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The Bellman-Gauss Principle for Constrained Motion

R. KALABA
Departments of Biomedical Engineering, Electrical Engineering and Economics
University of Southern California, Los Angeles, CA 90089, U.S.A.

H. NATSUYAMA AND S. UENO
Kyoto School of Computer Science
7 Tanaka Monzencho, Sakyo-ku, Kyoto 606, Japan

R. XU
School of Urban and Regional Planning
University of Southern California, Los Angeles, CA 90089, U.S.A.

Abstract—Gauss' principle of least constraint is solved in a sequential fashion via dynamic programming in this paper. The solution itself constitutes a new principle for constrained motion, which we may name the Bellman-Gauss principle for constrained motion.

Keywords—Principle of optimality, Dynamic programming, Gauss' principle of least constraint, Constrained motion.

1. INTRODUCTION

In 1829, C. F. Gauss stated his celebrated principle of least constraint, which subsumes all of analytical mechanics [1]. The principle takes the form of a constrained quadratic optimization problem. In this paper, Bellman's principle of optimality is used to solve Gauss' problem in a sequential fashion. The solution itself constitutes a new principle of analytical mechanics, which we may name the Bellman-Gauss principle for constrained motion.

2. GAUSS' PRINCIPLE OF LEAST CONSTRAINT

Consider a system of p particles, the i^{th} particle of which has mass m_i, displacement vector $\mathbf{x}_i$ in inertial Cartesian coordinates, velocity vector $\dot{\mathbf{x}}_i$, and acceleration vector $\ddot{\mathbf{x}}_i$. The external force on the i^{th} particle in rectangular coordinates in the inertial frame of reference is $\mathbf{f}_i$. If there were no constraints present, the free motion acceleration of the i^{th} particle would be $\mathbf{a}_i = \mathbf{f}/m_i$. Assume, though, that the particles are subject to equality constraints of both holonomic and nonholonomic type. A task of theoretical mechanics is to determine, at any time t, the actual accelerations $\ddot{\mathbf{x}}, \ddot{\mathbf{x}}_2, \dots, \ddot{\mathbf{x}}_p$ resulting from both the impressed forces and the constraint forces.

Let $\ddot{\mathbf{x}}$ be the $n \times 1$ actual acceleration vector obtained by stacking the three-dimensional acceleration vectors $\ddot{\mathbf{x}}_1, \ddot{\mathbf{x}}_2, \dots, \ddot{\mathbf{x}}_p$ in the usual fashion, $n = 3p$, and $\mathbf{a}$ be the $n \times 1$ free motion acceleration vector obtained by stacking the three-dimensional free motion acceleration vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_p$. Then, Gauss' principle of least constraint states that the actual acceleration vec-

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tor $\ddot{\mathbf{x}}$ is the one that minimizes G, where

$$G = (\ddot{\mathbf{x}} - \mathbf{a})^\top \mathbf{M} (\ddot{\mathbf{x}} - \mathbf{a}), \tag{1}$$

subject to whatever the constraints might be on the accelerations. As usual, superscript $\top$ denotes transposition, and $\mathbf{M}$ is the $n \times n$ diagonal mass matrix.

For all the bilateral constraints treated in Lagrangian mechanics, upon differentiating the holonomic constraints twice and the nonholonomic constraints once with respect to the time t, one is left with a consistent linear system of constraint equations for the acceleration vector $\ddot{\mathbf{x}}$

$$\mathbf{A}\ddot{\mathbf{x}} = \mathbf{b}, \tag{2}$$

where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{b}$ is an $m \times 1$ vector in which m is the number of constraints. The matrix $\mathbf{A}$ need not be of full rank. Given initial conditions on $\mathbf{x}$ and $\dot{\mathbf{x}}$, such differentiations of the constraints do not produce any loss in generality. In this case, Gauss' principle actually takes the form of minimizing G subject to the linear constraints (2). The matrix $\mathbf{A}$ and the vector $\mathbf{b}$ may be functions of t, $\mathbf{x}$, and $\dot{\mathbf{x}}$.

