New nonoscillation criteria for delay differential equations

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Abstract
This paper is concerned with the nonoscillation of first order delay differential equations of the form
\[ x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \]
where \( p, \tau \in C([t_0, \infty), \mathbb{R}^+) \), \( \mathbb{R}^+ = [0, \infty) \), \( \tau(t) \) is nondecreasing, \( \tau(t) < t \) for \( t \geq t_0 \) and \( \lim_{t \to \infty} \tau(t) = \infty \). New nonoscillation conditions are obtained. These conditions concern the case when the well-known nonoscillation condition \( \int_{\tau(t)}^{t} p(s) \, ds \leq 1/e \) is not satisfied.

Keywords: Oscillation; Nonoscillation; Delay differential equation

1. Introduction
The problem of establishing sufficient conditions for the oscillation of all solutions and the nonoscillation (i.e., the existence of a nonoscillatory solution) of the delay differential equation
\[ x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1.1} \]

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where \( p, \tau \in C([t_0, \infty), R^+) \), \( R^+ = [0, \infty) \), \( \tau(t) < t \) for \( t \geq t_0 \) and \( \lim_{t \to \infty} \tau(t) = \infty \), has been the subject of many investigations. See, for example, [1–11,13–28,30,31] and references cited therein.

By a solution of (1.1) we understand a continuously differentiable function defined on \( [\tau(t^*), \infty) \) for some \( t^* \geq t_0 \) and such that (1.1) is satisfied for \( t \geq t^* \). Such a solution is called oscillatory if it has arbitrarily large zeros, and otherwise it is called nonoscillatory. For Eq. (1.1), the existence of a nonoscillatory solution is equivalent to the existence of an eventually positive solution.

It is well known that in 1972 Ladas et al. [21] proved that all solutions of (1.1) oscillate if

\[
\limsup_{t \to \infty} \int_{\tau(t)}^{t} p(s) \, ds > 1.
\] (1.2)

While in 1979 Ladas [20] and in 1982 Koplatadze and Chanturija [16] proved that the same conclusion holds if

\[
\liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s) \, ds > \frac{1}{e}.
\] (1.3)

Concerning the constant \( 1/e \) in (1.3), Koplatadze and Chanturija [16] also proved that if the inequality

\[
\int_{\tau(t)}^{t} p(s) \, ds \leq \frac{1}{e}
\] (1.4)

holds eventually, then (1.1) has a nonoscillatory solution.

The condition (1.3) and (1.4) are sharp for the oscillation and the nonoscillation of (1.1) respectively in the sense that when \( p(t) \equiv p > 0 \) and \( \tau(t) = t - \tau \), \( \tau > 0 \), the condition (1.4) reduces to \( p \tau e \leq 1 \) which is a necessary and sufficient condition for the existence of a nonoscillatory solution of (1.1).

In the last 10 years, there are many papers which are devoted to the improvement of conditions (1.2) and (1.3), for example, see [2,5,7–10,13–15,17–19,24,25,27,28,30,31]. It should be mentioned that according to Theorem 2.1.4 and Remark 2.1.5 in [9] the condition (1.4) is not necessary for the existence of nonoscillatory solutions of (1.1). However, it seems that there exist few results which concern the existence of nonoscillatory solutions of (1.1) in the case when (1.4) is not satisfied. It should be emphasized that such results are important for both the study of nonoscillation of (1.1) and the further improvement of oscillation conditions (1.2) and (1.3).

The purpose of this paper is to derive the results on the existence of a nonoscillatory solution of (1.1) when condition (1.4) is not satisfied. We also indicate the direction for possible further study on the improvement of the oscillation conditions (1.2) and (1.3).
2. Main results

We first derive certain nonoscillation results for a simple form of Eq. (1.1) which are also interesting in their rights. We need the following lemmas.

Lemma 2.1 [31]. Equation (1.1) has an eventually positive solution if and only if the corresponding differential inequality

\[ x'(t) + p(t)x(\tau(t)) \leq 0 \]

has an eventually positive solution.

