MATHEMATICS

A FURTHER NOTE ON CERTAIN DUAL EQUATIONS INVOLVING FOURIER-LAGUERRE SERIES [1)]

BY

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Summary

The present note aims at providing a systematic investigation of the solution of a certain pair of dual equations involving Fourier-Laguerre series. It exhibits equivalence of the solutions obtained earlier (see [5] and [6]) by using a multiplying factor technique as well as by considering separately the equations when (i) $g(x)\equiv 0$, (ii) $f(x)\equiv 0$, and reducing the problem in each case to that of solving an Abel integral equation.

It is also shown how the solution of these dual series equations are affected when the parameters involved are constrained differently.

1. Introduction

In a series of recent papers the author discussed the problem of determining the sequence $\{A_n\}$ satisfying the dual series equations

$$(1.1)\qquad \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+n+1)} L_n^{(\nu)}(x) = f(x),\quad 0 \leqq x < y,$$

$$(1.2)\qquad \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta+n+1)} L_n^{(\sigma)}(x) = g(x),\quad y < x < \infty,$$

where

$$(1.3)\qquad L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!},\quad n=0, 1, 2, \ldots,$$

denotes the Laguerre polynomial [7, p. 101] of order α and degree n in x, the functions $f(x)$ and $g(x)$ are prescribed, and in general, $\alpha, \beta, \nu, \sigma > -1$. By using a generalization of the multiplying factor technique, which was developed by Noble [3] for solving dual equations involving Fourier-Jacobi series and applied subsequently by Lowndes [2] and Askey [1] for solving essentially the same special case, viz. $\nu=\sigma=\alpha$, of the dual equations (1.1) and (1.2), it was shown that the equations

$$(1.4)\qquad \sum_{n=0}^{\infty} \frac{B_n}{\Gamma(\alpha+n+1)} L_n^{(\alpha)}(x) = f(x),\quad 0 \leqq x < y,$$

$$(1.5)\qquad \sum_{n=0}^{\infty} \frac{B_n}{\Gamma(\beta+n+1)} L_n^{(\sigma)}(x) = g(x),\quad y < x < \infty,$$

[1)] See Abstract 70T–B67 in Notices Amer. Math. Soc. 17 (1970), 445–446. This work was supported in part by the National Research Council of Canada under Grant A7353.

which correspond to the special case $\nu=\alpha$ of (1.1) and (1.2), have for their exact solution [5, pp. 526–527]

$$(1.6)\quad \begin{cases} B_n=\dfrac{n!}{\Gamma(\beta-\alpha+m)}\displaystyle\int_0^{\nu} e^{-\xi}\,L_n^{(\beta)}(\xi)\left(\frac{d^m}{d\xi^m}\int_0^{\xi} x^{\alpha}(\xi-x)^{\beta-\alpha+m-1}f(x)\,dx\right)d\xi+\\ +\dfrac{n!}{\Gamma(\sigma-\beta)}\displaystyle\int_{\nu}^{\infty}\eta^{\beta}\,L_n^{(\beta)}(\eta)\left(\int_{\eta}^{\infty}e^{-x}(x-\eta)^{\sigma-\beta-1}\,g(x)\,dx\right)d\eta,\quad n=0,1,2,\ldots, \end{cases}$$

provided $\beta+m>\alpha>-1$ and $\sigma>\beta>-1$, m being a positive integer.

Subsequently, with a view to simplifying the calculations involved, we applied the method developed by SNEDDON and SRIVASTAV [4] which would require considering separately the equations (1.1) and (1.2) when (i) $g(x)\equiv 0$, (ii) $f(x)\equiv 0$, and adding the solutions of the individual problems (i) and (ii). The method did apply to equations (1.4) and (1.5), and the problem in each case was reduced to that of solving an Abel integral equation [8, p. 229]. Indeed the classes of functions $f(x)$ and $g(x)$ for which the problem under consideration could be solved in this manner must satisfy the following additional conditions:

(a) $F(x)=x^{\alpha}f(x)$ is continuously differentiable on $0\leqq x\leqq\delta<y$,

(b) $G(x)=\displaystyle\int_x^{\infty} e^{-t}g(t)\,dt$ is continuously differentiable for $y<1\leqq x<\infty$.

