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A Characterization of the McLaughlin's Simple Group

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1. INTRODUCTION

Recently, J. McLaughlin [6] discovered a new simple group Mc of order 898,128,000. From the character table of this group (which was computed by J. G. Thompson), we can see easily that the group Mc has precisely one conjugate class of involutions (elements of order 2) and that the centralizer of an involution in Mc is isomorphic to $\hat{A}_8$. Here $\hat{A}_8$ denotes the unique perfect central extension of the alternating group A_8 by a group of order 2. We shall prove the converse of this fact.

THEOREM. *Let G be a finite group of even order which possesses an involution z such that the centralizer H of z in G is isomorphic to $\hat{A}_8$. Assume further that $G \neq H \cdot O_{2'}(G)$, where $O_{2'}(G)$ denotes the maximal normal odd order subgroup of G. Then G is isomorphic to the McLaughlin's simple group* Mc *of order* 898,128,000.

Throughout the paper, G will denote a finite group satisfying the assumptions of our theorem. The other notation is standard (see [9]). The proof of the theorem is obtained in the following way: The structure of H which is isomorphic to $\hat{A}_8$ is known from the work of Schur [8]. In particular, we see that H has precisely two conjugate classes of involutions and if S is an S_2-subgroup of H, then $Z(S) = Z(H) = \langle z \rangle$ and so S is an S_2-subgroup of G. Since $G \neq H \cdot O_{2'}(G)$, a result of Glauberman [4] then forces that the group G has precisely one conjugate class of involutions. We then go into a detailed study of the 3-structure of G. In particular, we show that an S_3-subgroup W of G must have order 3^6 and is isomorphic to an S_3-subgroup of $U_4(3)$. The normalizer $N_G(W)$ of W in G has order $3^6 \cdot 2^3$, and an S_2-subgroup of $N_G(W)$ is a quaternion group. We then follow Phan [7] and construct a subgroup G_0 isomorphic to $U_4(3)$. We show that G_0 is a maximal subgroup of G. We then take an element v_1 (of order 4) in $N_G(W) - G_0$, and show that G consists of the three distinct (G_0, G_0)-double cosets G_0, $G_0v_1G_0$ and $G_0v_1a_1a_2v_1G_0$, with $a_1a_2 \in G_0$. It follows that the group G is a primitive transitive rank 3 permutation group on 275 points, for which the stabilizer of a point G_a is isomorphic to $U_4(3)$ with subdegrees 1, 112, and 162. We then construct a graph as in Wales [11] and we see that this graph is isomorphic to the graph in McLaughlin [6]. Since McLaughlin has shown that the group Mc is a subgroup of index 2 in the automorphism group of this graph, we must have that G is isomorphic to Mc. Finally, the following two properties of Mc seem interesting: The group $\langle v_1, a_1a_2\rangle$ is a semidihedral group of order 16 and so $G = G_0\langle v_1, a_1a_2\rangle G_0$, where $G_0 \simeq U_4(3)$ and $\langle v_1, a_1a_2\rangle$ is semidihedral of order 16. Further, there are no involutions in the double coset $G_0v_1G_0$.

2. Some Properties of H

We list here some properties of H which will be used in the proof of the theorem. These properties are established by using the fact that $H \simeq \hat{A}_8$ and also using the work of Schur [8].

Lemma 1. *The group H has precisely two conjugate classes of elements of order 3 with the representatives s_1 and s_2, such that $[s_1, s_2] = 1$. An S_2-subgroup of $C_H(s_1)$ is a quaternion group and an S_2-subgroup of $C_H(s_2)$ is a four-group. We put $P_i = \langle s_i\rangle$, $i = 1, 2$.*

Lemma 2. *We have $N_H(P_1) = P_1F$, where $F \cap P_1 = 1$, $F/\langle z\rangle \simeq S_5$ (the symmetric group in 5 letters) and an S_2-subgroup of F is a generalized quaternion group of order 16; and $N_H(P_2) = PD$, where $P = \langle P_1, P_2\rangle$ and D is a dihedral group of order 8. We also have $N_H(P) = PS$, where S is a semidihedral group of order 16. Finally, P possesses in $N_H(P)$ precisely 4 conjugates of s_1 and precisely 4 conjugates of s_2. The elements s_1 and s_2 are real in $N_H(P)$.*

