $\xi > \xi^*$ and $\phi^+(\xi) = e^{\Lambda_1(c)(\xi+\xi^+)} + qe^{\gamma\Lambda_1(c)(\xi+\xi^+)}$ for $\xi \leqslant \xi^*$.

For $\xi > \xi^*$, we have

$$N_c[\phi^+](\xi) = g(K) - h(K)\int_{\mathbb{R}} f\big(\phi^+(\xi - y - cr)\big)J(y)\,dy$$
$$\geqslant g(K) - h(K)f(K) = 0.$$

Notice that $f(u) \leqslant f'(0)u$, $g(u) \geqslant g'(0)u$ and $h(u) \leqslant h(0)$ for all $u \in (0, K)$. For $\xi < \xi^*$, we have

$$\begin{aligned} N_c[\phi^+](\xi) &= e^{\Lambda_1(c)(\xi+\xi^+)}\big[c\Lambda_1(c) - D\Lambda_1^2(c)\big] + qe^{\gamma\Lambda_1(c)(\xi+\xi^+)}\big[c\gamma\Lambda_1(c) - D\gamma^2\Lambda_1^2(c)\big] \\ &\quad + g\big(\phi^+(\xi)\big) - h\big(\phi^+(\xi)\big)\int_{\mathbb{R}} f\big(\phi^+(\xi - y - cr)\big)J(y)\,dy \\ &\geqslant qe^{\gamma\Lambda_1(c)(\xi+\xi^+)}\Delta\big(c, \gamma\Lambda_1(c)\big) - g'(0)\phi^+(\xi) \\ &\quad + h(0)f'(0)\int_{\mathbb{R}} \phi^+(\xi - y - cr)J(y)\,dy \\ &\quad + g\big(\phi^+(\xi)\big) - h\big(\phi^+(\xi)\big)\int_{\mathbb{R}} f\big(\phi^+(\xi - y - cr)\big)J(y)\,dy \\ &> 0. \end{aligned}$$

Therefore, ϕ^+ is a supersolution of (1.7).

Let $\xi_* = -\xi^- - \frac{1}{(\gamma-1)\Lambda_1(c)}\ln q$. If $q \geqslant 1$, then $\xi_* \leqslant -\xi^-$. Clearly, $\phi^-(\xi) = 0$ for $\xi > \xi_*$ and $\phi^-(\xi) = e^{\Lambda_1(c)(\xi+\xi^-)} - qe^{\gamma\Lambda_1(c)(\xi+\xi^-)}$ for $\xi \leqslant \xi_*$.

For $\xi > \xi_*$, we have

$$N_c[\phi^-](\xi) = g(0) - h(0)\int_{\mathbb{R}} f\big(\phi^-(\xi - y - cr)\big)J(y)\,dy \leqslant 0.$$

For $\xi < \xi_*$, we have $\xi + \xi^- < -\frac{1}{(\gamma-1)\Lambda_1(c)}\ln q$. By (H1), we can choose a positive number $M > 0$ so that $f'(0)u - f(u) \leqslant Mu^{1+\nu}$, $g(u) - g'(0)u \leqslant Mu^{1+\nu}$ and $h(0) - h(u) \leqslant Mu^{1+\nu}$ for all $u \in (0, K)$. Therefore, for $\xi < \xi_*$, we have

$$\begin{aligned} N_c[\phi^-](\xi) &= e^{\Lambda_1(c)(\xi+\xi^-)}\big[c\Lambda_1(c) - D\Lambda_1^2(c)\big] - qe^{\gamma\Lambda_1(c)(\xi+\xi^-)}\big[c\gamma\Lambda_1(c) - D\gamma^2\Lambda_1^2(c)\big] \\ &\quad + g\big(\phi^-(\xi)\big) - h\big(\phi^-(\xi)\big)\int_{\mathbb{R}} f\big(\phi^-(\xi - y - cr)\big)J(y)\,dy \\ &\leqslant -qe^{\gamma\Lambda_1(c)(\xi+\xi^-)}\Delta\big(c, \gamma\Lambda_1(c)\big) - g'(0)\phi^-(\xi) + g\big(\phi^-(\xi)\big) \\ &\quad + h(0)f'(0)\int_{\mathbb{R}} \phi^-(\xi - y - cr)J(y)\,dy \\ &\quad - h\big(\phi^-(\xi)\big)\int_{\mathbb{R}} f\big(\phi^-(\xi - y - cr)\big)J(y)\,dy \end{aligned}$$

$$\begin{aligned}
&\leqslant -qe^{\gamma\Lambda_1(c)(\xi+\xi^-)}\Delta\big(c,\gamma\Lambda_1(c)\big)+M\big(1+f(K)\big)\big[\phi^-(\xi)\big]^{1+\nu}\\
&\quad+Mh(0)\int_{\mathbb{R}}\big[\phi^-(\xi-y-cr)\big]^{1+\nu}J(y)\,dy\\
&\leqslant e^{\gamma\Lambda_1(c)(\xi+\xi^-)}\bigg\{-q\Delta\big(c,\gamma\Lambda_1(c)\big)\\
&\quad+M\bigg(1+f(K)+h(0)\int_{\mathbb{R}}e^{-(1+\nu)\Lambda_1(c)(y+cr)}J(y)\,dy\bigg)e^{(1+\nu-\gamma)\Lambda_1(c)(\xi+\xi^-)}\bigg\}\\
&\leqslant e^{\gamma\Lambda_1(c)(\xi+\xi^-)}\bigg\{-q\Delta\big(c,\gamma\Lambda_1(c)\big)\\
&\quad+M\bigg(1+f(K)+h(0)\int_{\mathbb{R}}e^{-(1+\nu)\Lambda_1(c)(y+cr)}J(y)\,dy\bigg)\bigg\}\\
&\leqslant 0,
\end{aligned}$$

provided that

$$q\geqslant Q(c,\gamma):=\max\bigg\{1,\frac{M}{\Delta(c,\gamma\Lambda_1(c))}\bigg(1+f(K)+h(0)\int_{\mathbb{R}}e^{-(1+\nu)\Lambda_1(c)(y+cr)}J(y)\,dy\bigg)\bigg\}.$$

Therefore, ϕ^- is a subsolution of (1.7). The proof is complete. □

When f is nondecreasing, the following result establish the existence and the behavior of nondecreasing traveling wavefronts of Eq. (1.2).