The solution thus obtained is [6, pp. 590–593]

$$(1.7)\quad \begin{cases} B_n=B_n^{(i)}+B_n^{(ii)}=\dfrac{n!}{\Gamma(\beta-\alpha+1)}\displaystyle\int_0^{\nu} e^{-\xi}\,L_n^{(\beta)}(\xi)\left(\frac{d}{d\xi}\int_0^{\xi}\frac{F(x)}{(\xi-x)^{\alpha-\beta}}\,dx\right)d\xi-\\ -\dfrac{n!}{\Gamma(\sigma-\beta)}\displaystyle\int_{\nu}^{\infty}\eta^{\beta}\,L_n^{(\beta)}(\eta)\left(\frac{d}{d\eta}\int_{\eta}^{\infty}\frac{G(x)}{(x-\eta)^{\beta-\sigma+1}}\,dx\right)d\eta,\quad n=0,1,2,\ldots, \end{cases}$$

provided $\beta+1>\alpha>-1$, $\sigma>\beta>-1$, and $F(x)$ and $G(x)$ satisfy the aforementioned conditions (a) and (b) respectively.

The object of the present note is manifold. First of all we discuss equivalence of the solutions (1.6) and (1.7). Then we proceed to show how the solution of the pair of dual equations (1.1) and (1.2) when $\beta=\alpha$ (instead of $\nu=\alpha$) can be obtained fairly easily from that of the equations (1.4) and (1.5). And finally we sum up our observations concerning the solvability of equations (1.1) and (1.2) under various restrictions on parameters α, β, ν and σ.

2. EQUIVALENCE OF (1.6) AND (1.7)

If we write $F(x)=x^{\alpha}f(x)$, then it is readily seen that the first part of the solution (1.7) is the same as the corresponding part of the solution (1.6) with $m=1$.

Next we recall the second part of the solution (1.7), viz.

$$(2.1)\qquad B_n^{(ii)}=-\frac{n!}{\Gamma(\sigma-\beta)}\int_{\nu}^{\infty}\eta^{\beta}\,L_n^{(\beta)}(\eta)\left(\frac{d}{d\eta}\int_{\eta}^{\infty}\frac{G(x)}{(x-\eta)^{\beta-\sigma+1}}\,dx\right)d\eta,\qquad n=0,1,2,\ldots,$$

and since

$$G(x) = \int_x^\infty e^{-t} g(t)\,dt,$$

we have

$$I \equiv \frac{d}{d\eta} \int_\eta^\infty \frac{G(x)}{(x-\eta)^{\beta-\sigma+1}}\,dx = \frac{d}{d\eta} \int_\eta^\infty (x-\eta)^{\sigma-\beta-1} \left[\int_x^\infty e^{-t} g(t)\,dt \right] dx.$$

On inverting the order of integration, we obtain

$$\begin{aligned} I &= \frac{d}{d\eta} \int_\eta^\infty e^{-t} g(t) \left[\int_\eta^t (x-\eta)^{\sigma-\beta-1}\,dx \right] dt \\ &= \frac{1}{\sigma-\beta} \frac{d}{d\eta} \int_\eta^\infty e^{-t}(t-\eta)^{\sigma-\beta} g(t)\,dt \\ &= - \int_\eta^\infty e^{-t}(t-\eta)^{\sigma-\beta-1} g(t)\,dt, \end{aligned}$$

it being provided that $\sigma > \beta$, and on substituting in equation (2.1) we find that

$$\text{(2.2)}\quad B_n^{(ii)} = \frac{n!}{\Gamma(\sigma-\beta)} \int_y^\infty \eta^\beta L_n^{(\beta)}(\eta) \left(\int_\eta^\infty e^{-t}(t-\eta)^{\sigma-\beta-1} g(t)\,dt \right) d\eta, \quad n = 0, 1, 2, \ldots,$$

which is the same as the second part of the solution (1.6).

Thus the solution (1.7) is in complete agreement with that given by (1.6) when $m = 1$.

3. The special case $\beta = \alpha$

Consider the pair of dual series equations

$$\text{(3.1)}\qquad \sum_{n=0}^\infty \frac{C_n}{\Gamma(\alpha+n+1)} L_n^{(\nu)}(x) = f(x), \quad 0 \leqq x < y,$$

$$\text{(3.2)}\qquad \sum_{n=0}^\infty \frac{C_n}{\Gamma(\alpha+n+1)} L_n^{(\sigma)}(x) = g(x), \quad y < x < \infty,$$

which correspond to the special case $\beta = \alpha$ of the original pair of equations (1.1) and (1.2).

