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SEMIGROUPS OF TOLERANCE RELATIONS

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We give an abstract algebraic characterization of semigroups of tolerance relations and semigroups of symmetric binary relations.

1. Introduction

A binary relation ρ between elements of a set A is called *symmetric* if $(a_1, a_2) \in \rho \Rightarrow (a_2, a_1) \in \rho$ for all $a_1, a_2 \in A$. It is called *partly reflexive* if

$$(a_1, a_2) \in \rho \Rightarrow (a_1, a_1) \in \rho \quad \text{and} \quad (a_2, a_2) \in \rho,$$

and *reflexive* if $(a, a) \in \rho$ for all $a_1, a_2, a \in A$. It is called a *partial tolerance relation* if it is both symmetric and partly reflexive; it is called a *tolerance relation* if it is symmetric and reflexive. Of course, if a binary relation ρ is represented by a (finite or infinite) square Boolean matrix consisting of 0 and 1 entries, then ρ is symmetric if and only if the matrix corresponding to ρ is symmetric, and ρ is reflexive if and only if the matrix has the main diagonal consisting of 1's.

A symmetric binary relation can be called a *graph,* in which case partial tolerance relations are precisely the graphs which have a loop at each non-isolated vertex. If ρ and σ are binary relations on A, then their product $\rho\sigma$ is defined as usual:

$$(a_1, a_2) \in \rho\sigma \Leftrightarrow (\exists a)[(a_1, a) \in \rho \quad \text{and} \quad (a, a_2) \in \sigma].$$

Clearly, $M_{\rho\sigma} = M_\rho M_\sigma$, where M_ρ, M_σ and $M_{\rho\sigma}$ are the 0, 1-matrices corresponding to ρ, σ, and $\rho\sigma$, and the product of matrices is the Boolean one (i.e., $1 + 1 = 1$). If ρ is a binary relation, then $\rho^{-1} = \{(a_2, a_1)\colon (a_1, a_2) \in \rho\}$ denotes the *converse* of ρ. For example, ρ is symmetric precisely when $\rho^{-1} = \rho$. While a product of reflexive relations is always reflexive, the product of two partly reflexive, or symmetric, or tolerance relations need not have the same property. However, a set of symmetric relations (or relations of another type) may be closed under multiplication, in which case it forms a semigroup of symmetric relations. We find algebraic conditions characterizing such semigroups, i.e., necessary and sufficient conditions under which a semigroup is isomorphic to a semigroup of symmetric binary relations, or to a semigroup of (partial) tolerance relations. We solve these problems also when the symmetric or tolerance relations are between elements of some algebras (say, groups) and they are compatible with the operations in the algebras.

Tolerance relations have attracted considerable attention in recent years (the corresponding publications are too numerous to be mentioned here), and they have various applications. Therefore, consideration of semigroups of tolerance relations may be interesting. Some of our results (in a weaker form) were announced without proof in [1].

Semigroups isomorphic to semigroups of reflexive binary relations, were characterized in [2], semigroups isomorphic to semigroups of partly reflexive binary relations, in [3].

2. Semigroups of symmetric binary relations

The following result gives an abstract characterization of semigroups of symmetric binary relations.

Theorem 1. *A semigroup S is isomorphic to a semigroup of symmetric binary relations if and only if*

(1) *S is commutative;*

(2) *S satisfies the condition $x = xy^2z^2 \Rightarrow x = xy^2$ for all $x, y, z \in S$.*

Proof. *Necessity.* Without loss of generality we may assume that S is a semigroup of symmetric binary relations on a set A. It is known (and trivial) that $(\rho\sigma)^{-1} = \sigma^{-1}\rho^{-1}$ for two relations ρ and σ. If $s, t \in S$, then $st \in S$, so that $st = (st)^{-1} = t^{-1}s^{-1} = ts$. Thus, S is commutative.

To check (2) assume that $x = xy^2z^2$ for some $x, y, z \in S$. Because of commutativity, $x = xz^2y^2$. Suppose that $(a_1, a_2) \in x$ for some $a_1, a_2 \in A$. Then $(a_1, a_2) \in xz^2y^2$. It follows that $(a_3, a_2) \in y$ for some $a_3 \in A$. Since y is symmetric, $(a_2, a_3) \in y$, so that $(a_2, a_2) \in y^2$. It follows that $(a_1, a_2) \in xy^2$.