Lemma 2.3. *Assume that* (H1), (H2) *hold, $K^*=K$, $f(u)$ is nondecreasing on* $[0,K]$ *and $f(u)>g(u)/h(u)$ for all $u\in(0,K)$. Let $c_*>0$ be as in Lemma* 2.1. *Then for each $c>c_*$, Eq.* (1.2) *admits a nondecreasing traveling wave solution $w(t,x)=U(x+ct)$ satisfying $U(+\infty)=K$ and*

$$\lim_{\xi\to-\infty}U(\xi)e^{-\Lambda_1(c)\xi}=1, \tag{2.5}$$

where $\lambda=\Lambda_1(c)>0$ is the smallest solution to the equation

$$\Delta(c,\lambda)=c\lambda-D\lambda^2+g'(0)-h(0)f'(0)\int_{\mathbb{R}}e^{-\lambda(y+cr)}J(y)\,dy=0.$$

Proof. For $c>c_*$, by virtue of Lemma 2.2, ϕ^+ and ϕ^- with $\xi^\pm=0$ are a supersolution and a subsolution to (1.7), respectively. Let $\varpi:=L_g+L_hf(K)$, here and in what follows, L_g and L_h are the Lipschitz constants of g and h on $[0,K]$, respectively. Let

$$H[\phi](\xi)=\varpi\phi(\xi)-g\big(\phi(\xi)\big)+h\big(\phi(\xi)\big)\int_{\mathbb{R}}f\big(\phi(\xi-y-cr)\big)J(y)\,dy,\quad \xi\in\mathbb{R}.$$

Then for any $\phi, \psi \in C(\mathbb{R}, [0, K])$ with $\phi(\xi) \geqslant \psi(\xi)$, $\xi \in \mathbb{R}$, we have

$$\begin{aligned}
&H[\phi](\xi) - H[\psi](\xi) \\
&\quad = \varpi\big[\phi(\xi) - \psi(\xi)\big] - \big[g\big(\phi(\xi)\big) - g\big(\psi(\xi)\big)\big] \\
&\qquad + h\big(\phi(\xi)\big)\int_{\mathbb{R}} f\big(\phi(\xi - y - cr)\big)J(y)\,dy - h\big(\psi(\xi)\big)\int_{\mathbb{R}} f\big(\psi(\xi - y - cr)\big)J(y)\,dy \\
&\quad \geqslant \varpi\big[\phi(\xi) - \psi(\xi)\big] - \big[g\big(\phi(\xi)\big) - g\big(\psi(\xi)\big)\big] \\
&\qquad + \big[h\big(\phi(\xi)\big) - h\big(\psi(\xi)\big)\big]\int_{\mathbb{R}} f\big(\psi(\xi - y - cr)\big)J(y)\,dy \\
&\quad \geqslant \big[\varpi - L_g - L_h f(K)\big]\big(\phi(\xi) - \psi(\xi)\big) = 0,
\end{aligned}$$

and hence

$$H[\phi](\xi) \geqslant H[\psi](\xi), \quad \xi \in \mathbb{R}.$$

Let

$$\lambda_1 := \frac{c - \sqrt{c^2 + 4D\varpi}}{2D}, \qquad \lambda_2 := \frac{c + \sqrt{c^2 + 4D\varpi}}{2D}.$$

Then each solution U of (1.7) satisfies

$$U(\xi) = \frac{1}{D(\lambda_2 - \lambda_1)}\left[\int_{-\infty}^{\xi} e^{\lambda_1(\xi - s)} H[U](s)\,ds + \int_{\xi}^{+\infty} e^{\lambda_2(\xi - s)} H[U](s)\,ds\right], \quad \xi \in \mathbb{R}.$$

For any $\lambda \in (0, \min\{\Lambda_1(c), \lambda_2\})$, let

$$X_\lambda = \left\{\phi \in C(\mathbb{R}, \mathbb{R}) \;\Big|\; \sup_{\xi \in \mathbb{R}} \big|\phi(\xi)\big| e^{-\lambda\xi} < +\infty\right\}, \qquad \|\phi\|_\lambda = \sup_{\xi \in \mathbb{R}} \big|\phi(\xi)\big| e^{-\lambda\xi}.$$

Then $(X_\lambda, \|\cdot\|_\lambda)$ is a Banach space. Since $\phi^-(\xi) \leqslant \phi^+(\xi)$ for all $\xi \in \mathbb{R}$ and $\phi^+(\xi)$ is nondecreasing on $\mathbb{R}$, it is easily known that the set

$$\Gamma := \left\{\phi \in C\big(\mathbb{R}, [0, K]\big) \;\middle|\; \begin{array}{c} \text{(i) } \phi(\xi) \text{ is nondecreasing on } \mathbb{R}; \\ \text{(ii) } \phi^-(\xi) \leqslant \phi(\xi) \leqslant \phi^+(\xi) \text{ for all } \xi \in \mathbb{R}; \\ \text{(iii) } |\phi(\xi_1) - \phi(\xi_2)| \leqslant 2K\sqrt{\frac{\varpi}{D}}|\xi_1 - \xi_2| \text{ for all } \xi_1, \xi_2 \in \mathbb{R} \end{array}\right\}$$

is nonempty, convex and compact in X_λ.

