TRIANGLES IN ARRANGEMENTS OF LINES

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A set of n nonconcurrent lines in the projective plane (called an arrangement) divides the plane into polygonal cells. It has long been a problem to find a nontrivial upper bound on the number of triangular regions. We show that \( \frac{1}{3}n(n - 1) \) is such a bound. We also show that if no three lines are concurrent, then the number of quadrilaterals, pentagons and hexagons is at least \( cn^2 \).

1. Introduction and notation

An arrangement \( A \) of \( n \) lines in the real projective plane \( P_2 \) is a set of \( n \) lines not all concurrent. Such an arrangement divides \( P_2 \) into cells of a complex which are convex polygons. We denote by \( f_0(A) \), \( f_1(A) \) and \( f_2(A) \) respectively the number of vertices, edges and faces of the complex. We denote by \( t_i(A) \) the number of vertices incident with precisely \( i \) lines, and by \( P_k(A) \) the number of \( k \)-sided faces. Euler’s relation for the projective plane states that \( f_0 - f_1 + f_2 = 1 \), and simple counting arguments give us also

\[
\begin{align*}
    f_0 &= \sum_{j \geq 2} t_j, \\
    f_2 &= \sum_{k \geq 3} p_k, \\
    \sum_{k \geq 3} kp_k &= 2 \sum_{j \geq 2} jt_j = 2f_1.
\end{align*}
\]

and

\[
\sum_{j \geq 2} \binom{j}{2} t_j = \binom{n}{2}.
\]

The following equations are useful consequences of Euler’s relation and the above and are given in [2]:

\[
\begin{align*}
    t_2 - 3 &= \sum_{j \geq 3} (j - 3)t_j + \sum_{k \geq 3} (k - 3)p_k, \quad (1) \\
    p_3 - 4 &= \sum_{k \geq 4} (k - 4)p_k + 2 \sum_{j \geq 3} (j - 2)t_j. \quad (2)
\end{align*}
\]

An arrangement is simple if \( f_0 = t_2 \), and is a near pencil if exactly \( n - 1 \) lines are concurrent. In this paper we are concerned mainly with two questions: What is the maximum of \( p_3 \) over all arrangements of \( n \) lines? What is the minimum value of \( p_4 \) over all simple arrangements of \( n \) lines? The identities and inequalities

157
developed to deal with these questions are also of interest. It is an easy consequence of the Burr et al. results in [3] that there exist arrangements for which \( p_3 \geq \frac{1}{3}n(n - 3) + 4 \) (see [6, Theorem 6]). It has been conjectured by Grünbaum [3] that \( p_3 \leq \frac{1}{3}n(n - 1) \) for \( n \) sufficiently large. Corollary 2.5 states that \( p_3 \leq \frac{1}{3}n(n - 1) \) for \( n \geq 6 \). This is an improvement on the trivial \( p_3 \leq f_2 \leq \left( \begin{array}{c} 3 \\ 2 \end{array} \right) + 1 \). Regarding the other problem, Corollary 3.3 states that \( 3p_a + 2p_3 + p_o \geq \left( \begin{array}{c} 4 \\ 2 \end{array} \right) + 7 \) for simple arrangements.

We also prove in Theorem 4.2 that for sufficiently large \( n \) there are gaps in the values which can be achieved by \( f_2 \), partially proving a conjecture of Grünbaum.

An arrangement of pseudolines is a collection of curves homotopic to lines such that any two intersect exactly once, and the intersection of all of them is empty. Most of the results in this paper are also true for arrangements of pseudolines.

An arrangement is said to be simplicial if \( p_3 = f_2 \), and quadrilateral if \( f_2 = p_3 + p_4 \).

2. A general upper bound on \( p_3 \)

**Lemma 2.1.** The following identities hold.

\[
p_3 + \frac{3}{2} t_2 + \frac{1}{3} \sum_{j=4} (j-3)(j-4) t_j = 4 + \left( \begin{array}{c} 3 \\ 2 \end{array} \right) + \sum_{k \geq 4} (k-4) p_k. \tag{3}
\]

\[
p_3 + \frac{3}{2} \sum_{k \geq 4} p_k + \frac{1}{3} \sum_{j=4} (j-3)(j-2) t_j = 2 + \left( \begin{array}{c} 3 \\ 2 \end{array} \right) + \frac{1}{3} \sum_{k \geq 4} (k-4) p_k. \tag{4}
\]