Conversely, suppose that $(a_1, a_2) \in xy^2$. Then $(a_1, a_4) \in x$ and $(a_4, a_2) \in y^2$ for some $a_4 \in A$. Since $x = xy^2z^2$, it follows that $(a_1, a_4) \in xy^2z^2$, hence $(a_5, a_4) \in z$ for some $a_5 \in A$. By symmetry of z, $(a_4, a_5) \in z$, whence $(a_4, a_4) \in z^2$ and $(a_1, a_4) \in xz^2$. Therefore, $(a_1, a_2) \in xz^2y^2 = x$.

We have proved that $(a_1, a_2) \in x \Leftrightarrow (a_1, a_2) \in xy^2$ for arbitrary $a_1, a_2 \in A$. Thus, $x = xy^2$ and (2) holds.

Sufficiency. Suppose that S satisfies both (1) and (2). If S has an identity element put $S^1 = S$. If S has none, add a new element e to S defining it to be the identity element of $S^1 = S \cup \{e\}$. S^1 is clearly commutative, and S^1 satisfies (2) for $x, y, z \in S$. If $y = e$ or $z = e$, (2) becomes trivial. If $e = x \notin S$, then $e = y^2z^2$. It follows that $y = z = e$ and (2) holds again. Clearly, it suffices to find a representation of S^1 by symmetric binary relations. This representation can be always restricted to S.

For $x, y \in S^1$ define $x \prec y$ whenever there exists $z \in S^1$ such that $y = xz^2$. The following lemma gives some properties of this relation.

Lemma 1. *The relation $\prec$ is a (partial) order relation compatible with the multiplication in S^1. Also $x \prec xy^2$ for all $x, y \in S^1$.*

Proof. Since $x = xe^2$, we see that $x \prec x$ for all $x \in S^1$, i.e., $\prec$ is reflexive. Suppose that $x \prec y$ and $y \prec z$ for some $x, y, z \in S^1$. Then $y = xu^2$ and $z = yv^2$ for some $u, v \in S^1$. It follows that $z = xu^2v^2 = x(uv)^2$, hence $x \prec z$ and $\prec$ is transitive. Here we used the identity $u^2v^2 = (uv)^2$ which follows from commutativity.

Now, if $x \prec y$ and $y \prec x$ for some $x, y \in S^1$, then $y = xu^2$ and $x = yv^2$ for some $u, v \in S^1$. It follows that $x = xu^2v^2$. Applying (2) we obtain $x = xu^2 = y$, i.e., $\prec$ is antisymmetric. Thus, $\prec$ is an order relation on S^1.

Suppose that $x_i \prec y_i$ for some $x_i, y_i \in S^1$, $i = 1, 2$. Then $y_i = x_iz_i^2$ for some $z_i \in S^1$. It follows that $y_1y_2 = x_1z_1^2x_2z_2^2 = x_1x_2(z_1z_2)^2$, hence $x_1x_2 \prec y_1y_2$, i.e., $\prec$ is compatible with multiplication. The inequality $x \prec xy^2$ follows from the definition of $\prec$. Lemma 1 is proved. □

We say that a subset $T \subset S^1$ is *majorant* if $x \in T$ and $x \prec y$ imply $y \in T$ for all $x, y \in S^1$. Here and elsewhere the inclusion $\subset$ is reflexive, i.e., $T \subset T$. Majorant subsets have been considered in the literature under various names ("up-set dual order ideals" is one of them).

Let M denote the set of all majorant subsets of S^1. For any subset $T \subset S^1$ let $\bar{T} = \{y \in S^1: (\exists x)[x \in T \text{ and } x \prec y]\}$. Clearly, $T \subset \bar{T}$ and $\bar{T} \in M$. For every $s \in S$ define a binary relation $\rho_s \in M \times M$:

$$(A, B) \in \rho_s \Leftrightarrow As \subset B \quad \text{and} \quad Bs \subset A$$

for all $A, B \in M$. Here $As = \{as: a \in A\}$. Clearly, ρ_s is a symmetric binary relation.

Next we prove that $s \mapsto \rho_s$ is an isomorphism of S onto a semigroup of symmetric binary relations.