To recast Gauss' principle, let

$$\mathbf{y} = \mathbf{M}^{1/2}(\ddot{\mathbf{x}} - \mathbf{a}), \tag{3}$$

so that

$$\ddot{\mathbf{x}} = \mathbf{M}^{-1/2}\mathbf{y} + \mathbf{a}. \tag{4}$$

Consequently, equations (1) and (2) are equivalent to

$$G = \mathbf{y}^\top \mathbf{y} \tag{5}$$

and

$$\mathbf{A}\mathbf{M}^{-1/2}\mathbf{y} = \mathbf{b} - \mathbf{A}\mathbf{a}. \tag{6}$$

Gauss' principle is then reduced to the problem of finding the shortest length vector $\mathbf{y}$ such that the consistent equation (6) is satisfied. As is known [2], the solution to this problem is

$$\mathbf{y} = \left(\mathbf{A}\mathbf{M}^{-1/2}\right)^+ (\mathbf{b} - \mathbf{A}\mathbf{a}), \tag{7}$$

or equivalently [3,4],

$$\ddot{\mathbf{x}} = \mathbf{a} + \mathbf{M}^{-1/2}\left(\mathbf{A}\mathbf{M}^{-1/2}\right)^+ (\mathbf{b} - \mathbf{A}\mathbf{a}), \tag{8}$$

where $(\mathbf{A}\mathbf{M}^{-1/2})^+$ is the Moore-Penrose generalized inverse [5] of the matrix $\mathbf{A}\mathbf{M}^{-1/2}$.

In the following, we present an alternative approach to solving this shortest length problem via Bellman's principle of optimality [6,7].

3. SOLUTION VIA BELLMAN'S PRINCIPLE OF OPTIMALITY

Let $\mathbf{C} = \mathbf{A}\mathbf{M}^{-1/2}$ and $\mathbf{d} = \mathbf{b} - \mathbf{A}\mathbf{a}$. Apparently, $\mathbf{C}$ is an $m \times n$ matrix and $\mathbf{d}$ is an $m \times 1$ vector. Suppose that the rank of the matrix $\mathbf{C}$ is r, and its first r columns, $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r$, are linearly independent (if $\mathbf{C}$ is not in this form, it can be transformed into this form in various ways [8]). Let $\mathbf{c}_k$ be the k^{th} column of the matrix $\mathbf{C}$, $k = 1, 2, \dots, n$. Gauss' principle is then equivalent to the problem of finding the set of scalars $y_1, y_2, \dots, y_n$ such that $y_1^2 + y_2^2 + \cdots + y_n^2$ is minimum, and concurrently the consistent set of linear equations $\mathbf{c}_1 y_1 + \mathbf{c}_2 y_2 + \cdots + \mathbf{c}_n y_n = \mathbf{d}$ is satisfied.

The imbedding procedure of dynamic programming suggests the following. Let $g_k(\mathbf{d}_k)$ be the value of $y_1^2 + y_2^2 + \cdots + y_k^2$ when using an optimal set of scalars $y_1, y_2, \dots, y_k$, where $\mathbf{c}_1 y_1 + \mathbf{c}_2 y_2 + \cdots + \mathbf{c}_k y_k = \mathbf{d}_k$ is a consistent set of linear algebraic equations, $\mathbf{d}_k$ is an $m \times 1$ vector, and $k = r, r+1, \dots, n$. Optimal is in the sense that this particular set of scalars $y_1, y_2, \dots, y_k$ makes the value of $y_1^2 + y_2^2 + \cdots + y_k^2$ be minimum. Then, Bellman's principle of optimality[1] leads to the basic recurrence relationship

$$g_k(\mathbf{d}_k) = \min_{y_k} \left\{ y_k^2 + g_{k-1}(\mathbf{d}_k - \mathbf{c}_k y_k) \right\}, \qquad k = r+1, r+2, \ldots, n. \tag{9}$$

This is because once y_k is chosen, there is an immediate cost of y_k^2, and $y_1, y_2, \ldots, y_{k-1}$ have to be chosen so that $y_1^2 + y_2^2 + \cdots + y_{k-1}^2$ is minimum subject to the consistent constraint

$$\mathbf{c}_1 y_1 + \mathbf{c}_2 y_2 + \cdots + \mathbf{c}_{k-1} y_{k-1} = \mathbf{d}_k - \mathbf{c}_k y_k. \tag{10}$$

But, by definition, this minimum is $g_{k-1}(\mathbf{d}_k - \mathbf{c}_k y_k)$. Therefore, y_k must be chosen to minimize the sum in equation (9). The reason k starts from $r+1$ is that there is one and only one set of scalars $y_1, y_2, \ldots, y_r$ that fulfills the constraint $\mathbf{c}_1 y_1 + \mathbf{c}_2 y_2 + \cdots + \mathbf{c}_r y_r = \mathbf{d}_r$, and hence there is no freedom in choosing the scalars $y_1, y_2, \ldots, y_k$, when $k \le r$, in view of the independence assumption.