Define $F : \Gamma \to \Gamma$ by

$$F[\phi](\xi) = \frac{1}{D(\lambda_2 - \lambda_1)}\left[\int_{-\infty}^{\xi} e^{\lambda_1(\xi - s)} H[\phi](s)\,ds + \int_{\xi}^{+\infty} e^{\lambda_2(\xi - s)} H[\phi](s)\,ds\right].$$

By using an argument as used in [7], it is easily seen that F is well defined and a fixed point of F is a solution of (1.7) and (1.8).

For any $\phi, \psi \in \Gamma$, we have

$$\begin{aligned}
&\big|H[\phi](\xi) - H[\psi](\xi)\big|e^{-\lambda\xi} \\
&\quad\leqslant \varpi\big|\phi(\xi) - \psi(\xi)\big|e^{-\lambda\xi} + \big|g\big(\phi(\xi)\big) - g\big(\psi(\xi)\big)\big|e^{-\lambda\xi} \\
&\qquad + \big|h\big(\phi(\xi)\big) - h\big(\psi(\xi)\big)\big|e^{-\lambda\xi}\int_{\mathbb{R}} f\big(\psi(\xi - y - cr)\big)J(y)\,dy \\
&\qquad + h\big(\psi(\xi)\big)\int_{\mathbb{R}}\big|f\big(\phi(\xi - y - cr)\big) - f\big(\psi(\xi - y - cr)\big)\big|e^{-\lambda\xi}J(y)\,dy \\
&\quad\leqslant L\|\phi - \psi\|_{\lambda},
\end{aligned}$$

where $L := \varpi + L_g + f(K)L_h + h(0)L_f\int_{\mathbb{R}} e^{-\lambda(y+cr)}J(y)\,dy$, where L_f is the Lipschitz constant of f on $[0, K]$. Therefore, we have

$$\begin{aligned}
&D(\lambda_2 - \lambda_1)\big|F[\phi](\xi) - F[\psi](\xi)\big|e^{-\lambda\xi} \\
&\quad\leqslant e^{-\lambda\xi}\left\{\int_{-\infty}^{\xi} e^{\lambda_1(\xi-s)}\big|H[\phi](s) - H[\psi](s)\big|\,ds + \int_{\xi}^{+\infty} e^{\lambda_2(\xi-s)}\big|H[\phi](s) - H[\psi](s)\big|\,ds\right\} \\
&\quad\leqslant \left\{\int_{-\infty}^{\xi} e^{(\lambda_1-\lambda)(\xi-s)}\,ds + \int_{\xi}^{+\infty} e^{(\lambda_2-\lambda)(\xi-s)}\right\}\sup_{\xi\in\mathbb{R}}\big|H[\phi](\xi) - H[\psi](\xi)\big|e^{-\lambda\xi} \\
&\quad\leqslant \frac{\lambda_1 - \lambda_2}{(\lambda_1 - \lambda)(\lambda_2 - \lambda)}L\|\phi - \psi\|_{\lambda},
\end{aligned}$$

which yields

$$\big\|F[\phi] - F[\psi]\big\|_{\lambda} \leqslant \frac{L}{D(\lambda - \lambda_1)(\lambda_2 - \lambda)}\|\phi - \psi\|_{\lambda}. \tag{2.6}$$

That is, $F : \Gamma \to \Gamma$ is continuous. By virtue of the Schauder's fixed point theorem, it follows that F has a fixed point U in X_{λ}. Moreover, U satisfies $U(+\infty) = K$ and

$$e^{\Lambda_1(c)\xi} - qe^{\gamma\Lambda_1(c)\xi} \leqslant U(\xi) \leqslant e^{\Lambda_1(c)\xi} + qe^{\gamma\Lambda_1(c)\xi}, \quad \xi \in \mathbb{R},$$

which implies that

$$\lim_{\xi\to-\infty}\big|U(\xi)e^{-\Lambda_1(c)\xi} - 1\big| \leqslant \lim_{\xi\to-\infty} qe^{(\gamma-1)\Lambda_1(c)\xi} = 0.$$

This completes the proof. □

3. Proof of the main theorem

In this section, we shall apply the results obtained in the previous section to prove our main Theorem 1.1. To this end, we firstly need to construct two auxiliary reaction–diffusion equations with quasi-monotonicity.

In what follows, we always assume that (H1) and (H2) hold. Since $f'(0) > g'(0)/h(0)$ and $f(u) > 0$ for $u \in (0, K^*]$, it follows from the definition of K_* that $g(K_*)/h(K_*) > 0$. Therefore, there is a small $\epsilon_0 \in (0, K_*)$ such that $g(K_* - \epsilon)/h(K_* - \epsilon) > 0$ for every $\epsilon \in [0, \epsilon_0]$. For any $\epsilon \in (0, \epsilon_0)$, define two continuous functions as follows:

$$f^*(u) = \begin{cases} \min\{f'(0)u, g(K^*)/h(K^*)\}, & \text{for } u \in [0, K^*], \\ \max\{g(K^*)/h(K^*), f(u)\}, & \text{for } u > K^*, \end{cases} \tag{3.1}$$

and

$$f_\epsilon(u) = \begin{cases} \inf_{\eta \in [u, K^*]}\{f(\eta), g(K_* - \epsilon)/h(K_* - \epsilon)\}, & \text{for } u \in [0, K^*], \\ \min\{f(u), g(K_* - \epsilon)/h(K_* - \epsilon)\}, & \text{for } u > K^*. \end{cases} \tag{3.2}$$

Then we have the following observations:

Lemma 3.1. *The following statements hold true*:

