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Note

A difficulty in particular Shannon-like games*

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Abstract

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We have a simple theory of strategy for generalized Shannon switching games (on edges). This theory naturally gives us an algorithm of polynomial order. On the other hand Reisch states the PSPACE completeness of judging who can win in all the positions of Hex.

Our purpose in this paper is to discuss the difficulty not of a class but of each particular game. We shall see that any strategy theory of a certain type cannot be applied either to Hex of 4^2 vertices or to a certain 3-pair game, though it can be applied to all 2-pair games. Here, an *n-pair game* is a game played on the edge set of a graph where a designated player intends to connect at least one of the given n pairs of terminals.

1. Parity-preserving games

In this paper we fix a set Π consisting of two players and another element θ. An *occupation game* is a pair $\Gamma=(X,F)$ of a finite set X called the *basic set* and a subfamily F of 2^X. We generally denote a player by π and his opponent by $\hat{\pi}$.

Let Γ be an occupation game. In this section we deal with a so-called *regular play* Γ^+ held on Γ as follows. The two players choose alternately an element of X which

* Dedicated to C. Berge for his 60th birthday.

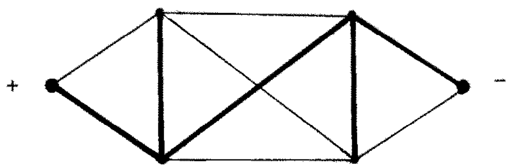

Fig. 1.

is not occupied yet. A player called *Short* wins if he covers an element of F with the elements of X which he has occupied and the other player called *Cut* wins if he prevents it. There are three possibilities as follows:

- (S^+) Short can win Γ^+ even playing second,
- (θ^+) the first player can win Γ^+,
- (C^+) Cut can win Γ^+ even playing second.

We define an element $\pi^+(\Gamma)$ as Short, θ or Cut respectively. A *position* is a game obtained from Γ when several elements of X are occupied by the two players. A position is denoted by $\Gamma_{x \cdot \pi}$ if a player π occupies a point x and $\hat{\pi}$ nowhere. Suppose that $\pi^+(\Gamma)$ is a player π. Then we call Γ a *π-game*. If $\pi^+(\Gamma_{x \cdot \hat{\pi}})$ coincides with θ for any element x of X, then Γ is called a *minimal π-game*.

An occupation game Γ is said to be *parity-preserving* if the basic set of any position Δ including itself is of an even size whenever Δ is a minimal π-game for a player π.

Example 1. Several generalized Shannon switching games are parity-preserving (cf. [7, 8]). For a classical Shannon switching game, we define X as the set of edges and F as the set of paths between the terminals. In this case, any Short-game, for instance, has a co-spanning pair of trees joining two terminals. Then we obtain a Short-game again even if Cut occupies all the edges except these two trees. Here a matching strategy does not neccessarily exist (see Fig. 1).

Example 2. A 2-pair game is parity-preserving (cf. [7, 8]). However, an n-pair game

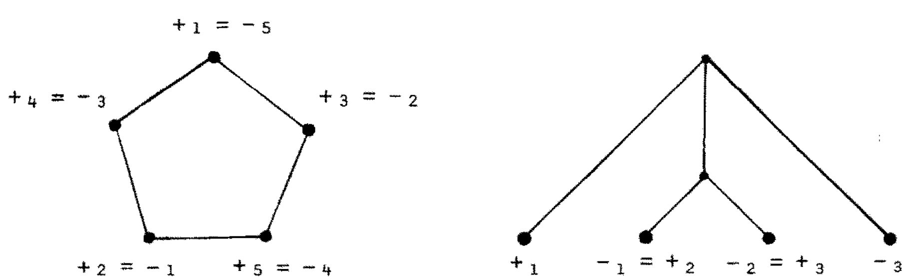

Fig. 2.

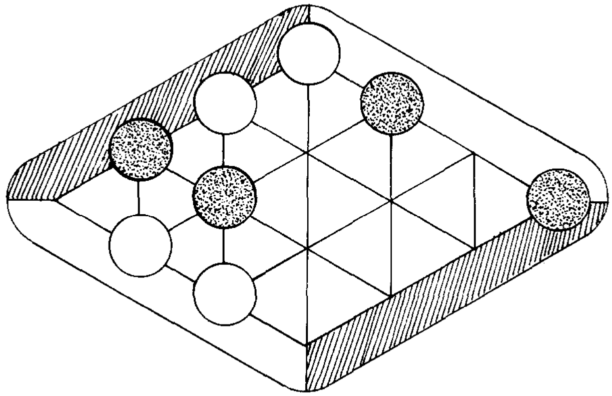

Fig. 3.

is not always parity-preserving. In fact, Fig. 2 on the left side and Fig. 2 on the right side yield a minimal Short-game and a minimal Cut-game, respectively. In each case, we define X as the set of 5 edges and F as the set of paths between one of the pairs $(+_i, -_i)$. One can separate the coinciding terminals by adding double edges.

Example 3. The Hex game of 4^2 vertices is not parity-preserving. Figure 3 yields a minimal Short-game, where we identify Short with Black (cf. [7, p. 61–63]). Here we define X as the set of remaining 7 cross points and F as the set consisting of all the subsets of X each of which forms a chain between the two terminals, together with several black pieces.

2. Reversibility

Let $\Gamma = (X, F)$ be an occupation game. In this section we consider also the *misère play* Γ^- obtained from Γ^+ by the reversed decision rule of the winner. There are three possibilities as follows:

- (S^-) Short is to lose Γ^-,
- (θ^-) a player is to lose Γ^- if he is to play the $|X|$-th move,
- (C^-) Cut is to lose Γ^-.

We define an element $\pi^-(\Gamma)$ as Short, θ or Cut respectively.

An occupation game Γ is said to be *reversible* if $\pi^+(\Delta)$ coincides with $\pi^-(\Delta)$ for any position Δ including itself. A Shannon switching game is parity-preserving and then reversible because of the following theorem:

Theorem. *Let $\Gamma = (X, F)$ be an occupation game. Then it is reversible if and only if it is parity-preserving.*

Proof. We prove this theorem by induction on the size of the basic set. We assume that any position except Γ is parity-preserving and reversible.

Assume that Γ is parity-preserving. First suppose that a player π can win Γ^+ playing second. Then $\hat{\pi}$ can win Γ^- even if he is to play the $|X|$-th move, always playing out of a subset of X of an even size. Next, suppose that a player π can win Γ^+ playing first on a point x. Then $\hat{\pi}$ can win Γ^- if he is not to play the $|X|$-th move, reserving x for π's last move (cf. [6, Theorem 1]).

We assume that Γ is reversible. Suppose that $|X|$ is odd and that a player π can win Γ^+ playing second. Then, by the reversibility of Γ, $\hat{\pi}$ can win Γ^- playing first on a point x. Therefore π can win $\Gamma^+_{x \cdot \hat{\pi}}$ playing second, by the reversibility of Γ. Now Γ is not minimal and our theorem is verified (cf. [6, Theorem 1]). □

Remarks. The property of "parity-preserving" has been introduced from the regular play theory. We can define the corresponding concept in the misère play theory. However, it gives no limitation on occupation games.

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