Denote $\mathbf{C}_r$ to be the $m \times r$ matrix whose columns are $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_r$ and $\mathbf{y}^{(r)}$ to be the $r \times 1$ vector whose elements are $y_1, y_2, \ldots, y_r$. Since $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_r$ are assumed to be independent, so that the matrix $\mathbf{C}_r^\top \mathbf{C}_r$ is of dimension $r \times r$ with rank r, it follows that the inverse $(\mathbf{C}_r^\top \mathbf{C}_r)^{-1}$ exists. Thus, the unique solution to the consistent set of linear algebraic equations $\mathbf{C}_r \mathbf{y}^{(r)} = \mathbf{d}_r$, in vector form, is

$$\mathbf{y}^{(r)} = \left(\mathbf{C}_r^\top \mathbf{C}_r\right)^{-1} \mathbf{C}_r^\top \mathbf{d}_r. \tag{11}$$

Consequently, by definition,

$$\begin{aligned} g_r(\mathbf{d}_r) &= y_1^2 + y_2^2 + \cdots + y_r^2 = {\mathbf{y}^{(r)}}^\top \mathbf{y}^{(r)} \\ &= \mathbf{d}_r^\top \mathbf{C}_r \left[\left(\mathbf{C}_r^\top \mathbf{C}_r\right)^{-1}\right]^\top \left(\mathbf{C}_r^\top \mathbf{C}_r\right)^{-1} \mathbf{C}_r^\top \mathbf{d}_r \\ &= \mathbf{d}_r^\top \mathbf{C}_r \left(\mathbf{C}_r^\top \mathbf{C}_r\right)^{-1} \left(\mathbf{C}_r^\top \mathbf{C}_r\right)^{-1} \mathbf{C}_r^\top \mathbf{d}_r. \end{aligned} \tag{12}$$

Denote

$$\mathbf{R}_r = \mathbf{C}_r \left(\mathbf{C}_r^\top \mathbf{C}_r\right)^{-1} \left(\mathbf{C}_r^\top \mathbf{C}_r\right)^{-1} \mathbf{C}_r^\top. \tag{13}$$

Then

$$g_r(\mathbf{d}_r) = \mathbf{d}_r^\top \mathbf{R}_r \mathbf{d}_r, \tag{14}$$

where $\mathbf{R}_r$ is an $m \times m$ positive semidefinite symmetric matrix.

Assuming $g_{k-1}(\mathbf{d}_{k-1}) = \mathbf{d}_{k-1}^\top \mathbf{R}_{k-1} \mathbf{d}_{k-1}$, where $\mathbf{R}_{k-1}$ is an $m \times m$ positive semidefinite symmetric matrix, we now prove that $g_k(\mathbf{d}_k)$ has the form $g_k(\mathbf{d}_k) = \mathbf{d}_k^\top \mathbf{R}_k \mathbf{d}_k$, where $\mathbf{R}_k$ is an $m \times m$ positive semidefinite symmetric matrix and $k = r+1, r+2, \ldots, n$.

From the recurrence relation (9), we see that for $k = r+1, r+2, \ldots, n$,

$$\begin{aligned} g_k(\mathbf{d}_k) &= \min_{y_k} \left\{ y_k^2 + g_{k-1}(\mathbf{d}_k - \mathbf{c}_k y_k) \right\} = \min_{y_k} \left\{ y_k^2 + (\mathbf{d}_k - \mathbf{c}_k y_k)^\top \mathbf{R}_{k-1} (\mathbf{d}_k - \mathbf{c}_k y_k) \right\} \\ &= \min_{y_k} \left\{ y_k^2 + \mathbf{d}_k^\top \mathbf{R}_{k-1} \mathbf{d}_k - 2\mathbf{d}_k^\top \mathbf{R}_{k-1} \mathbf{c}_k y_k + \mathbf{c}_k^\top \mathbf{R}_{k-1} \mathbf{c}_k y_k^2 \right\} \\ &= \min_{y_k} \left\{ \left(1 + \mathbf{c}_k^\top \mathbf{R}_{k-1} \mathbf{c}_k\right) y_k^2 + \mathbf{d}_k^\top \mathbf{R}_{k-1} \mathbf{d}_k - 2\mathbf{d}_k^\top \mathbf{R}_{k-1} \mathbf{c}_k y_k \right\}. \end{aligned} \tag{15}$$