(i) f^* *and* f_ϵ *are continuous on* $[0, +\infty)$ *and nondecreasing on* $[0, K^*]$;
(ii) $f^*(u) \geqslant f(u) \geqslant f_\epsilon(u)$ *for all* $u \geqslant 0$;
(iii) $f'(0)u \geqslant f^*(u) > 0$ *and* $f'(0)u \geqslant f_\epsilon(u) > 0$ *for all* $u \in (0, K^*]$;
(iv) $f^*(0) = g(K^*)/h(K^*) - f^*(K^*) = 0$ *and* $f^*(u) > g(u)/h(u)$ *for all* $u \in (0, K^*)$;
(v) $f_\epsilon(0) = g(K_* - \epsilon)/h(K_* - \epsilon) - f_\epsilon(K_* - \epsilon) = 0$ *and* $f_\epsilon(u) > g(u)/h(u)$ *for all* $u \in (0, K_* - \epsilon)$;
(vi) $f_\epsilon'(0) = f'(0)$ *and* $\limsup_{u\to 0^+}[f_\epsilon'(0) - f_\epsilon(u)/u]u^{-\nu} < +\infty$.

Proof. We only prove (vi). The others are obvious and their proofs are omitted.

Since $K_* - \epsilon > 0$, it follows from the definition of $f_\epsilon(\cdot)$ that $f_\epsilon(u) = \inf_{\eta \in [u, \delta_0]} f(\eta)$ for some small $\delta_0 > 0$ and all $u \in [0, \delta_0]$.

Since $f(u) \leqslant f'(0)u$ for all $u \in [0, K^*]$ and $\limsup_{u\to 0^+}[f_\epsilon'(0) - f_\epsilon(u)/u]u^{-\nu} < +\infty$, there exists $M > 0$ such that

$$0 \leqslant f'(0) - f(u)/u \leqslant M u^\nu \quad \text{for } u \in (0, \delta_0],$$

and hence

$$f'(0)u - M u^{1+\nu} \leqslant f(u) \leqslant f'(0)u \quad \text{for } u \in [0, \delta_0].$$

We can choose $\delta_0 > 0$ small enough so that $f'(0)u - Mu^{1+\nu}$ is increasing on $[0, \delta_0]$, so it follows that

$$f'(0)u - M u^{1+\nu} \leqslant f_\epsilon(u) \leqslant f'(0)u \quad \text{for } u \in [0, \delta_0],$$

from which we conclude that $f_\epsilon'(0) = f'(0)$ and $\limsup_{u\to 0^+}[f_\epsilon'(0) - f_\epsilon(u)/u]u^{-\nu} < +\infty$. This completes the proof. □

Consider the following two auxiliary delayed diffusion equations

$$w_t(t,x) = Dw_{xx}(t,x) - g\big(w(t,x)\big) + h\big(w(t,x)\big)\int_{\mathbb{R}} f^*\big(w(t-r,y)\big)J(x-y)\,dy \tag{3.3}$$

and

$$w_t(t,x) = Dw_{xx}(t,x) - g\big(w(t,x)\big) + h\big(w(t,x)\big)\int_{\mathbb{R}} f_\epsilon\big(w(t-r,y)\big)J(x-y)\,dy. \tag{3.4}$$

Clearly, the wave equations corresponding to (3.3) and (3.4) read as

$$cU'(\xi) - DU''(\xi) + g\big(U(\xi)\big) - h\big(U(\xi)\big)\int_{\mathbb{R}} f^*\big(U(\xi - i - cr)\big)J(y)\,dy = 0 \tag{3.5}$$

and

$$cU'(\xi) - DU''(\xi) + g\big(U(\xi)\big) - h\big(U(\xi)\big)\int_{\mathbb{R}} f_\epsilon\big(U(\xi - i - cr)\big)J(y)\,dy = 0, \tag{3.6}$$

respectively.

The following lemma is a direct consequence of Lemmas 2.3 and 3.1.

Lemma 3.2. *Assume that* (H1) *and* (H2) *hold. Let $c_* > 0$ be as in Lemma* 2.1. *Then for each $c > c_*$, Eqs.* (3.3) *and* (3.4) *have nondecreasing traveling wave solutions $U^*(x+ct)$ and $U_\epsilon(x+ct)$, respectively, satisfying $U^*(+\infty) = K^*$, $U_\epsilon(+\infty) = K_* - \epsilon$ and*

$$\lim_{\xi\to-\infty} U^*(\xi)e^{-\Lambda_1(c)\xi} = \lim_{\xi\to-\infty} U_\epsilon(\xi)e^{-\Lambda_1(c)\xi} = 1, \tag{3.7}$$

where $\lambda = \Lambda_1(c) > 0$ is the smallest solution to the equation

$$\Delta(c,\lambda) = c\lambda - D\lambda^2 + g'(0) - h(0)f'(0)\int_{\mathbb{R}} e^{-\lambda(y+cr)}J(y)\,dy = 0.$$

We are now in a position to give a proof of our main result.

Proof of Theorem 1.1. For $c > c_*$, let $U^*(x+ct)$ and $U_\epsilon(x+ct)$ be the nondecreasing traveling wavefronts of (3.3) and (3.4), respectively, which are given in Lemma 3.2. Let $a_1 > 0$ be such that $e^{\Lambda_1(c)a_1} \geqslant 3$. Then

$$\lim_{\xi\to-\infty} U^*(\xi + a_1)e^{-\Lambda_1(c)\xi} = e^{\Lambda_1(c)a_1} \geqslant 3. \tag{3.8}$$

Therefore, there exists $M_1 > 0$ such that

$$U^*(\xi + a_1)e^{-\Lambda_1(c)\xi} > 2 > U_\epsilon(\xi)e^{-\Lambda_1(c)\xi} \quad \text{for all } \xi \leqslant -M_1. \tag{3.9}$$