Lemma 2.2 [12,29]. Let \( p(t) \in C([t_0, \infty), \mathbb{R}^+) \) and consider the second order ordinary differential equation

\[ y''(t) + p(t)y(t) = 0, \quad t \geq t_0. \] (*)

Then Eq. (*) has an eventually positive solution if for large \( t \)

\[ \int_{\infty}^{t} p(s) \, ds \leq \frac{1}{4}. \] (**) 

Theorem 2.1. Consider a special equation of (1.1) of the form

\[ x'(t) + x(\tau(t)) = 0, \quad t \geq t_0, \] (2.1)

where

\[ t - \tau(t) \geq 1/e. \] (2.2)

Assume that

\[ \lim_{t \to \infty} \left( t - \tau(t) - \frac{1}{e} \right) e^{e(t-\tau(t))} < a < 1, \] (2.3)

and that the second order ordinary differential equation

\[ y''(t) + \frac{2}{1-a} e^{2 + e(t-\tau(t))} \left( t - \tau(t) - \frac{1}{e} \right) y(t) = 0, \quad t \geq t_0, \] (2.4)

has an eventually positive solution. Then (2.1) has also an eventually positive solution.

Proof. Let \( y(t) \) be an eventually positive solution of (2.4). From (2.2)–(2.4), we see that there exists a \( T \geq t_0 \) such that

\[
\begin{cases}
0 \leq \left( t - \tau(t) - \frac{1}{e} \right) e^{e(t-\tau(t))} < a, \\
y(t) > 0, \quad y'(t) > 0, \quad y''(t) \leq 0, \quad t \geq T.
\end{cases}
\] (2.5)

Set \( u(t) = y'(t) \). Then \( u(t) > 0 \) and \( u'(t) \leq 0 \) for \( t \geq T \), and

\[ y(t) = \int_{T}^{t} u(s) \, ds + y(T), \quad t \geq T. \] (2.6)
Let
\[ w(t) = 2e \left[ \int_{T}^{t} u(s) \, ds + y(T) \right], \quad t \geq T. \] (2.7)

Then, for \( t \geq T + 1/e \), we have
\[
e^{t - t - 1/e} w(s) \, ds = 2e^{2} \int_{T}^{t} \left[ \int_{T}^{s} u(\xi) \, d\xi + y(T) \right] \, ds \\
= 2e^{2} \int_{T}^{t} \int_{T}^{s} u(\xi) \, d\xi \, ds + 2ey(T) \\
= 2e \int_{T}^{t} u(s) \, ds - 2e^{2} \int_{T}^{t} \left( s + \frac{1}{e} - t \right) u(s) \, ds + 2ey(T) \\
\leq w(t) - 2e^{2}u(t) \int_{T}^{t} \left( s + \frac{1}{e} - t \right) \, ds \\
= w(t) - u(t),
\]
or
\[ w(t) \geq u(t) + e \int_{T}^{t} w(s) \, ds, \quad t \geq T + \frac{1}{e}. \] (2.8)

Define a sequence \( \{v_{n}(t)\}_{n=0}^{\infty} \) of functions as follows:
\[ v_{0}(t) = w(t), \quad t \geq T, \]
and for \( n = 1, 2, \ldots \)
\[ v_{n}(t) = \begin{cases} u(t) + e \int_{T}^{t} \left[ v_{n-1}(s) \right] \, ds, & t \geq T + \frac{1}{e}, \\
u(t) + \frac{w(T + 1/e) - u(T + 1/e)}{w(t + 1/e) - u(t + 1/e)} [w(t) - u(t)], & T < t < T + \frac{1}{e}. \end{cases} \]

From (2.8), by induction, we have
\[ u(t) < v_{n}(t) \leq v_{n-1}(t) \leq w(t), \quad t \geq T, \; n = 1, 2, \ldots. \]