Indeed, if $(A, B) \in \rho_s\rho_t$ for some $A, B \in M$ and $s, t \in S$, then $(A, C) \in \rho_s$ and $(C, B) \in \rho_t$ for some $C \in M$. It follows that $As \subset C$, $Cs \subset A$, $Ct \subset B$, and $Bt \subset C$. Therefore, $Ast \subset Ct \subset B$ and $Bst = Bts \subset Cs \subset A$, hence $(A, B) \in \rho_{st}$. Thus, $\rho_s\rho_t \subset \rho_{st}$.

Conversely, suppose that $(A, B) \in \rho_{st}$ for some $A, B \in M$. Let $C = \overline{As \cup Bt}$. Then $C \in M$, $As \subset C$ and $Bt \subset C$. To prove that $Cs \subset A$ and $Ct \subset B$, assume that $y \in C$. Then $x \prec y$ for some $x \in As \cup Bt$. By symmetry we may assume that $x \in As$, then $x = as$ for some $a \in A$. Therefore, $a \prec as^2 = xs \prec ys$. Since A is majorant, $ys \in A$. Also $xt = ast \in Ast \subset B$. Since $xt \prec yt$ and B is majorant, we obtain $yt \in B$.

It follows that $Cs \subset A$ and $Ct \subset B$, i.e., $(A, C) \in \rho_s$ and $(C, B) \in \rho_t$. Therefore, $(A, B) \in \rho_s\rho_t$ which proves that $\rho_{st} \subset \rho_s\rho_t$. Thus, $\rho_s\rho_t = \rho_{st}$, i.e., the mapping $s \mapsto \rho_s$ is a homomorphism.

Next suppose that $\rho_s = \rho_t$. We write $\bar{s}$ for $\overline{\{s\}}$. We prove that $(\bar{e}, \bar{s}) \in \rho_s$. Let $x \in \bar{e}$. Then $e < x$ and $s < es < xs$. Therefore, $xs \in \bar{s}$, i.e., $\bar{e}s \subset \bar{s}$. Now, if $y \in \bar{s}$, then $s < y$ and $e < es^2 = s^2 < ys$, so that $ys \in \bar{e}$ and $\bar{s}s \subset \bar{e}$. Therefore, $(\bar{e}, \bar{s}) \in \rho_s$. It follows that $(\bar{e}, \bar{s}) \in \rho_t$, hence $t = et \in \bar{e}t \subset \bar{s}$. Thus, $s < t$. By symmetry $t < s$. Therefore $s = t$ and $s \mapsto \rho_s$ is an isomorphism of S onto the semigroup $\{\rho_s : s \in S\}$ of symmetric binary relations on the set M. □

In fact, we have proved more than promised. A binary relation $\rho \subset A \times A$ is called a *multipermutation* if, for every $a \in A$, there exist $a_1, a_2 \in A$ such that $(a_1, a) \in \rho$ and $(a, a_2) \in \rho$. All the relations ρ_s are multipermutations of M. Indeed, for every $A \in M$ and $s \in S$, we have $As \subset \overline{As}$ and $(\overline{As})s \subset A$. The second inclusion follows from Lemma 1. Thus, $(A, \overline{As}) \in \rho_s$. Since ρ_s is symmetric, $(\overline{As}, A) \in \rho_s$; therefore, ρ_s is a multipermutation.

A binary relation ρ between the elements of a universal algebra A is called *stable* if ρ is a (possibly, empty) subalgebra of $A \times A$. Stable binary relations are sometimes called compatible, invariant, preserved by the operations, etc. For example, if A is a group (considered as an algebra with two operations: a binary multiplication and a unary inversion), then $\rho \subset A \times A$ is stable if and only if $(a_1, a_2), (a_3, a_4) \in \rho \Rightarrow (a_1a_3, a_2a_4) \in \rho$ and $(a_1, a_2) \in \rho \Rightarrow (a_1^{-1}, a_2^{-1}) \in \rho$. If A is a vector space, then ρ is stable when it is a subspace of $A \times A$.

It is easily seen that the set M introduced in the proof of sufficiency of Theorem 1 is a complete distributive lattice under the set-theoretical operations. Our relations $\rho_s \subset M \times M$ are stable (in fact, they are complete sublattices of $M \times M$). We omit an easy proof of this statement. It follows from the obvious properties:

$$A \subset B \Rightarrow As \subset Bs \quad \text{and} \quad (A \cup B)s = As \cup Bs,$$

which hold for any $A, B \subset S^1$ and $s \in S$.