Since $U^*(+\infty) = K^* > K_* - \epsilon = U_\epsilon(+\infty)$, we can choose $a_2 > 0$ sufficiently large, so that

$$U^*(\xi + a_2) > U_\epsilon(\xi) \quad \text{for all } \xi \geqslant -M_1. \tag{3.10}$$

Let $a_0 = \max\{a_1, a_2\}$. Since $U^*(\cdot)$ is nondecreasing, it follows from (3.9) and (3.10) that

$$U^*(\xi + a_0) > U_\epsilon(\xi) \quad \text{for all } \xi \in \mathbb{R}. \tag{3.11}$$

Let $\varpi^* := L_g^* + L_h^* g(K^*)/h(K^*)$, here and in what follows, L_g^* and L_h^* are the Lipschitz constants of g and h on $[0, K^*]$, respectively. Define

$$H^*[\phi](\xi) = \varpi^*\phi(\xi) - g\big(\phi(\xi)\big) + h\big(\phi(\xi)\big)\int_{\mathbb{R}} f^*\big(\phi(\xi - y - cr)\big)J(y)\,dy, \quad \xi \in \mathbb{R},$$

and

$$H_\epsilon[\phi](\xi) = \varpi^*\phi(\xi) - g\big(\phi(\xi)\big) + h\big(\phi(\xi)\big)\int_{\mathbb{R}} f_\epsilon\big(\phi(\xi - y - cr)\big)J(y)\,dy, \quad \xi \in \mathbb{R},$$

then for any $\phi, \psi \in C(\mathbb{R}, [0, K^*])$ with $\phi(\xi) \geqslant \psi(\xi)$, $\xi \in \mathbb{R}$, we have

$$H^*[\phi](\xi) \geqslant H^*[\psi](\xi) \quad \text{and} \quad H_\epsilon[\phi](\xi) \geqslant H_\epsilon[\psi](\xi) \quad \text{for all } \xi \in \mathbb{R}. \tag{3.12}$$

Set

$$\Gamma^* := \left\{\phi \in C\big(\mathbb{R}, [0, K^*]\big) \;\middle|\; \begin{array}{l} \text{(i) } U_\epsilon(\xi) \leqslant \phi(\xi) \leqslant U^*(\xi + a_0) \text{ for all } \xi \in \mathbb{R}; \\ \text{(ii) } |\phi(\xi_1) - \phi(\xi_2)| \leqslant 2K^*\sqrt{\frac{\varpi^*}{D}}|\xi_1 - \xi_2| \text{ for all } \xi_1, \xi_2 \in \mathbb{R} \end{array}\right\}.$$

Then, we can show that Γ^* is nonempty, convex and compact in X_λ, where for any $\lambda \in (0, \min\{\Lambda_1(c), \lambda_2^*\})$, X_λ is the Banach space as given in the proof of Lemma 2.3.

Define $F^* : \Gamma^* \to C(\mathbb{R}, [0, K^*])$ by

$$F^*(\phi)(\xi) = \frac{1}{D(\lambda_2^* - \lambda_1^*)}\left[\int_{-\infty}^{\xi} e^{\lambda_1^*(\xi - s)}H(\phi)(s)\,ds + \int_{\xi}^{+\infty} e^{\lambda_2^*(\xi - s)}H(\phi)(s)\,ds\right],$$

where $H(\phi)(\xi) = \varpi^*\phi(\xi) - g(\phi(\xi)) + h(\phi(\xi))\int_{\mathbb{R}} f(\phi(\xi - y - cr))J(y)\,dy$, $\xi \in \mathbb{R}$, and

$$\lambda_1^* := \frac{c - \sqrt{c^2 + 4D\varpi^*}}{2D}, \qquad \lambda_2^* := \frac{c + \sqrt{c^2 + 4D\varpi^*}}{2D}.$$

Clearly, for any $\phi \in \Gamma^* \subset C(\mathbb{R}, [0, K^*])$, it follows from (3.12) that

$$0 \leqslant H_\epsilon[\phi](\xi) \leqslant H(\phi)(\xi) \leqslant H^*[\phi](\xi) \leqslant \varpi^* K^* - g(K^*) + h(K^*) f^*(K^*)$$
$$\leqslant \varpi^* K^*, \tag{3.13}$$

for all $\xi \in \mathbb{R}$. So it follows that

$$0 \leqslant F^*(\phi)(\xi) \leqslant \frac{\varpi^* K^*}{D(\lambda_2^* - \lambda_1^*)} \left[\int_{-\infty}^{\xi} e^{\lambda_1^*(\xi - s)}\, ds + \int_{\xi}^{+\infty} e^{\lambda_2^*(\xi - s)}\, ds \right] = K^*,$$

and hence, $F^* : \Gamma^* \to C(\mathbb{R}, [0, K^*])$ is well defined. Furthermore, it is easily seen that a fixed point of F^* is a solution of (1.7).

For any $\phi, \psi \in \Gamma^*$, an argument similar to (2.6) shows that

$$\left\| F^*(\phi) - F^*(\psi) \right\|_\lambda \leqslant \frac{L^*}{D(\lambda - \lambda_1^*)(\lambda_2^* - \lambda)} \|\phi - \psi\|_\lambda,$$

where $L^* := \varpi^* + L_g^* + f(K^*) L_h^* + h(0) L_f^* \int_{\mathbb{R}} e^{-\lambda(y + cr)} J(y)\, dy$, where L_f^* is the Lipschitz constant of f on $[0, K^*]$. Therefore, $F^* : \Gamma^* \to C(\mathbb{R}, [0, K^*])$ is continuous.