**Proof.** We prove (3) first. From (2) we have

\[
p_3 = \frac{3}{2} \left( \begin{array}{c} 3 \\ 2 \end{array} \right) - 4 = \sum_{k \geq 4} (k-4)p_k + 2 \sum_{j=2} (j-2) t_j - \frac{1}{3} \sum_{j=2} (j-3) (j-4) t_j
\]

\[
= \sum_{k \geq 4} (k-4)p_k - \frac{1}{3} \sum_{j=3} (j-3)(j-4) t_j
\]

\[
= \sum_{k \geq 4} (k-4)p_k - \frac{3}{2} t_2 - \frac{1}{3} \sum_{j=3} (j-3)(j-4) t_j
\]

and (3) follows. To prove (4), we use (1) to obtain

\[
-\frac{3}{2} t_2 = -2 - \frac{3}{2} \sum_{j=3} (j-3) t_j - \frac{3}{2} \sum_{k \geq 3} (k-3) p_k.
\]

Hence, by (3),

\[
p_3 - 4 - \sum_{k \geq 4} (k-4)p_k = \frac{3}{2} \left( \begin{array}{c} 3 \\ 2 \end{array} \right) - 2 - \frac{1}{3} \sum_{j=3} (j-3)(j-2) t_j - \frac{1}{3} \sum_{k \geq 3} (k-3) p_k,
\]

and (4) follows.
Remark 2.2. Identity (3) implies \( p_3 \leq \frac{1}{3} n(n - 1) + 4 + \sum_{k=4}^n (k - 4) p_k - \frac{2}{3} t_2 \), which is [6, Theorem 1], and (4) implies \( p_3 \leq \frac{1}{3} n(n - 1) + 2 + \frac{1}{3} \sum_{k=4}^n (k - 6) p_k \) which is [6, Corollary 3]. If \( A \) is quadrilateral, \( p_3 \leq \frac{1}{3} n(n - 1) \div O(n) \).

Lemma 2.3. If \( A \) is not a near pencil \((n - 1) \) lines concurrent), then

\[
p_3 \leq \frac{1}{3} n(n - 1) - \frac{1}{3} \sum_{j=3} j(j - 4) t_j
\]  

(5)

Proof. Let \( l \) be a line of the arrangement \( A \), let \( t_k(l) \) be the number of points on \( l \) incident with exactly \( k \) lines, and let \( p_3(l) \) be the number of triangles having \( l \) as one side. Let \( P_1, \ldots, P_m \) be the vertices on \( l \) in order. Since \( A \) is not a near pencil it is clear that if \( P_i \) and \( P_{i+1} \) are both simple (incident with only two lines), then the line segment \( P_iP_{i+1} \) is not the side of two triangles.

Since a nonsimple point is an endpoint of only two intervals, it follows that

\[
p_3(l) \leq m + 2 \sum_{j=3} t_j(l)
\]

\[
= n - 1 - \sum_{j=3} (j - 2) t_j(l) + 2 \sum_{j=3} t_j(l)
\]

\[
= n - 1 - \sum_{j=3} (j - 4) t_j(l).
\]

If we now add these inequalities for all lines \( l \), we obtain

\[
3p_3 \leq n(n - 1) - \sum_{j=3} j(j - 4) t_j
\]

and (5) follows.

Theorem 2.4. If \( A \) is not a near pencil, then

\[
p_3 + \frac{1}{2} \sum_{k=4} p_k \leq \frac{1}{3} + \frac{5}{12} n(n - 1) - \frac{1}{3} \sum_{j=3} (j - 3)(5j - 2) t_j.
\]

Proof. By (5), we have

\[
p_3 \leq \frac{1}{3} n(n - 1) + \sum_{j=3} (j - 2) t_j - \frac{1}{3} \sum_{j=3} (j + 2)(j - 3) t_j.
\]

and by (2),

\[
p_3 \leq \frac{1}{3} n(n - 1) + \frac{1}{2} p_3 - 2 - \frac{1}{3} \sum_{k=4} (k - 4) p_k - \frac{1}{3} \sum_{j=3} (j + 2)(j - 3) t_j.
\]
Hence
\[ \frac{1}{3} \sum_{k=4} (k-4)p_k \leq \frac{2}{3}n(n-1) - \frac{1}{3}p_3 - \frac{2}{3} \sum_{j=3} (j+2)(j-3)t_j. \]

Applying the last inequality to (4), we obtain
\[ p_3 + \frac{3}{3} \sum_{k=4} p_k \leq 2 + \frac{1}{3}n(n-1) + \frac{1}{3}n(n-1) - \frac{1}{3}p_3 - \frac{1}{3} \sum_{j=4} (j-3)(5j-2)t_j. \]

Hence
\[ \frac{1}{3}p_3 + \frac{3}{3} \sum_{k=4} p_k \leq \frac{2}{3}n(n-1) - \frac{1}{3} \sum_{j=4} (j-3)(5j-2)t_j, \]

and the theorem follows.