Thus, we arrive at

Corollary 1. *If a semigroup S is isomorphic to a semigroup of symmetric binary relations, it is also isomorphic to a semigroup of stable and symmetric multipermutations of a distributive lattice. If S is finite, the distributive lattice can be chosen to be finite.*

Since every lattice is a semilattice with respect to any one of its operations, and any semilattice is a (commutative) semigroup, we can replace "distributive lattice" in Corollary 1 either by "semigroup" or by "commutative semigroup".

Corollary 2. *A semigroup S is isomorphic to a semigroup of symmetric binary relations if and only if S is commutative and there exists an order relation $\leqslant$ on S*

such that $s \leqslant st^2$ *for all* $s, t \in S$ *(i.e., all squares* t^2 *are 'positive elements' of S with respect to* $\leqslant$*).*

Proof. *Necessity*. Suppose that S is isomorphic to a semigroup of symmetric binary relations. Then S satisfies the conditions (1) and (2) of Theorem 1 and, by Lemma 1, the relation $<$ on S satisfies the conditions of Corollary 2.

Sufficiency. Supose that S is commutative and has an ordering $\leqslant$ such that $s \leqslant st^2$ for all $s, t \in S$. Suppose that $x = xy^2z^2$ for some $x, y, z \in S$. Then $x \leqslant xy^2 \leqslant xy^2z^2 = x$, hence $x = xy^2$. Thus, S satisfies conditions (1) and (2) of Theorem 1. □

Remark. We could require the order relation $\leqslant$ in Corollary 2 to be stable.

If S is a semigroup of binary relations, then it is ordered by the set-theoretical inclusion $\subset$. Clearly, $\subset$ is stable on S. If S consists of symmetric multipermutations then it is easy to check that $s \subset st^2$ for all $s, t \in S$, i.e. that $\subset$ satisfies the conditions for $\leqslant$ stated in Corollary 2. However, if S consists of symmetric relations which are not necessarily multipermutations, than $s \subset st^2$ may not hold. Nevertheless, by Corollary 1, S is always isomorphic to a semigroup T of symmetric multipermutations. The inclusion relation on T does satisfy the conditions for $\leqslant$ of Corollary 2.

As we have mentioned in Corollary 1, if a semigroup S is representable by symmetric binary relations, the relations can be chosen to be stable multipermutations on lattices. An immediate question is: when can we choose stable symmetric relations or stable multipermutations on structures of such importance as groups, rings, modules, vector spaces, or Boolean algebras? It turns out that S must satisfy an additional condition.

Theorem 2. *The following conditions are equivalent for any semigroup S: S is isomorphic to a semigroup of stable symmetric binary relations on*

(A) *a Boolean algebra;* (B) *a ring;* (C) *a group;*
(D) *a vector space;* (E) *a module.*

The stable symmetric binary relations on these structures can be chosen to be multipermutations and, if S is finite, the structures (A)–(E) *can be chosen to be finite.*

A semigroup S satisfies any (or all) of the equivalent conditions (A)–(E) *if and only if it satisfies the following conditions* (1) *and* (3)*:*

$$S \text{ is commutative,} \tag{1}$$

$$x = x^3 \text{ for every } x \in S. \tag{3}$$

Proof. Every Boolean algebra can be considered as a (Boolean) ring, and every ring is an abelian group with respect to addition. Thus, the implications (A)$\Rightarrow$(B)$\Rightarrow$(C) are obvious. Every Boolean algebra can be considered as a vector space (take the two element field GF(2) for scalars, and define an obvious multiplication of scalars and vectors). Every vector space is a module and every module is an (additive) abelian group. Thus, the implications (A)$\Rightarrow$(D)$\Rightarrow$(E)$\Rightarrow$(C) are obvious. The implication (C)$\Rightarrow$(1) follows from Theorem 1.