Thus, for \( t \geq T, v(t) = \lim_{n \to \infty} v_{n}(t) \) exists and satisfies
\[ u(t) \leq v(t) \leq w(t) = 2ey(t), \quad t \geq T, \] (2.9)

and
\[ v(t) = u(t) + e \int_{T}^{t} v(s) \, ds, \quad t \geq T + \frac{1}{e}. \] (2.10)
Substituting (2.10) into (2.4), we obtain
\[
\left( v(t) - e^{\int_{t-1/e}^t v(s) \, ds} \right)' + q(t)y(t) = 0, \quad t \geq T + \frac{1}{e},
\]
or
\[
v'(t) - ev(t) + ev(t - 1/e) + q(t)y(t) = 0, \quad t \geq T + \frac{1}{e},
\]
where
\[
q(t) = \frac{2}{1-a}e^{2+\epsilon(t-\tau(t))} \left( t - \tau(t) - \frac{1}{e} \right).
\]
Set \( z(t) = v(t)e^{-et} \) for \( t \geq T \). From (2.11), it follows that
\[
z' + z(t - 1/e) + q(t)e^{-et}y(t) = 0, \quad t \geq T + \frac{1}{e},
\]
Integrating (2.12) from \( \tau(t) \) to \( t - 1/e \) and using (2.2), (2.5) and (2.9), we have
\[
z(t) = z(\tau(t)) \leq z(t - 1/e) + \int_{\tau(t)}^{t-1/e} \left[ z(s - 1/e) + q(s)e^{-es}y(s) \right] ds
\]
\[
\leq z(t - 1/e) + \int_{\tau(t)}^{t-1/e} \left[ 2e^2 + q(s) \right] e^{-es}y(s) ds
\]
\[
\leq z(t - 1/e) + \int_{\tau(t)}^{t-1/e} \frac{2e^2}{1-a} e^{-es}y(s) ds
\]
\[
\leq z(t - 1/e) + \frac{2e^2}{1-a} y(t)e^{-et(t)} \left( t - \tau(t) - \frac{1}{e} \right)
\]
\[
= z(t - 1/e) + q(t)e^{-et}y(t).
\]
Substituting this into (2.12) yields
\[
z'(t) + z(\tau(t)) \leq 0, \quad \text{for large } t.
\]
(2.13)
This shows that the inequality (2.13) has an eventually positive solution. By Lemma 2.1, the corresponding equation (2.1) has also an eventually positive solution. The proof is complete. \( \Box \)

From Theorem 2.1 and Lemma 2.2, it is easy to see that the following Theorem 2.2 is true.

**Theorem 2.2.** Assume that (2.2) and (2.3) hold, and that
holds eventually. Then (2.1) has an eventually positive solution.

Let \( \lambda_0 = 0.576 \ldots \) be the root of the transcendental equation
\[
(e^x - 1)e^x = e.
\]

Example. Consider the delay differential equation
\[
x'(t) + x(\tau(t)) = 0, \quad t \geq 1,
\]
where \( \tau(t) = t - \delta(t) - 1/e \) and for \( n = 1, 2, \ldots \)
\[
\delta(t) = \begin{cases} 
0, & t \in [n + n^{-3}, n + 1), \\
2n^3 \lambda(t - n), & t \in [n, n + 0.5n^{-3}), \\
2n^3 \lambda(n + n^{-3} - t), & t \in [n + 0.5n^{-3}, n + n^{-3}),
\end{cases}
\]
with \( 0 < \lambda < \lambda_0 - 1/e. \)

It is easy to see that the condition (1.4) cannot be applied to derive the nonoscillation of (2.16), and
\[
\lim_{t \to \infty} (t - \tau(t)) = \frac{1}{e}, \quad \lim_{t \to \infty} (t - \tau(t)) = 1/e < \lambda < \lambda_0 = 0.576 \ldots .
\]

On the other hand, since there exists a constant \( a > 0 \) such that
\[
\lim_{t \to \infty} \left( t - \tau(t) - \frac{1}{e} \right) e^{e(t - \tau(t))} \leq \lambda e^{1 + \phi_0} < a < \left( \lambda_0 - \frac{1}{e} \right) e^{\lambda_0} = 1,
\]
and for large \( t \)
\[
t \int_0^t e^{e(s - \tau(s))} \left( s - \tau(s) - \frac{1}{e} \right) ds \leq \lambda_0 e^{e^2} \sum_{n=\lfloor t \rfloor}^{\infty} \frac{1}{n^3} \leq \lambda_0 e^{e^2} < \frac{1 - a}{8e^2},
\]
it follows that (2.2), (2.3) and (2.14) hold. By Theorem 2.2, Eq. (2.16) has an eventually positive solution.