Lemma 3. *Let S be an S_2-subgroup of H; then $Z(S) = Z(H) = \langle z\rangle$, and so S is also an S_2-subgroup of G. The group H has precisely two conjugate classes of involutions with the representatives z and t. Also $|C_H(t)| = 2^6 \cdot 3$, $C_H(t)$ is 2-closed and an S_3-subgroup of $C_H(t)$ is conjugate in H to P_2. The group H has precisely two conjugate classes of elements of order 4 with the representatives w and y, so that $w^2 = z$ and $y^2 = t$; and H has precisely one conjugate class of elements of order 8 with the representative $\tilde{w}$ such that $\tilde{w}^2 = w$.*

LEMMA 4. *Let ψ be the permutation representation of H on a subgroup of index* 8. *Then* $\ker \psi = \langle z \rangle$ *and we can assume the following*:

(i) $\psi(t) = (12)(34)(56)(78)$;

(ii) $\psi(w) = (12)(34)$;

(iii) $\psi(\tilde{w}) = (1234)(56)$;

(iv) $\psi(y) = (1234)(5678)$.

LEMMA 5. *Let R and Q be an S_5-subgroup and an S_7-subgroup of H, respectively. Then $N_H(R) = R_1 Z$, where Z is a cyclic group of order* 8 *and* $R_1 = R \times P_1$, *a cyclic group of order* 15; *and $\langle z \rangle$ is an S_2-subgroup of $N_H(Q)$.*

Since $G \neq H \cdot O_{2'}(G)$ we have with Lemma 3 and a result of Glauberman [4] the following result.

LEMMA 6. *The group G has precisely one conjugate class of involutions.*

3. THE 3-STRUCTURE OF G

Let t be an involution in H which is distinct from z. Let $\bar{P}_2$ be an S_3-subgroup of $C_H(t)$. Then by Lemma 3, $\bar{P}_2$ is conjugate in H to P_2. Let X be an S_3-subgroup of $C_G(t)$ containing $\bar{P}_2$. Since t is conjugate in G to z, we have $|X| = 3^2$ and so $C_X(\bar{P}_2) = X$. This implies that $C_G(\bar{P}_2) \not\subseteq H$.

LEMMA 7. *The centralizer $C_G(P_2)$ of P_2 in G is not contained in H.*

It follows from Lemmas 1 and 2, that an S_2-subgroup of $C_G(s_1)$ is a quaternion group and an S_2-subgroup of $C_G(s_2)$ is a four-group. Hence we have the following:

LEMMA 8. *The elements s_1 and s_2 are not conjugate in G. Hence the group G has precisely two conjugate classes of elements of order* 3 *which are centralized by some involution in G.*

From Lemmas 2 and 8, it follows that if X is any 3-subgroup of G which normalizers P, then X centralizes P, and so we have

LEMMA 9. *The factor group $N_G(P)/C_G(P)$ has order prime to* 3. *If $N_G(P) \not\subseteq H$, then $C_G(P_1) \not\subseteq H$. In particular, if P is not an S_3-subgroup of G, then $C_G(P_1) \not\subseteq H$.*

We assume from now on that P is an S_3-subgroup of G. We now use the fact that $C_G(P_2) \not\subseteq H$.

Assume at first that $C_G(P_2)$ has no normal 2-complement. Since $\langle t, z\rangle$ is an S_2-subgroup of $C_G(P_2)$, it follows that $C_G(P_2)$ has precisely one conjugate class of involutions. We have $C(z) \cap C(P_2) = P\langle t, z\rangle$ and so by a result of Gorenstein and Walter [5], $C_G(P_2)/O_{2'}(C_G(P_2)) \simeq L_2(q)$, q odd prime power. We have $O_{2'}(C_G(P_2)) \supseteq P_2$ and since $O_{2'}(C_G(P_2))$ is a 3-group and P is an S_3-subgroup of $C_G(P_2)$, we get $O_{2'}(C_G(P_2)) \subseteq P$. If $O_{2'}(C_G(P_2)) = P$, then since z centralizes P and $z \underset{C_G(P_2)}{\sim} t \underset{C_G(P_2)}{\sim} tz$, we get that $\langle t, z\rangle$ centralizes P which is not the case.