(C)$\Rightarrow$(3). An easy argument belonging to the folklore shows that every stable binary relation ρ on a group is difunctional, i.e., $\rho\rho^{-1}\rho = \rho$. Indeed, $\rho \subset \rho\rho^{-1}\rho$ holds for every relation ρ: if $(g, h) \in \rho$, then $(g, h) \in \rho$, $(h, g) \in \rho^{-1}$, and $(g, h) \in \rho$ which implies that $(g, h) \in \rho\rho^{-1}\rho$. Conversely, supose that $(g, h) \in \rho\rho^{-1}\rho$, i.e., $(g, u) \in \rho$, $(u, v) \in \rho^{-1}$, and $(v, h) \in \rho$ for some $u, v \in G$. Then $(v, u) \in \rho$, hence $(v^{-1}, u^{-1}) \in \rho$ and $(g, h) = (gv^{-1}v, uu^{-1}h) \in \rho$, i.e., $\rho\rho^{-1}\rho \subset \rho$. If ρ is symmetric, then $\rho^{-1} = \rho$ and the difunctionality of ρ means that $\rho^3 = \rho$. Thus, (C) implies (3).

Now suppose that S is a semigroup which satisfies (1) and (3). Note that conditions (1) and (3) imply condition (2) of Theorem 1. Indeed, if S satisfies (1) and (3) and if $xy^2z^2 = x$ for some $x, y, z \in S$, then $xy^2 = xy^2z^2y^2 = xy^2z^2 = x$ because, by (3), $y^4 = y^2$. For every $s \in S$ and every subset A of S define $A_s = \{a \in A: as^2 = a\}$. Let $\mathcal{P}(S)$ denote the set of all subsets of S. For every $A \in \mathcal{P}(S)$ we obtain $(As)_s = As$. Indeed, $(As)_s \subset As$. Conversely, if $a \in A$, then $as \cdot s^2 = as^3 = as$ so that $as \in (As)_s$ which implies $As \subset (As)_s$. Also, $A \subset B \Rightarrow A_s \subset B_s$ for all $A, B \in \mathcal{P}(S)$ and $s \in S$. Therefore, if $As \subset B$, then $As = (As)_s \subset B_s$.

For every $s \in S$ define a binary relation σ_s on $\mathcal{P}(S)$ as follows:

$$(A, B) \in \sigma_s \Leftrightarrow A_s s = B_s \quad \text{for all } A, B \in \mathcal{P}(S).$$

We find some properties of these relations.

(i) σ_s is a symmetric binary relation.

Indeed, if $(A, B) \in \sigma_s$, then $A_s s = B_s$. Multiplying this equality by s we obtain $A_s s^2 = B_s s$. However, $A_s s^2 = A_s$ by the definition of A_s. Therefore $B_s s = A_s$, i.e., $(B, A) \in \sigma_s$.

(ii) σ_s is a multipermutation of $\mathcal{P}(S)$.

By (i), it suffices to prove that, for every $A \in \mathcal{P}(S)$, there exists $B \in \mathcal{P}(S)$ such that $(B, A) \in \sigma_s$. Indeed, $(A_s s)_s s = (A_s s)s = A_s s^2 = A_s$, so that $(A_s s, A) \in \sigma_s$.

(iii) σ_s is stable on the Boolean algebra $\mathcal{P}(S)$.

We shall prove even more: σ_s is a *complete* Boolean subalgebra of $\mathcal{P}(S) \times \mathcal{P}(S)$.

Suppose that $(A_i, B_i) \in \sigma_s$ for all i from some index set I. Then $(A_i)_s s = (B_i)_s$. Obviously, $(\bigcup A_i)_s = \bigcup (A_i)_s$. Therefore,

$$(\bigcup A_i)_s s = [\bigcup (A_i)_s]s = \bigcup (A_i)_s s = \bigcup (B_i)_s = (\bigcup B_i)_s,$$

hence $(\bigcup A_i, \bigcup B_i) \in \sigma_s$.

Now suppose that $(A, B) \in \sigma_s$, i.e., $A_s s = B_s$. Assume that $a \notin A$ and $as^2 = a$, i.e., that $a \in (A')_s$, where A' is the complement of A in S. If $as \in B$, then $as \in B_s$

and $a = as^2 \in B_s s = A_s s \cdot s = A_s s^2 = A_s \subset A$, i.e., $a \in A$ contrary to our assumption. Thus, $(A')_s s \subset B'$ an so $(A')_s s \subset (B')_s$. Analogously, $(B')_s s \subset (A')_s$. Multiplying the last inclusion by s we obtain $(B')_s = (B')_s s^2 \subset (A')_s s$. Thus, $(A')_s s = (B')_s$, i.e., $(A', B') \in \sigma_s$. It follows that σ_s is a complete Boolean subalgebra of $\mathcal{P}(S) \times \mathcal{P}(S)$.