**Theorem 2.3.** Assume that \( p(t) \neq 0 \) on any subinterval of \([t_0, \infty)\). Furthermore suppose that there exist \( a \in (0, 1) \) and \( T > t_0 \) such that
\[
\int_{\tau(t)}^t p(s) \, ds \geq \frac{1}{e}, \quad t \geq T,
\]
\[
\lim_{t \to \infty} \left[ \left( \int_{\tau(t)}^t p(s) \, ds - \frac{1}{e} \right) e^{e \int_{\tau(t)}^t p(s) \, ds} \right] < a < 1,
\]
Then Eq. (1.1) has an eventually positive solution.

**Proof.** Let

\[ u = y(t) = \int_{T}^{t} p(s) ds, \quad \sigma(t) = \int_{\tau(y^{-1}(t))}^{y^{-1}(t)} p(s) ds, \quad (2.20) \]

where \( y^{-1}(t) \) is the inverse of the function \( y(t) \). Then \( \sigma(t) \geq 1/e \) for \( t \geq y(T) \) and

\[ \limsup_{t \to \infty} \left( \sigma(t) - \frac{1}{e} \right) e^{\sigma(t)} < a < 1. \]

Since

\[ \int_{T}^{t} p(s) ds \int_{\tau(s)}^{s} \left( \int_{\tau(\xi)}^{\xi} p(\xi) d\xi - \frac{1}{e} \right) \exp \left( e \int_{\tau(\xi)}^{\xi} p(\xi) d\xi \right) d\xi \]

\[= y(t) \int_{t}^{\infty} \left( \int_{\tau(s)}^{s} p(\xi) d\xi - \frac{1}{e} \right) \exp \left( e \int_{\tau(\xi)}^{\xi} p(\xi) d\xi \right) dy(s) \]

\[= y(t) \int_{y(t)}^{\infty} e^{\sigma(u)} \left( \sigma(u) - \frac{1}{e} \right) du, \]

by (2.19), it follows that (2.14) holds. By Theorem 2.2, the delay differential equation

\[ x'(t) + x(t - \sigma(t)) = 0 \]

has an eventually positive solution \( x(t) \). Set \( z(t) = x(y(t)) \). Then, we have

\[ z'(t) + p(t)z(\tau(t)) = 0, \quad t \geq T. \]

This shows that Eq. (1.1) has an eventually positive solution. The proof is complete. \( \square \)

3. Remark

In this section, we conclude with the following remarks which also indicate the directions for possible further study on the improvement of the oscillation conditions (1.2) and (1.3) and further investigation for the nonoscillation of (1.1).
Remark. It is well known that there is a gap between the conditions (1.2) and (1.3) when the limit \( \lim_{t \to \infty} \int_{\tau(t)}^{t} p(s) \, ds \) does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors. In 1988, Erbe and Zhang [10] first derived such results by employing the upper bound of the ratio \( x(\tau(t))/x(t) \) for possible nonoscillatory solutions \( x(t) \) of (1.1). Since then, several authors tried to obtain better results by improving the upper bound for \( x(\tau(t))/x(t) \). As one can see from the introduction in a recent paper [13], their results, when formulated in terms of the numbers \( k \) and \( L \) defined by

\[
\begin{align*}
  k &= \liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s) \, ds \\
  L &= \limsup_{t \to \infty} \int_{\tau(t)}^{t} p(s) \, ds,
\end{align*}
\]

yield the oscillation conditions for (1.1) by giving the lower bound \( B(k) \) of \( L \) when \( 0 \leq k \leq 1/e \). As pointed out in [13], the known best value of the lower bound \( B(1/e) \) obtained in these conditions is 0.599216. We can now conclude by the above example that the best possible value of the lower bound \( B(1/e) \) is 0.576 . . . .

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