Hence $O_{2'}(C_G(P_2)) = P_2$ and the centralizer of an involution in $C_G(P_2)/P_2$ is a dihedral group of order 12. It follows that

$$C_G(P_2)/P_2 \simeq L_2(11) \qquad \text{or} \qquad L_2(13).$$

Since P_2 splits in P, we get by a result of Gaschütz [3] that P_2 splits in $N_G(P_2)$.

We get that $N_G(P_2) = P_2L$, $P_2 \cap L = 1$ and $C_G(P_2) = P_2 \times L_1$, where L_1 is a subgroup of index 2 in L and $L_1 \simeq L_2(11)$ or $L_2(13)$. Also, an S_2-subgroup of L is dihedral of order 8. It follows that $L = \operatorname{aut} L_1$ and so $L \simeq PGL(2, 11)$ or $PGL(2, 13)$. Let t' be an involution in $L\backslash L_1$. Then $C_{L_1}(t')$ is a dihedral group of order 10 or 14 which contradicts Lemma 5.

Assume now that $C_G(P_2)$ has a normal 2-complement X. Then $X \supset P$ and since $N_X(P) = C_X(P)$, X has a normal 3-complement $X_1 \neq 1$. Since

$$X_1 \lhd N_G(P_2) \qquad \text{and} \qquad t \underset{N_G(P_2)}{\sim} tz,$$

we get $X_1 = C_{X_1}(t) \times C_{X_1}(tz)$, $C_{X_1}(t) \simeq C_{X_1}(tz) \neq 1$ and z acts fixed-point-free and so invertingly on X_1. Since $C_X(t) = P_2 \times C_{X_1}(t)$, we get that $|C_{X_1}(t)| = 5$. But then $\langle t, z\rangle$ normalizes the group $C_{X_1}(t)$ of order 5 and $\langle t, z\rangle C_{X_1}(t) \subseteq C_G(t)$, which contradicts Lemma 5. We have proved the following result.

LEMMA 10. *The group P is not an S_3-subgroup of G. In particular, an S_3-subgroup of $C_G(P_1)$ has order $\geqslant 3^3$.*

Since an S_2-subgroup of $C_G(P_1)$ is quaternion, it follows by a result of Brauer and Suzuki [2], that $C_G(P_1) = MF$, $M \cap F = 1$ and $M = O_{2'}(C_G(P_1)) \supset P_1$. The involution z acts fixed-point-free on the group M/P_1 and so M/P_1 is Abelian, $M' \subseteq P_1$, $P_1 \subseteq Z(M)$, M is nilpotent of class $\leqslant 2$. Let M_3 be the S_3-subgroup of M. Then $M_3 \supset P_1$ and the group $\bar{F}$ acts faithfully on M_3/P_1. This implies that $|M_3/P_1| \geqslant 3^4$ and so an S_3-subgroup of G has order $\geqslant 3^6$.

LEMMA 11. *An S_3-subgroup of G has order $\geqslant 3^6$.*

We shall determine now $N_G(P_2)$. Assume at first that $C_G(P_2)$ has no normal 2-complement. Since $\langle t, z\rangle$ is an S_2-subgroup of $C_G(P_2)$, it follows that $C_G(P_2)$ has precisely one conjugate class of involutions. We have $C(z) \cap C(P_2) = P\langle t, z\rangle$ and so by a result of Gorenstein and Walter [5], we have

$$C_G(P_2)/O_{2'}(C_G(P_2)) \simeq L_2(q),$$

q odd. We have $O_{2'}(C_G(P_2)) \supseteq P_2$ and $O_{2'}(C_G(P_2))$ is a 3-group. If $O_{2'}(C_G(P_2)) = P_2$, then $C_G(P_2)/P_2 \simeq L_2(11)$ or $L_2(13)$ which contradicts the fact that an S_3-subgroup of $C_G(P_2)$ has order $\geqslant 27$.