(iv) $\sigma_x \sigma_y = \sigma_{xy}$ for all $x, y \in S$.

Suppose that $(A, B) \in \sigma_x \sigma_y$, i.e., $(A, C) \in \sigma_x$ and $(C, B) \in \sigma_y$ for some $C \in \mathcal{P}(S)$. Then $A_x x = C_x$ and $C_y y = B_y$. If $a \in A_{xy}$, i.e., if $a(xy)^2 = a$, then $ax^2 = a$ by (2). Therefore $a \in A_x$, i.e., $A_{xy} \subset A_x$. Because of commutativity of multiplication, $A_{xy} \subset A_y$. Let $a \in A_{xy}$. Then $a \in A_x$, hence $ax \in A_x x \subset C$. Now, $axy^2 = ay^2x = ax$, because $a \in A_y$. Thus, $ax \in C_y$. It follows that $axy \in C_y y \subset B$, whence $axy \in B_{xy}$. Thus, $A_{xy} xy \subset B_{xy}$. If $b \in B_{xy}$, then $b \in B_y = C_y y$, hence $b = cy$ for some $x \in C_y$. Therefore, $by = cy^2 = c \in C$. Since $by \cdot x^2 = bx^2 y = by$ follows from $B_{xy} \subset B_x$, we see that $by \in C_x = A_x x$, so that $by = ax$ for some $a \in A_x$. Therefore, $bxy = byx = ax^2 = a \in A$, whence $bxy \in A_{xy}$. It follows that $b = b(xy)^2 \in A_{xy} xy$. Thus, $A_{xy} xy = B_{xy}$, i.e., $(A, B) \in \sigma_{xy}$. We have proved that $\sigma_x \sigma_y \subset \sigma_{xy}$.

Conversely, suppose that $(A, B) \in \sigma_{xy}$, i.e., $A_{xy} xy = B_{xy}$. Let $C = A_x x \cup B_y y$. Then $A_x x \subset C$, therefore, $A_x x \subset C_x$. If $c \in C_x$, then either $c \in A_x x$ or $c \in B_y y$. In the latter case $c = by$ for some $b \in B_y$. Therefore, $b(xy)^2 = byx^2 y = cx^2 y = cy = by^2 = b$, i.e., $b \in B_{xy} = A_{xy} xy$. Thus, $b = axy$ for some $a \in A_{xy} \subset A_y$. It follows that $c = by = axy^2 = ay^2 x = ax \in A_{xy} x \subset A_x x$. Thus, $C_x = A_x x$. Analogously, $C_y = B_y y$, whence $C_y y = B_y y^2 = B_y$. Therefore, $(A, C) \in \sigma_x$ and $(C, B) \in \sigma_y$ which implies $(A, B) \in \sigma_x \sigma_y$. So $\sigma_{xy} \subset \sigma_x \sigma_y$.

(v) $\sigma_x = \sigma_y \Rightarrow x = y$.

Let $\sigma_x = \sigma_y$. As we have seen in the proof of (ii), $(A_s s, A) \in \sigma_s$ for every $A \in \mathcal{P}(S)$. Thus, $(S_x x, S) \in \sigma_x$. Obviously, $S_x x \subset S$, so that $S_x x \subset S_x$. On the other hand, using $S_x \subset S$ and $Sx \subset S$, we obtain that $S_x = S_x x^2 \subset Sx^2 = Sx \cdot x = (Sx)_x x \subset S_x x$. Therefore, $S_x x = S_x$. It follows that $(S_x, S) \in \sigma_x$, whence $(S_x, S) \in \sigma_y$. By the symmetry of σ_y we have $(S, S_x) \in \sigma_y$, i.e., $S_y y = (S_x)_y$. However, $S_y y = S_y$ as we have just seen. Also $(S_x)_y \subset S_x$. Therefore $S_y \subset S_x$. Obviously, $y \in S_y$. Thus, $y \in S_x$. Therefore, $yx^2 = y$, i.e., $\{y\}_x = \{y\}$. Also $\{y\}_y = \{y\}$ and $\{y^2\}_y = \{y^2\}$, so that $\{y\}_y y = \{y\} y = \{y^2\} = \{y^2\}_y$ and $(\{y\}, \{y^2\}) \in \sigma_y$. It follows that $(\{y\}, \{y^2\}) \in \sigma_x$, i.e., $\{y\}_x x = \{y^2\}_x$. However, $\{y\}_x = \{y\}$, so that $\{y\}_x x = \{y\} x = \{yx\}$. Thus, $\{yx\} = \{y^2\}_x \subset \{y^2\}$, i.e., $yx = y^2$. Interchanging x and y, we can obtain $xy = x^2$ in an analogous way. Therefore, $x = x \cdot x^2 = x^2 y = yx^2 = y$.