Since the first order condition for the minimizing value of y_k is

$$\frac{\partial\{\,\}}{\partial y_k} = 2\left(1 + \mathbf{c}_k^\top \mathbf{R}_{k-1} \mathbf{c}_k\right) y_k - 2\mathbf{d}_k^\top \mathbf{R}_{k-1} \mathbf{c}_k = 0, \tag{16}$$

[1]Bellman's principle of optimality states: "An optimal policy has the property that whatever the initial decision is, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

we have

$$y_k^{\text{opt}} = \frac{\mathbf{d}_k^\top \mathbf{R}_{k-1}\mathbf{c}_k}{1+\mathbf{c}_k^\top \mathbf{R}_{k-1}\mathbf{c}_k} \tag{17}$$

or

$$y_k^{\text{opt}} = \frac{\mathbf{c}_k^\top \mathbf{R}_{k-1}}{1+\mathbf{c}_k^\top \mathbf{R}_{k-1}\mathbf{c}_k}\mathbf{d}_k. \tag{18}$$

Substituting equation (17) or (18) into equation (15) gives

$$\begin{aligned} g_k(\mathbf{d}_k) &= \left(1+\mathbf{c}_k^\top \mathbf{R}_{k-1}\mathbf{c}_k\right)\left[y_k^{\text{opt}}\right]^2 - 2\mathbf{c}_k^\top \mathbf{R}_{k-1}\mathbf{d}_k y_k^{\text{opt}} + \mathbf{d}_k^\top \mathbf{R}_{k-1}\mathbf{d}_k \\ &= \left(1+\mathbf{c}_k^\top \mathbf{R}_{k-1}\mathbf{c}_k\right)\frac{\mathbf{d}_k^\top \mathbf{R}_{k-1}\mathbf{c}_k\mathbf{c}_k^\top \mathbf{R}_{k-1}\mathbf{d}_k}{(1+\mathbf{c}_k^\top \mathbf{R}_{k-1}\mathbf{c}_k)^2} - 2\mathbf{d}_k^\top \frac{\mathbf{R}_{k-1}\mathbf{c}_k\mathbf{c}_k^\top \mathbf{R}_{k-1}}{1+\mathbf{c}_k^\top \mathbf{R}_{k-1}\mathbf{c}_k}\mathbf{d}_k + \mathbf{d}_k^\top \mathbf{R}_{k-1}\mathbf{d}_k \\ &= \mathbf{d}_k^\top \mathbf{R}_{k-1}\mathbf{d}_k - \mathbf{d}_k^\top \frac{\mathbf{R}_{k-1}\mathbf{c}_k\mathbf{c}_k^\top \mathbf{R}_{k-1}}{1+\mathbf{c}_k^\top \mathbf{R}_{k-1}\mathbf{c}_k}\mathbf{d}_k \\ &= \mathbf{d}_k^\top \left[\mathbf{R}_{k-1} - \frac{\mathbf{R}_{k-1}\mathbf{c}_k\mathbf{c}_k^\top \mathbf{R}_{k-1}}{1+\mathbf{c}_k^\top \mathbf{R}_{k-1}\mathbf{c}_k}\right]\mathbf{d}_k \\ &= \mathbf{d}_k^\top \mathbf{R}_k \mathbf{d}_k, \end{aligned} \tag{19}$$

where

$$\mathbf{R}_k = \mathbf{R}_{k-1} - \frac{\mathbf{R}_{k-1}\mathbf{c}_k\mathbf{c}_k^\top \mathbf{R}_{k-1}}{1+\mathbf{c}_k^\top \mathbf{R}_{k-1}\mathbf{c}_k}, \qquad k = r+1, r+2, \dots, n. \tag{20}$$