Hence, we must have $O_{2'}(C_G(P_2)) \supset P_2$. Put $N = O_{2'}(C_G(P_2))$. Since $C_N(\langle t, z\rangle) = P_2$ and $N \supset P_2$, we must have $|C_N(z)| = 3^2$ and so $P \subseteq N$. But z, t and tz are conjugate in $C_G(P_2)$, and so $C_N(t) \simeq C_N(tz)$ are elementary Abelian of order 9. It follows that $|N| = 3^4$, and since the centralizer of an involution in $C_G(P_2)/N$ is a four-group, we get that $C_G(P_2)/N \simeq A_4$ or A_5. But if $C_G(P_2)/N \simeq A_5$, then an S_5-subgroup Y of $C_G(P_2)$ acts trivially on N, since Y centralizes P_2 and $|N| = 3^4$. Thus $C(P_2) = N \cdot (C(N) \cap C(P_2))$, and so a four-subgroup in $C_G(P_2)$ centralizes N, a contradiction. Hence, we must have $C_G(P_2)/N \simeq A_4$, and since an S_2-subgroup of $N_G(P_2)/N$ is dihedral of order 8, $N_G(P_2)/N \simeq S_4$.

Assume now that $C_G(P_2)$ has a normal 2-complement N. Since $N \supset P$ and P is not an S_3-subgroup of N, it follows (since t is conjugate to tz in $N_G(P_2)$) that $|N| = 3^4$. We have proved the following result:

LEMMA 12. *Let N be the maximal normal odd order subgroup of $N_G(P_2)$. Then $|N| = 3^4$ and $N_G(P_2)/N$ is either a dihedral group of order 8 or is isomorphic to S_4. In any case, $C_G(N) = Z(N)$.*

We want to determine the structure of N. Put $\langle t, z, t'\rangle = D \simeq D_8$. Since $P = C_N(z)$ is D-admissible, it follows that $N_N(P)(\supset P)$ is D-admissible. Since $t \underset{D}{\sim} tz$, we get that $|C(t) \cap N_N(P)| = |C(tz) \cap N_N(P)|$ and so $P \lhd N$. Hence $P \subseteq Z(N)$ (from Lemma 9).

Assume that N is non-Abelian. Then $Z(N) = P$. If $C_G(P_2)/N \simeq A_4$, then the three involutions z, t and tz are conjugate in $C_G(P_2)$ and since $C_N(z) \subseteq Z(N)$, we get $C_N(t) \subseteq Z(N)$ and $C_N(tz) \subseteq Z(N)$; and so N is Abelian, a contradiction. Thus $C_G(P_2)$ has the normal 2-complement N. Let N_1 be a subgroup of order 3^5 of G containing N. Then $N_1 \cap N_G(P_2) = N$ and since P is characteristic in N, we get $P \lhd N_1$, and so by Lemma 9, $P \subseteq Z(N_1)$, a contradiction.

Hence, N is Abelian, and since it is generated by the elementary Abelian subgroups P, $C_N(t)$ and $C_N(tz)$, it follows that N is elementary Abelian. We have proved the following result:

LEMMA 13. *The group N is elementary Abelian of order 3^4, $C_G(N) = N$ and $C_G(P) = N\langle z\rangle$, with $N \lhd N\langle z\rangle$.*

We have $N_H(P) = PB$, where $B \supset D$ and B is a semidihedral group of order 16. Hence B normalizes $N\langle z\rangle$ and so normalizes N. It follows from Lemma 12 that $N_G(N) \supset N_G(P_2)$ and $B \subseteq N_G(N)$. Assume that B is not an S_2-subgroup of $N_G(N)$. Let B_1 be a 2-group in $N_G(N)$ which contains B as a subgroup of index 2. Then $\langle z\rangle \lhd B_1$, and so B_1 normalizes $P = C_N(z)$ and $PB_1 \subseteq C_G(z) = H$, a contradiction. It follows that the semidihedral group B of order 16 is an S_2-subgroup of $N_G(N)$, and $C(z) \cap N(N) = PB$.

Assume now that $N_G(P_2)/N$ is a dihedral group of order 8. Acting with $\langle z, t\rangle$ on $O_{2'}(N_G(N))$, we get that $N = O_{2'}(N_G(N))$. But N is not an S_3-subgroup of G and so 3 divides the order of $N_G(N)/N$, which implies that $N_G(N)$ does not have a normal 2-complement. Since the centralizer of an involution in $N_G(N)/N$ is semidihedral of order 16, it follows that $N_G(N)$ has a normal subgroup L of index 2, where L has no subgroup of index 2, and an S_2-subgroup of L is dihedral of order 8. Since the centralizer of an involution in L/N is dihedral of order 8, it follows that $L/N \sim L_2(7)$ or $L_2(9)$ and so we have $|N_G(N) : C_G(P_2)| = 2^2 \cdot 3 \cdot 7$ or $2^2 \cdot 3^2 \cdot 5$, which is a contradiction.