Next, we shall show that $F^*(\Gamma^*) \subseteq \Gamma^*$. Since $U_\epsilon(\xi)$ is a solution of (3.6), we have

$$U_\epsilon(\xi) = \frac{1}{D(\lambda_2^* - \lambda_1^*)} \left[\int_{-\infty}^{\xi} e^{\lambda_1^*(\xi - s)} H_\epsilon[U_\epsilon](s)\, ds + \int_{\xi}^{+\infty} e^{\lambda_2^*(\xi - s)} H_\epsilon[U_\epsilon](s)\, ds \right]. \tag{3.14}$$

For any $\phi \in \Gamma^*$, we have $0 \leqslant U_\epsilon(\xi) \leqslant \phi(\xi) \leqslant U^*(\xi + a_0) \leqslant K^*$ for all $\xi \in \mathbb{R}$. Therefore, it follows from (3.12)–(3.14) that

$$\begin{aligned} F^*(\phi)(\xi) &= \frac{1}{D(\lambda_2^* - \lambda_1^*)} \left[\int_{-\infty}^{\xi} e^{\lambda_1^*(\xi - s)} H(\phi)(s)\, ds + \int_{\xi}^{+\infty} e^{\lambda_2^*(\xi - s)} H(\phi)(s)\, ds \right] \\ &\geqslant \frac{1}{D(\lambda_2^* - \lambda_1^*)} \left[\int_{-\infty}^{\xi} e^{\lambda_1^*(\xi - s)} H_\epsilon[\phi](s)\, ds + \int_{\xi}^{+\infty} e^{\lambda_2^*(\xi - s)} H_\epsilon[\phi](s)\, ds \right] \\ &\geqslant \frac{1}{D(\lambda_2^* - \lambda_1^*)} \left[\int_{-\infty}^{\xi} e^{\lambda_1^*(\xi - s)} H_\epsilon[U_\epsilon](s)\, ds + \int_{\xi}^{+\infty} e^{\lambda_2^*(\xi - s)} H_\epsilon[U_\epsilon](s)\, ds \right] \\ &= U_\epsilon(\xi). \end{aligned}$$

Since $U^*(\xi + a_0)$ is a solution of (3.5), by using a similar argument, we can show that $F^*(\phi)(\xi) \leqslant U^*(\xi + a_0)$ for all $\xi \in \mathbb{R}$.

For any $\phi \in \Gamma^*$ and $\xi_1, \xi_2 \in \mathbb{R}$ with $\xi_1 < \xi_2$, it follows from (3.13) that

$$
\begin{aligned}
&D(\lambda_2^*-\lambda_1^*)\big|F^*(\phi)(\xi_1)-F^*(\phi)(\xi_2)\big|\\
&=\left|\int_{-\infty}^{\xi_1} e^{\lambda_1^*(\xi_1-s)}H(\phi)(s)\,ds-\int_{\xi_2}^{+\infty} e^{\lambda_2^*(\xi_2-s)}H(\phi)(s)\,ds\right|\\
&\leqslant \big(e^{\lambda_1^*\xi_2}-e^{\lambda_1^*\xi_1}\big)\int_{-\infty}^{\xi_1} e^{-\lambda_1^*s}H(\phi)(s)\,ds+\int_{\xi_1}^{\xi_2} e^{\lambda_1^*(\xi_2-s)}H(\phi)(s)\,ds\\
&\quad+\int_{\xi_1}^{\xi_2} e^{\lambda_2^*(\xi_1-s)}H(\phi)(s)\,ds+\big(e^{\lambda_2^*\xi_2}-e^{\lambda_2^*\xi_1}\big)\int_{\xi_2}^{+\infty} e^{-\lambda_2^*s)}H(\phi)(s)\,ds\\
&=\left\{-\lambda_1^*e^{\lambda_1^*\xi_1}\int_{-\infty}^{\xi_1} e^{-\lambda_1^*s}\,ds+2+\lambda_2^*e^{\lambda_2\xi_2}\int_{\xi_2}^{+\infty} e^{-\lambda_2^*s}\,ds\right\}|\xi_1-\xi_2|\sup_{\xi\in\mathbb{R}}H(\phi)(\xi)\\
&\leqslant 4\varpi^*K^*|\xi_1-\xi_2|.
\end{aligned}
$$

Since $D(\lambda_2^*-\lambda_1^*)=\sqrt{c^2+4D\varpi^*}\geqslant 2\sqrt{D\varpi^*}$, the last inequality implies that

$$\big|F^*(\phi)(\xi_1)-F^*(\phi)(\xi_2)\big|\leqslant 2K^*\sqrt{\frac{\varpi^*}{D}}|\xi_1-\xi_2|.$$

Therefore, we conclude that $F^*(\phi)\in\Gamma^*$ for all $\phi\in\Gamma^*$.

By virtue of the Schauder's fixed point theorem, it follows that F^* has a fixed point U in $\Gamma^*\subset X_\lambda$, which satisfies

$$U(\xi)=\frac{1}{D(\lambda_2^*-\lambda_1^*)}\left[\int_{-\infty}^{\xi} e^{\lambda_1^*(\xi-s)}H(U)(s)\,ds+\int_{\xi}^{+\infty} e^{\lambda_2^*(\xi-s)}H(U)(s)\,ds\right] \tag{3.15}$$

and

$$U_\epsilon(\xi)\leqslant U(\xi)\leqslant U^*(\xi+a_0)\quad\text{for all }\xi\in\mathbb{R}. \tag{3.16}$$

Sending $\xi\to-\infty$ and $\xi\to+\infty$ in (3.16), respectively, we get $U(-\infty)=0$ and

$$K_*-\epsilon\leqslant\liminf_{\xi\to+\infty}U(\xi)\leqslant\limsup_{\xi\to+\infty}U(\xi)\leqslant K^*.$$

Since $U(\xi)$ is independent of ϵ, taking the limit as $\epsilon\to 0^+$ in the last inequality, we get

$$K_*\leqslant\liminf_{\xi\to+\infty}U(\xi)\leqslant\limsup_{\xi\to+\infty}U(\xi)\leqslant K^*. \tag{3.17}$$