Corollary 2.5. If $A$ is not simplicial, then $A$ is not a near pencil and $\sum_{k=4} p_k > 1$, so that $p_3 \leq \frac{1}{3}n(n-1)$. On the other hand if $A$ is simplicial, then Strommer has shown in [6] that $p_3 \leq \frac{1}{3}n(n-1) + 4 - \frac{1}{n}$, and this is at most $\frac{1}{3}n(n-1)$ for $n \geq 6$.

3. Some more inequalities

In this section we prove some more inequalities which hold if the arrangement is not a near pencil.

Theorem 3.1. If $A$ is not a near pencil, then
\[ 4t_3 + 3(p_4 + t_4) + 2(p_5 + t_5) + p_6 + t_6 \geq 7 + \frac{1}{3}t_2 + \sum_{k=8} (k-7)(p_k + t_k) \geq 7 + \frac{1}{3}t_2, \quad (6) \]

and
\[ 8t_3 + 6t_4 + 4t_5 + 2t_6 + 5p_4 + \frac{1}{4}p_5 + \frac{1}{4}p_6 \geq \]
\[ \geq 14 + \frac{1}{3}p_3 + 2 \sum_{k=8} (k-7)t_k + \frac{2}{3} \sum_{k=7} (k-7)p_k + \frac{2}{3} \sum_{k=7} p_k. \quad (7) \]

Proof. We first prove (6). From (4) and (5) we have
\[ \frac{1}{3}n(n-1) - \frac{1}{3} \sum_{j=4} j(j-4)t_j + t_3 \geq \]
\[ \geq 2 + \frac{1}{3}n(n-1) + \frac{1}{3} \sum_{k=4} (k-6)p_k - \frac{1}{3} \sum_{j=4} (j-3)(j-2)t_j. \]

Hence
\[ 3t_3 + 2(p_4 + t_4) + p_5 + t_5 \geq 6 + \sum_{k=7} (k-6)(p_k + t_k). \quad (8) \]
Adding $\frac{1}{3}(p_4 + t_4) + \frac{1}{3}(p_5 + t_5) + \frac{3}{4}(p_6 + t_6)$ to both sides, and using (1), we obtain

$$3t_3 + \frac{1}{3}(p_4 + t_4) + \frac{3}{4}(p_5 + t_5) + \frac{3}{4}(p_6 + t_6) \Rightarrow$$

$$\geq 6 + \frac{1}{3} \sum_{k=3}^{10} (k-3)(p_k + t_k) + \frac{3}{4} \sum_{k=7}^{10} (k-7)(p_k + t_k)$$

$$= 6 + \frac{1}{3}t_2 - \frac{3}{4} + \frac{3}{4} \sum_{k=7}^{10} (k-7)(p_k + t_k),$$

from which (6) follows.

We now prove (7). Adding

$$\frac{1}{3}t_3 + \frac{2}{3}t_4 + \frac{3}{4}t_5 + \frac{3}{4}t_6 + \frac{1}{10}p_5 + \frac{1}{4}p_6$$

to both sides of (8), and using (2), we obtain

$$(3 + \frac{1}{3})t_3 + (2 + \frac{2}{3})t_4 + (1 + \frac{3}{4})t_5 + \frac{3}{4}t_6 + 2p_4 + (1 + \frac{1}{10})p_5 + \frac{1}{4}p_6 \Rightarrow$$

$$\geq 6 + \frac{3}{10} \sum_{k=3}^{10} (k-2)t_k + \frac{4}{10} \sum_{k=7}^{10} (k-7)t_k + \frac{1}{10} \sum_{k=4}^{10} (k-4)p_k$$

$$+ \frac{8}{10} \sum_{k=7}^{10} (k-7)p_k + \frac{7}{10} \sum_{k=7}^{10} p_k$$

$$= 6 + \frac{1}{10}(p_3 - 4) + \frac{8}{10} \sum_{k=8}^{10} (k-7)t_k + \frac{9}{10} \sum_{k=7}^{10} (k-7)p_k + \frac{7}{10} \sum_{k=7}^{10} p_k,$$

and (7) follows.

**Remark 3.2.** Strommer proved the surprising fact in [6] that $p_4 + p_5 > 0$ in simple arrangements. In the corollaries below we give some related results, but we are unable to prove Grünbaum's conjecture that $p_4 > 0$.