Notice that the denominator $1+\mathbf{c}_k^\top \mathbf{R}_{k-1}\mathbf{c}_k \neq 0$ always holds, since $\mathbf{R}_{k-1}$ is positive semidefinite. Let

$$\beta_k = \mathbf{R}_{k-1}\mathbf{c}_k, \qquad k = r+1, r+2, \dots, n. \tag{21}$$

Then, equation (20) is equivalent to

$$\mathbf{R}_k = \mathbf{R}_{k-1} - \frac{\beta_k \beta_k^\top}{1+\mathbf{c}_k^\top \beta_k}, \qquad k = r+1, r+2, \dots, n. \tag{22}$$

Substituting equation (22) into equations (17) and (18) gives

$$y_k^{\text{opt}} = \frac{\mathbf{d}_k^\top \beta_k}{1+\mathbf{c}_k^\top \beta_k} \tag{23}$$

and

$$y_k^{\text{opt}} = \frac{\beta_k^\top}{1+\mathbf{c}_k^\top \beta_k}\mathbf{d}_k, \tag{24}$$

respectively, where

$$\begin{aligned} \mathbf{d}_n &= \mathbf{d}, \\ \mathbf{d}_k &= \mathbf{d}_{k+1} - \mathbf{c}_{k+1} y_{k+1}^{\text{opt}}, \qquad k = n-1, n-2, \dots, r. \end{aligned} \tag{25}$$

Therefore, the sequence of $\mathbf{R}_r, \mathbf{R}_{r+1}, \dots, \mathbf{R}_n$ is now obtainable through formulas (13) and (22). The optimal values of $\mathbf{y}^{(r)}$, $y_{r+1}, y_{r+2}, \dots, y_n$ can be calculated via equation (11) and equation (23) or (24).

In brief, to find the shortest length solution to the consistent set of linear algebraic equations $\mathbf{C}\mathbf{y} = \mathbf{d}$, where $\mathbf{C}$ is a matrix with rank r whose first r columns are linearly independent, we may use the following $\beta - \mathbf{R}$ algorithm. We carry out the following sequence of calculations:

$$\mathbf{R}_r = \mathbf{C}_r \left(\mathbf{C}_r^\top \mathbf{C}_r\right)^{-1} \left(\mathbf{C}_r^\top \mathbf{C}_r\right)^{-1} \mathbf{C}_r^\top, \tag{26}$$

$$\beta_{r+1} = \mathbf{R}_r \mathbf{c}_{r+1},$$
$$\mathbf{R}_{r+1} = \mathbf{R}_r - \frac{\beta_{r+1}\beta_{r+1}^\top}{1 + \mathbf{c}_{r+1}^\top \beta_{r+1}}, \tag{27}$$

$$\beta_{r+2} = \mathbf{R}_{r+1} \mathbf{c}_{r+2},$$
$$\mathbf{R}_{r+2} = \mathbf{R}_{r+1} - \frac{\beta_{r+2}\beta_{r+2}^\top}{1 + \mathbf{c}_{r+2}^\top \beta_{r+2}}, \tag{28}$$

$$\vdots$$

$$\beta_n = \mathbf{R}_{n-1} \mathbf{c}_n,$$
$$\mathbf{R}_n = \mathbf{R}_{n-1} - \frac{\beta_n \beta_n^\top}{1 + \mathbf{c}_n^\top \beta_n}. \tag{29}$$

Only $\beta_{r+1}, \beta_{r+2}, \dots, \beta_n$ need to be stored. The optimal set of scalars $y_n, y_{n-1}, \dots, y_1$ is then obtained by

$$y_n^{\text{opt}} = \frac{\beta_n^\top}{1 + \mathbf{c}_n^\top \beta_n} \mathbf{d}, \tag{30}$$

$$y_{n-1}^{\text{opt}} = \frac{\beta_{n-1}^\top}{1 + \mathbf{c}_{n-1}^\top \beta_{n-1}} \left(\mathbf{d} - \mathbf{c}_n y_n^{\text{opt}}\right), \tag{31}$$

$$\vdots$$

$$y_{r+1}^{\text{opt}} = \frac{\beta_{r+1}^\top}{1 + \mathbf{c}_{r+1}^\top \beta_{r+1}} \left(\mathbf{d} - \sum_{i=r+2}^{n} \mathbf{c}_i y_i^{\text{opt}}\right), \tag{32}$$

$$\begin{bmatrix} y_1^{\text{opt}} \\ y_2^{\text{opt}} \\ \vdots \\ y_r^{\text{opt}} \end{bmatrix} = \left(\mathbf{C}_r^\top \mathbf{C}_r\right)^{-1} \mathbf{C}_r^\top \left(\mathbf{d} - \sum_{i=r+1}^{n} \mathbf{c}_i y_i^{\text{opt}}\right). \tag{33}$$

Once $\mathbf{y}$ is determined, $\ddot{\mathbf{x}}$ is obtained from equation (4).