In what follows, we assume that (H3) also holds. Set $\alpha:=\liminf_{\xi\to+\infty}U(\xi)$ and $\beta:=\limsup_{\xi\to+\infty}U(\xi)$. Then $K_*\leqslant\alpha\leqslant\beta\leqslant K^*$. We shall show that $\alpha=\beta$. Suppose for the contrary that $\alpha<\beta$. If there is a large number $M>0$ such that $U'>0$ on $[M,+\infty)$ or $U'<0$

on $[M, +\infty)$, then $\lim_{\xi \to +\infty} U(\xi)$ exists and hence $\alpha = \beta$, a contradiction. So there exists a sequence $\{\xi_j\}_{j \in N}$, with $\xi_j \to +\infty$ as $j \to +\infty$, such that $U'(\xi_j) = 0$, $U''(\xi_j) \leqslant 0$ and $U(\xi_j) \to \beta$ as $j \to +\infty$. Then we have

$$0 = cU'(\xi_j) = DU''(\xi_j) - g\big(U(\xi_j)\big) + h\big(U(\xi_j)\big) \int_{\mathbb{R}} f\big(U(\xi_j - y - cr)\big) J(y)\, dy$$

$$\leqslant -g\big(U(\xi_j)\big) + h\big(U(\xi_j)\big) \int_{\mathbb{R}} f\big(U(\xi_j - y - cr)\big) J(y)\, dy,$$

and hence

$$g\big(U(\xi_j)\big)/h\big(U(\xi_j)\big) \leqslant \int_{\mathbb{R}} f\big(U(\xi_j - y - cr)\big) J(y)\, dy. \tag{3.18}$$

For any $\epsilon > 0$, there is a sufficiently large number $N > 0$, such that

$$g(K^*)/h(K^*) \int_{|y|>N} J(y)\, dy < \epsilon. \tag{3.19}$$

Since f is continuous, we can choose $\delta > 0$ such that $\delta < \epsilon$ and

$$\max\big\{ f(u) \bigm| u \in [\alpha - \delta, \beta + \delta]\big\} < \max\big\{ f(u) \bigm| \alpha \leqslant u \leqslant \beta\big\} + \epsilon. \tag{3.20}$$

For such a $\delta > 0$, we take a large number $M_2 > 0$ such that

$$U(\xi) \in [\alpha - \delta, \beta + \delta] \quad \text{for all } \xi \geqslant M_2. \tag{3.21}$$

Choose a positive integer $J_0 > 0$ such that

$$\xi_j \geqslant M_2 + N + cr \quad \text{for all } j \geqslant J_0. \tag{3.22}$$

Therefore, for $j \geqslant J_0$, it follows from (3.18)–(3.21) that

$$g\big(U(\xi_j)\big)/h\big(U(\xi_j)\big) \leqslant \int_{|y| \leqslant N} f\big(U(\xi_j - y - cr)\big) J(y)\, dy + \int_{|y|>N} f\big(U(\xi_j - y - cr)\big) J(y)\, dy$$

$$\leqslant \max\big\{ f(u) \bigm| u \in [\alpha - \delta, \beta + \delta]\big\} + g(K^*)/h(K^*) \int_{|y|>N} J(y)\, dy$$

$$\leqslant \max\big\{ f(u) \bigm| \alpha \leqslant u \leqslant \beta\big\} + 2\epsilon.$$

Taking the limit as $j \to +\infty$ in the last inequality, we get

$$g(\beta)/h(\beta) \leqslant \max\big\{ f(u) \bigm| \alpha \leqslant u \leqslant \beta\big\} + 2\epsilon,$$

which yields, by sending $\epsilon \to 0^+$, that

$$g(\beta)/h(\beta) \leqslant \max\bigl\{f(u) \bigm| \alpha \leqslant u \leqslant \beta\bigr\}. \tag{3.23}$$

In a similar way, we can obtain

$$g(\alpha)/h(\alpha) \geqslant \min\bigl\{f(u) \bigm| \alpha \leqslant u \leqslant \beta\bigr\}. \tag{3.24}$$

If $\alpha < \beta \leqslant K$, then (3.23) implies that $g(\alpha)/h(\alpha) \geqslant \min\{f(u) \mid \alpha \leqslant u \leqslant \beta\} > g(\alpha)/h(\alpha)$, a contradiction, and if $K \leqslant \alpha < \beta$, then (3.22) implies that $g(\beta)/h(\beta) \leqslant \max\{f(u) \mid \alpha \leqslant u \leqslant \beta\} < g(\beta)/h(\beta)$, a contradiction. Therefore, we conclude that $\alpha < K < \beta$.

Take $u_1, u_2 \in [\alpha, \beta]$ so that $f(u_1) = \max\{f(u) \mid \alpha \leqslant u \leqslant \beta\}$ and $f(u_2) = \min\{f(u) \mid \alpha \leqslant u \leqslant \beta\}$. We distinguish among three cases:

Case (i). $K \leqslant u_1 \leqslant \beta$. If $u_1 = \beta$, then (3.22) yields $g(\beta)/h(\beta) \leqslant f(\beta)$, which is impossible since $\beta > K$. Therefore, we have $u_1 < \beta$ and hence

$$g(\beta)/h(\beta) \leqslant \max\bigl\{f(u) \bigm| \alpha \leqslant u \leqslant \beta\bigr\} = f(u_1) \leqslant g(u_1)/h(u_1) < g(\beta)/h(\beta),$$

which is a contradiction.