**Corollary 3.3.** If $A$ is simple and $n \geq 4$, then $A$ is not a near pencil and (6) gives

$$3p_4 + 2p_5 + p_6 \geq \frac{3}{2}(2) + 7,$$

whereas (7) gives

$$5p_4 + \frac{11}{4}p_5 + \frac{1}{4}p_6 \geq 14 + \frac{1}{4}p_3.$$

**Corollary 3.4.** If $A$ is simplicial, but not a near pencil, then (6) gives

$$4t_3 + 3t_4 + 2t_5 + t_6 \geq 7 + \frac{1}{3}t_2$$

and (7) gives

$$8t_3 + 6t_4 + 4t_5 + 2t_6 \geq 14 + \frac{1}{4}p_3.$$

**Corollary 3.5.** For general $A$ that are not near pencils, Kelly and Moser's result that $t_3 \geq \frac{3}{10}n$ [4] and the well-known result that $p_3 \geq n$ when applied to (6) and (7) give
respectively
\[ 4t_3 + 3(p_4 + t_4) + 2(p_5 + 7t_5) + p_6 + t_6 \geq 7 + \frac{1}{3}n, \]
and
\[ 8t_3 + 6t_4 + 4t_5 + 2t_6 + 5p_4 + \frac{11}{4}p_5 + \frac{1}{2}p_6 \geq 14 + \frac{1}{4}n. \]

**Corollary 3.6.** Using the inequality \( t_3 + 2t_2 \geq 3 + f_0 \) proved in [2], (6) gives us
\[-\frac{25}{4}t_3 + 3(p_4 + t_4) + 2(p_5 + t_5) + p_6 + t_6 \geq \frac{15}{2} + \frac{1}{8}f_0. \]
The inequality \( 2p_4 \geq 4 + f_2 \) given in the proof of [5, Theorem 2] together with (7) gives us
\[ 8t_3 + 6t_4 + 4t_5 + 2t_6 + \frac{11}{4}p_4 + \frac{11}{4}p_5 + \frac{1}{2}p_6 \geq \frac{29}{2} + \frac{1}{8}f_2. \]
Both of these inequalities are incidentally trivially true for near pencils.

### 4. The proof of a conjecture of Grünbaum

**Lemma 4.1.** Given \( m \geq 2 \) lines through a point \( P \) and \( n \geq 1 \) lines not through \( P \),
\[ f_2 \leq (n + 1)m + \binom{n}{2}. \]

**Proof.** The result is true if \( n = 1 \), since then \( f_2 = 2m \). Let \( n \geq 1 \), and use induction on \( n \). Let \( f'_2 \) be the number of regions in the arrangement obtained by removing one of the \( n \) lines. Then
\[ f'_2 \leq nm + \binom{n}{2}. \]
The number of new regions created by adding the \( n \)th line is at most \( m + n - 1 \), and so
\[ f_2 \leq f'_2 + m + n + 1 \leq (n + 1)m + \binom{n}{2} \]
and the lemma is proved.

**Theorem 4.2.** If \( A \) is an arrangement of lines (or pseudolines),
\[ n \geq 4k^2 + k + 1, \]
and
\[ f_2 \geq k(n - k + 1) + \binom{k - 1}{2}, \]
then
\[ f_2 \geq (k + 1)(n - k). \]

**Proof.** If no more than \( n - k \) lines are concurrent, then by [5, Theorem 1], \( f_2 \geq (k + 1)(n - k) \). We may therefore suppose that exactly \( n - r \) lines are concurrent, where \( 1 \leq r \leq k - 1 \). By Lemma 4.1, \( f_2 \leq g(r) \), where \( g(r) = (r + 1)(n - r) + \binom{r}{2} \),
and $g'(r) = n - r - \frac{1}{2} \geq n - k - \frac{1}{2} > 0$. Hence $f_2 \leq g(k - 1) = k(n - k + 1) + \binom{k-1}{2}$ contrary to (10), and the theorem is proved.

**Corollary 4.3.** Putting $k = 3$, we see that if $f_2 > 3n - 5$, then $f_2 \geq 4n - 12$, proving for $n \geq 40$ (see [3, Conjecture 2.4]). The conjecture is still open for $9 \leq n \leq 39$.

**Remark 4.9.** The theorem is trivially true if $n \leq \frac{1}{2}(k^2 + k + 2)$.

**Note added in proof**

The author has recently shown that $p_3 \leq \frac{7}{18} n(n - 1) + \frac{1}{3}$ for $n \geq 6$.

**References**