On setting

$$\text{(3.3)}\qquad C_n = \frac{\Gamma(\alpha+n+1)}{\Gamma(\nu+n+1)} D_n, \quad n = 0, 1, 2, \ldots,$$

these equations will reduce to their equivalent form

$$\text{(3.4)}\qquad \sum_{n=0}^\infty \frac{D_n}{\Gamma(\nu+n+1)} L_n^{(\nu)}(x) = f(x), \quad 0 \leqq x < y,$$

$$\text{(3.5)}\qquad \sum_{n=0}^\infty \frac{D_n}{\Gamma(\nu+n+1)} L_n^{(\sigma)}(x) = g(x), \quad y < x < \infty.$$

It is interesting to note that the solution of the dual equations (3.4) and (3.5) is contained in that of (1.4) and (1.5). Indeed if in (1.4) and (1.5) we set $\beta=\alpha$ and then replace α by ν and B_n by D_n, $n=0, 1, 2, \ldots$, we shall at once be led to equations (3.4) and (3.5) respectively. Therefore, on using (1.6) in conjunction with (3.3), we obtain the solution of the pair of dual series equations (3.1) and (3.2) in the form

$$(3.6)\quad \begin{cases} C_n = \dfrac{n!\,\Gamma(\alpha+n+1)}{(m-1)!\,\Gamma(\nu+n+1)} \displaystyle\int_0^\nu e^{-\xi} L_n^{(\nu)}(\xi)\cdot \\ \qquad\qquad \cdot\left(\dfrac{d^m}{d\xi^m}\displaystyle\int_0^\xi x^\nu(\xi-x)^{m-1} f(x)\,dx\right) d\xi + \\ + \dfrac{n!\,\Gamma(\alpha+n+1)}{\Gamma(\sigma-\nu)\Gamma(\nu+n+1)} \cdot \displaystyle\int_\nu^\infty \eta^\nu L_n^{(\nu)}(\eta)\left(\int_\eta^\infty e^{-x}(x-\eta)^{\sigma-\nu-1} g(x)\,dx\right) d\eta, \\ \qquad\qquad n=0, 1, 2, \ldots, \end{cases}$$

provided that $\sigma>\nu>-1$, m being a positive integer.

4. Remarks

We now return to the question of solvability of the pair of dual series equations (1.1) and (1.2). It does not seem possible to solve them, by using one or the other known technique, in the most general form, i.e. when the parameters α, β, ν and σ are all free. The least restricted situation would occur when three of these parameters remain free. Of six such possibilities, the case $\alpha=\beta$ is trivial in the sense that it is contained in the more general case $\nu=\alpha$, as we have observed in the preceding section. In the remaining four obviously distinct circumstances, it would seem necessary to have an additional parametric restriction in order to deduce the solution of the resulting problem from that of equations (1.4) and (1.5).

It would seem worthwhile to mention here that in an earlier discussion with the author at the University of Strathclyde, as well as in a subsequent private communication to the author, Dr. John S. Lowndes has pointed out an interesting connection between dual Laguerre series equations and certain dual integral equations involving Bessel functions. Unfortunately, however, the alternative technique of solving dual series equations involving Laguerre polynomials, provided by this connection, does not seem to apply to the dual equations (1.1) and (1.2) except possibly in their special case $\nu=\alpha$ and $\sigma=\beta$, which evidently is already covered by equations (1.4) and (1.5).

It may be of interest to conclude with the remark that the method of solving dual equations (1.4) and (1.5) by reduction to Abel integral equations has a distinct advantage over that by the multiplying factor technique. Indeed it enables us to obtain, without any additional effort, the values of the series in (1.4) and (1.5) where they are not already specified. For instance, formulas (4.1) and (4.9) of [6, pp. 592–593] will readily yield the value of the series in (1.4) above when $y<x<\infty$, while the value of the series in (1.5) when $0 \leqq x < y$ is given by formulas (3.1) and (3.10) in [6, pp. 590–591].

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