Properties (i)–(v) show that the mapping $s \mapsto \sigma_s$ is an isomporphism of S onto the semigroup $\{\sigma_x : x \in S\}$ of stable symmetric multipermutations of the Boolean algebra $\mathcal{P}(S)$, i.e., (A) holds. It remains to notice that if S is finite, then $\mathcal{P}(S)$ is finite. Theorem 2 is proved. □

Remark. The readers familiar with the rudiments of the theory of semigroups would instantly recognize that semigroups satisfying (1) and (3) are precisely semilattices of groups of exponent 2. Also they are precisely subdirect products of sufficiently many copies of the two-element group and the same group with an adjoined zero. This fact can yield an alternative proof of Theorem 2.

The semigroups satisfying (1) and (2) are precisely the commutative semigroups for which every Schützenberger group is of exponent 2 (i.e., it satisfies the identity $x^2 = 1$). Since this paper is not aimed at semigroup theorists, here is a short definition of Schützenberger groups of a semigroup S. Elements $s, t \in S$ are *$\mathcal{R}$-related* if $sx = t$ and $ty = s$ for some $x, y \in S^1$ (i.e., when s and t divide each other on the left, or equivalently, when s and t generate the same principal right ideals of S). Dually, one defines *$\mathcal{L}$-related* elements. Elements are *$\mathcal{H}$-related* if they are both $\mathcal{R}$- and $\mathcal{L}$-related. Let H be an equivalence class modulo $\mathcal{H}$. For every $t \in S$ one of the two possibilities holds: either $Ht = H$ or $Ht \cap H = \emptyset$. In the former case the mapping $h \mapsto ht$, $h \in H$, is a permutation of H. The set of all such permutations forms a simply transitive group $\Gamma(H)$ of transformations of H. This group is called the *Schützenberger group* of H. Clearly, $|\Gamma(H)| = |H|$ and if H is a subgroup of S (which is true exactly when H contains an idempotent), then $\Gamma(H)$ is isomorphic to H.

3. Semigroups of tolerance relations

Theorem 3. *The following conditions are equivalent for every semigroup S:*

(A) *S is isomorphic to a semigroup of tolerance relations,*

(B) *S is isomorphic to a semigroup of partial tolerance relations,*

(C) *S is commutative and satisfies the condition*

$$x = xyz \Rightarrow x = xy \quad \textit{for all } x, y, z \in S. \tag{4}$$

Proof. (A)$\Rightarrow$(B) is obvious.

(B)$\Rightarrow$(C). Without loss of generality we may assume that S is a semigroup of partial tolerance relations between elements of a set A. By Theorem 1 S is commutative. Let $x = xyz$ for some $x, y, z \in S$. If $(a_1, a_2) \in x$ for some $a_1, a_2 \in A$, then $(a_1, a_2) \in xzy$, and so $(a, a_2) \in y$ for some $a \in A$. Since y is partly reflexive, $(a_2, a_2) \in y$, whence $(a_1, a_2) \in xy$. It shows that $x \subset xy$.

Now suppose that $(a_1, a_2) \in xy$ for some $a_1, a_2 \in A$. Then $(a_1, a) \in x$ and $(a, a_2) \in y$ for some $a \in A$. Thus, $(a_1, a) \in xyz$. It follows that $(a_3, a) \in z$ for some $a_3 \in A$. Since z is partly reflexive, $(a, a) \in z$. Thus, $(a_1, a) \in x$, $(a, a) \in z$, and $(a, a_2) \in y$, hence $(a_1, a_2) \in xzy = x$ and $xy \subset x$. Therefore, $x = xy$ and (4) holds. Note that in the proof of (4) we used commutativity of S and reflexivity of ρ_s, $s \in S$. Thus, (4) holds for commutative semigroups of reflexive binary relations.