4. THE BELLMAN-GAUSS PRINCIPLE FOR CONSTRAINED MOTION

Gauss' principle describes the relationship between the actual accelerations and the free motion accelerations. The algorithm derived from Bellman's principle of optimality gives an algorithm for converting a set of free motion accelerations to a set of actual accelerations. It is thus seen that equations (26)–(33) plus equation (4) constitute a free standing principle of analytical mechanics, which we may call the Bellman-Gauss principle for constrained motion.

5. AN APPLICATION

Consider a simple pendulum problem in mechanics, in which we want to find the equations of motion for a material point with mass m and coordinates (x_1, x_2). See Figure 1.

According to Gauss, the actual acceleration vector $\ddot{\mathbf{x}}$ for this pendulum is the one that minimizes

$$G = (\ddot{\mathbf{x}} - \mathbf{a})^\top \mathbf{M} (\ddot{\mathbf{x}} - \mathbf{a}), \tag{34}$$

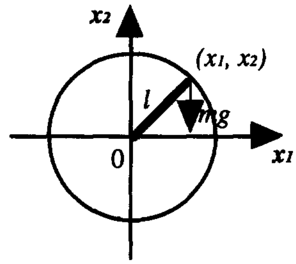

Figure 1. The motion of a pendulum.

subject to the constraint

$$x_1^2 + x_2^2 = l^2, \tag{35}$$

where

$$\ddot{\mathbf{x}} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}, \qquad \mathbf{a} = \begin{bmatrix} 0 \\ -g \end{bmatrix}, \qquad \text{and} \qquad \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}.$$

To apply the Bellman-Gauss principle, we first identify the matrix $\mathbf{C}$ and vectors $\mathbf{d}$ and $\mathbf{y}$. As before, $\mathbf{c}_k$ denotes the k^{th} column of matrix $\mathbf{C}$ and y_k denotes the k^{th} element of vector $\mathbf{y}$, $k = 1, 2$. Differentiating the holonomic constraint (35) twice gives

$$x_1\ddot{x}_1 + x_2\ddot{x}_2 = -\left(\dot{x}_1^2 + \dot{x}_2^2\right). \tag{36}$$

In matrix form, equation (36) is equivalent to

$$\mathbf{A}\ddot{\mathbf{x}} = \mathbf{b}, \tag{37}$$

where $\mathbf{A}_{1\times 2} = [\,x_1 \quad x_2\,]$ and $\mathbf{b}_{1\times 1} = -(\ddot{x}_1^2 + \dot{x}_2^2)$. Therefore,

$$\mathbf{C}_{1\times 2} = \mathbf{A}\mathbf{M}^{-1/2} = [\,x_1 \quad x_2\,] \begin{bmatrix} m^{-1/2} & 0 \\ 0 & m^{-1/2} \end{bmatrix} = [\,m^{-1/2}x_1 \quad m^{-1/2}x_2\,], \tag{38}$$

$$\mathbf{d}_{1\times 1} = \mathbf{b} - \mathbf{A}\mathbf{a} = -\left(\dot{x}_1^2 + \dot{x}_2^2\right) - [\,x_1 \quad x_2\,] \begin{bmatrix} 0 \\ -g \end{bmatrix} = -\left(\dot{x}_1^2 + \dot{x}_2^2\right) + gx_2, \tag{39}$$

and

$$\mathbf{y}_{2\times 1} = \mathbf{M}^{1/2}(\ddot{\mathbf{x}} - \mathbf{a}) = \begin{bmatrix} m^{1/2} & 0 \\ 0 & m^{1/2} \end{bmatrix} \left(\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} - \begin{bmatrix} 0 \\ -g \end{bmatrix} \right) = \begin{bmatrix} m^{1/2}\ddot{x}_1 \\ m^{1/2}(\ddot{x}_2 + g) \end{bmatrix}. \tag{40}$$