Case (ii). $\alpha \leqslant u_2 \leqslant K$. By using a similar argument as used in (i), we get $\alpha < u_2$ and

$$g(\alpha)/h(\alpha) \geqslant \min\bigl\{f(u) \bigm| \alpha \leqslant u \leqslant \beta\bigr\} = f(u_2) \geqslant g(u_2)/h(u_2) > g(\alpha)/h(\alpha),$$

which is also a contradiction.

Case (iii). $u_1 < K < u_2$. In this case, we have $u_1 = \alpha$ and $u_2 = \beta$. Otherwise, we have

$$\begin{aligned} g(\beta)/h(\beta) - g(\alpha)/h(\alpha) &\leqslant f(u_1) - f(u_2) \\ &\leqslant 2g(K)/h(K) - g(u_1)/h(u_1) - \bigl[2g(K)/h(K) - g(u_2)/h(u_2)\bigr] \\ &= g(u_2)/h(u_2) - g(u_1)/h(u_1) \\ &< g(\beta)/h(\beta) - g(\alpha)/h(\alpha), \end{aligned}$$

which is a contradiction. Therefore, we have $g(\alpha)/h(\alpha) \geqslant f(\beta)$ and $g(\beta)/h(\beta) \leqslant f(\alpha)$. So we have

$$\begin{aligned} g(\beta)/h(\beta) - g(\alpha)/h(\alpha) &\leqslant f(\alpha) - f(\beta) \\ &< 2g(K)/h(K) - g(\alpha)/h(\alpha) - \bigl[2g(K)/h(K) - g(\beta)/h(\alpha)\bigr] \\ &= g(\beta)/h(\beta) - g(\alpha)/h(\alpha), \end{aligned}$$

which is impossible.

Thus, $\alpha = \beta$, and hence the limit $\lim_{\xi \to +\infty} U(\xi) = \alpha \in [K_*, K^*]$ exists. Taking the limit as $\xi \to +\infty$ in (3.15) and applying the Lebesgue's dominated convergence theorem to get

$$
\begin{aligned}
\alpha &= \frac{1}{D(\lambda_2^* - \lambda_1^*)} \lim_{\xi \to +\infty} \left[\int_{-\infty}^{\xi} e^{\lambda_1^*(\xi - s)} H(U)(s)\, ds + \int_{\xi}^{+\infty} e^{\lambda_2^*(\xi - s)} H(U)(s)\, ds \right] \\
&= \frac{1}{D(\lambda_2^* - \lambda_1^*)} \left[\frac{1}{\lambda_2^*} - \frac{1}{\lambda_1^*} \right] \lim_{\xi \to +\infty} H(U)(\xi) \\
&= \frac{1}{-D\lambda_1^* \lambda_2^*} \big[\varpi^* \alpha - g(\alpha) + h(\alpha) f(\alpha) \big] \\
&= \frac{1}{\varpi^*} \big[\varpi^* \alpha - g(\alpha) + h(\alpha) f(\alpha) \big],
\end{aligned}
$$

which yields $g(\alpha)/h(\alpha) = f(\alpha)$, and hence $\lim_{\xi \to +\infty} U(\xi) = \alpha = K$. This completes the proof. □

References

[1] J. Al-Omari, S.A. Gourley, Monotone travelling fronts in an age-structured reaction–diffusion model of a single species, J. Math. Biol. 45 (2002) 294–312.

[2] X. Chen, J.S. Guo, Existence and asymptotic stability of traveling waves of discrete quasilinear monostable equations, J. Differential Equations 184 (2002) 549–569.

[3] T. Faria, W. Huang, J. Wu, Traveling waves for delayed reaction–diffusion equations with global response, Proc. R. Soc. Lond. Ser. A 462 (2065) (2006) 229–261.

[4] S.A. Gourley, Y. Kuang, Wavefronts and global stability in a time-delayed population model with stage structure, Proc. R. Soc. Lond. Ser. A 459 (2034) (2003) 1563–1579.

[5] S.A. Gourley, J.W.-H. So, J. Wu, Non-locality of reaction–diffusion equations induced by delay: Biological modeling and nonlinear dynamics, preprint.

[6] J. Huang, X. Zou, Existence of traveling wavefronts of delayed reaction–diffusion systems without monotonicity, Discrete Contin. Dyn. Syst. Ser. A 9 (2003) 925–936.

[7] S.W. Ma, Traveling wavefronts for delayed reaction–diffusion systems via a fixed point theorem, J. Differential Equations 171 (2001) 294–314.

[8] S.W. Ma, J.H. Wu, Existence, uniqueness and asymptotic stability of traveling wavefronts in a non-local delayed diffusion equation, J. Dynam. Differential Equations (2006), in press, DOI: 10.1007/s10884-006-9065-7.

[9] J.W.-H. So, J.H. Wu, X.F. Zou, A reaction–diffusion model for a single species with age structure. I. Traveling wavefronts on unbounded domains, Proc. R. Soc. Lond. Ser. A 457 (2001) 1841–1853.

[10] H.R. Thieme, X.-Q. Zhao, Asymptotic speed of spread and traveling waves for integral equations and delayed reaction–diffusion models, J. Differential Equations 195 (2003) 430–470.

[11] A.I. Volpert, V.A. Volpert, V.A. Volpert, Traveling Wave Solutions of Parabolic Systems, Transl. Math. Monogr., vol. 140, Amer. Math. Soc., Providence, RI, 1994.

[12] J. Wu, X. Zou, Asymptotic and periodic boundary value problems of mixed FDEs and wave solutions of lattice differential equations, J. Differential Equations 135 (1997) 315–357.

[13] J. Wu, X. Zou, Traveling wave fronts of reaction–diffusion systems with delay, J. Dynam. Differential Equations 13 (2001) 651–687.

[14] J.H. Wu, Theory and Applications of Partial Functional-Differential Equations, Appl. Math. Sci., vol. 119, Springer-Verlag, New York, 1996.