(C)$\Rightarrow$(A). Let S be a commutative semigroup which satisfies (4). In all essential features the proof which follows is similar to our proof of sufficiency in Theorem 1, so we give it here in less detail.

It is easy to see that the semigroup S^1 is commutative and satisfies (4). For $x, y \in S^1$ define $x \mid y$ if $y = xz$ for some $z \in S^1$. Thus, '$|$' is the divisibility relation. As in Lemma 1, we may prove that '$|$' is a (partial) order compatible with multiplication in S^1. Obviously, '$|$' is positive, i.e., $x \mid xy$ for any $x, y \in S^1$. Condition (4) means that '$|$' is antisymmetric, i.e., $x \mid y$ and $y \mid x$ imply that $x = y$.

Next we introduce the set M of all majorant subsets of S^1 with respect to '$|$', precisely as we did in the proof of Theorem 1 with respect to $<$. In fact, M is the set of all (including the empty one) ideals of S^1. For every $s \in S$ consider a binary relation ρ_s as defined in the proof of Theorem 1. This time ρ_s is not only symmetric but also reflexive. Indeed, if $A \in M$ then $a \mid as$ for every $a \in A$. Therefore, $as \in A$, i.e., $As \subset A$ and $(A, A) \in \rho_s$. Thus, ρ_s is a tolerance relation on M.

As in the previous proof, we check that $\rho_s\rho_t = \rho_{st}$ for any $s, t \in S$. Let $\rho_s = \rho_t$. For $a \in S$ put $\bar{a} = Sa$. Now $(\bar{e}, \bar{s}) \in \rho_s$, therefore, $(\bar{e}, \bar{s}) \in \rho_t$ and $\bar{e}t \subset \bar{s}$. It follows that $t \in \bar{s}$, i.e., $s \mid t$. Analogously, $t \mid s$, so that by (4) $s = t$.

Thus, $s \mapsto \rho_s$ is an isomorphism of S onto a semigroup of tolerance relations on A. □

Remark. A commutative semigroup satisfies (4) precisely when all of its Schützenberger groups are trivial. Analogously to Corollary 1 to Theorem 1 we obtain now

Corollary 1. *If a semigroup S is isomorphic to a semigroup of tolerance relations, it is also isomorphic to a semigroup of stable tolerance relations on a distributive lattice. If S is finite, the lattice can be chosen to be finite.*

The words "distributive lattice" can be replaced by "semigroup" or "commutative semigroup".

Corollary 2. *A semigroup is isomorphic to a semigroup of tolerance relations if and only if it is commutative and isomorphic to a semigroup of reflexive binary relations.*

Proof. The "only if" part is trivial, and for the "if" part see our proof of (B) ⇒ (C) in Theorem 3. □

Remark. It was proved in [2] that a commutative semigroup is isomorphic to a semigroup of reflexive binary relations if and only if it satisfies (4).

Again, we can ask which semigroups are representable as semigroups of stable tolerance relations on structures listed in Theorem 2. The answer is very simple: the semigroups are semilattices (i.e., commutative and idempotent semigroups). Indeed, if ρ is a stable tolerance relation on, say a group, then $\rho = \rho^3$. By (4), it follows that $\rho = \rho^2$, i.e., ρ is an equivalence relation. Since ρ is stable, it is a congruence relation on the group. The fact that stable tolerance relations on groups are congruences is both trivial and known. Of course, congruences of groups (Boolean algebras, rings, vector spaces, modules) are in one-to-one correspondence with normal subgroups (ideals of Boolean algebras or rings, subspaces of vector spaces, submodules of modules). Therefore, every semilattice can be embedded either in the semigroup of all congruences on these structures or in the (join-) semilattice of all normal subgroups, ideals, subspaces, submodules. We do not state this result formally because it is very simple, its proof is a greatly simplified version of our proof of Theorem 2. It is easy to prove that if a tolerance relation is stable with respect to a group multiplication, then it is also stable with respect to this group inversion operation.

References

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