Obviously, the rank of the matrix $\mathbf{C}$ is 1, namely, $r = 1$. For simplicity, assume $x_1 \neq 0$. It follows that $\mathbf{C}_1 = \mathbf{c}_1 = m^{-1/2}x_1$, $\mathbf{c}_2 = m^{-1/2}x_2$, $y_1 = m^{1/2}\ddot{x}_1$, and $y_2 = m^{1/2}(\ddot{x}_2 + g)$. Therefore, equations (26)–(33) provide

$$\mathbf{R}_1 = \mathbf{C}_1\left(\mathbf{C}_1^{\top}\mathbf{C}_1\right)^{-1}\left(\mathbf{C}_1^{\top}\mathbf{C}_1\right)^{-1}\mathbf{C}_1^{\top} = \frac{m^{-1}x_1^2}{[(m^{-1}x_1^2)(m^{-1}x_1^2)]} = mx_1^{-2}, \tag{41}$$

$$\beta_2 = \mathbf{R}_1\mathbf{c}_2 = mx_1^{-2}m^{-1/2}x_2 = m^{1/2}x_1^{-2}x_2. \tag{42}$$

The optimal scalars y_2 and y_1 are then obtained as

$$\begin{aligned} y_2^{\text{opt}} &= \frac{\beta_2^{\top}}{1 + \mathbf{c}_2^{\top}\beta_2}\mathbf{d} = \frac{m^{1/2}x_1^{-2}x_2}{1 + m^{-1/2}x_2m^{1/2}x_1^{-2}x_2}\left[-\left(\dot{x}_1^2 + \dot{x}_2^2\right) + gx_2\right] \\ &= \frac{m^{1/2}x_2}{x_1^2 + x_2^2}\left[gx_2 - \left(\dot{x}_1^2 + \dot{x}_2^2\right)\right] \end{aligned} \tag{43}$$

and

$$\begin{aligned} y_1^{\text{opt}} &= \left(\mathbf{C}_1^{\mathsf T}\mathbf{C}_1\right)^{-1}\mathbf{C}_1^{\mathsf T}\left(\mathbf{d}-\mathbf{c}_2 y_2^{\text{opt}}\right) \\ &= m^{1/2}x_1^{-1}\left\{-\left(\dot{x}_1^2+\dot{x}_2^2\right)+gx_2-\frac{m^{-1/2}x_2 m^{1/2}x_2}{x_1^2+x_2^2}\left[gx_2-\left(\dot{x}_1^2+\dot{x}_2^2\right)\right]\right\} \\ &= \frac{m^{1/2}x_1\left[gx_2-\left(\dot{x}_1^2+\dot{x}_2^2\right)\right]}{x_1^2+x_2^2}. \end{aligned} \tag{44}$$

From equation (4), it is seen that the equations of motion for the pendulum are

$$\begin{aligned} \ddot{x}_1 &= m^{-1/2}y_1^{\text{opt}} = \frac{gx_1x_2-x_1\left(\dot{x}_1^2+\dot{x}_2^2\right)}{x_1^2+x_2^2}, \\ \ddot{x}_2 &= m^{-1/2}y_2^{\text{opt}} - g = \frac{x_2}{x_1^2+x_2^2}\left[gx_2-\left(\dot{x}_1^2+\dot{x}_2^2\right)\right]-g = \frac{-gx_1^2-x_2\left(\dot{x}_1^2+\dot{x}_2^2\right)}{x_1^2+x_2^2}, \end{aligned} \tag{45}$$

which are seen to be correct by standard methods. These equations were derived for $x_1 \neq 0$. However, they hold even when $x_1 = 0$.

6. DISCUSSION

The Bellman-Gauss principle introduced in this paper provides new insights into the nature of constrained motion. Not only is the relationship between the actual accelerations and free motion accelerations specified in a new way, but also the interrelationship among the actual accelerations of all the particles is prescribed. Since Gauss' principle holds even when the system is described in generalized coordinates [8], the possibilities are far wider than is indicated above. In practical applications, the FEED (Fast and Efficient Evaluation of Derivatives [9]) procedure would be employed to automatically calculate all the needed derivatives for the holonomic and nonholonomic constraints. The potential utility of the Bellman-Gauss principle, both conceptually and computationally, remains to be explored further.

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