Hence by Lemma 12, we must have $C_G(P_2)/N \simeq A_4$ and $N_G(P_2)/N \sim S_4$. Since $C_G(P_2)$ has precisely one conjugate class of involutions, it follows that the group $N_G(N)$ must have precisely one conjugate class of involutions. Now, the centralizer of an involution in $N_G(N)/N$ is semidihedral of order 16, and so it follows that $N_G(N)$ has a subgroup L of index 2, where L has no subgroup of index 2, and an S_2-subgroup of L is dihedral of order 8. Since the centralizer of an involution in L/N is dihedral of order 8, we have $L/N \sim L_2(7)$ or $L_2(9)$. But $C_G(N) = N$, and so $L/N \simeq L_2(9) \simeq A_6$. There are no involutions in $N_G(N)\backslash L$, and $N_G(N)/N$ is a subgroup of the automorphism group of L/N with a semidihedral group of order 16 as an S_2-subgroup. We have proved:

LEMMA 14. *We have $N_G(P_2)/N \simeq S_4$, $|C_G(P_2)| = 2^2 \cdot 3^5$ and $N_G(N)$ has a subgroup L of index 2 such that $L/N \simeq A_6$. An S_2-subgroup of $N_G(N)$ is semidihedral of order 16. Thus $N_G(N)/N$ is an automorphism group of L/N. There are no involutions in $N_G(N)\backslash L$.*

Put $P_i = \langle s_i\rangle$, $i = 1, 2$. Since $|N_G(N) : C_G(s_2)| = 60$, there are precisely 60 conjugates of s_2 in $N_G(N)$. There are $80 - 60 = 20$ nontrivial elements left in N. An S_5-subgroup of $N_G(N)$ acts fixed-point-free on N and so $|N_G(N) : C(s_1) \cap N(N)| = n$ is divisible by 5. Also, there is no four-subgroup in $C_G(s_1)$ and so $10 \mid n$. Hence $3 \nmid n$. Assume that $n = 10$. Then, an S_2-subgroup of $C(s_1) \cap N(N)$ has order 8. But there is an element in B which inverts P_1 and so an S_2-subgroup of $N(P_1) \cap N(N)$ has order 16 and so is semidihedral. This contradicts the fact that $N_G(P_1)$ does not contain four-subgroup. Hence $n = 20$. We have proved:

LEMMA 15. *We have* $|C(s_1) \cap N(N)| = 2^2 \cdot 3^6$ *and an* S_2*-subgroup of* $C(s_1) \cap N(N)$ *is cyclic (of order* 4*). We have that* $C(s_2) \cap N(N) = C_G(s_2)$ *has order* $2^2 \cdot 3^5$. *Hence* $N^{\#}$ *consists of precisely* 20 *conjugates in* $N_G(N)$ *of* s_1 *and* 60 *conjugates in* $N_G(N)$ *of* s_2, *where* $P_i = \langle s_i \rangle$, $i = 1, 2$. *Finally, (since an element of order* 4 *in* B *inverts* P_1*)* $N(P_1) \cap N(W) = WQ$ *is* 3*-closed, where* W *is an* S_3*-subgroup of* $N_G(N)$ *of order* 3^6 *and* Q *is a quaternion group. Also* $N(W) \cap N(N) = WQ$, *where the quaternion group* Q *acts fixed-point-free on* W/N *and so* Q *acts transitively on* $(W/N)^{\#}$; *also* $Z(Q) \subseteq L$.

All elements of order 3 in $N_G(N)/N$ lie in a single conjugate class in $N_G(N)/N$. Let X/N be a subgroup of L/N isomorphic to A_4 and $x \in X \backslash N$, where x has 3-power order. There is an element y in X such that $\langle x, x^y \rangle N/N = X/N$, and so $\langle x, x^y \rangle$ contains a four-subgroup.

Assume now that there is an element of 3-power order $x \in N_G(N) \backslash N$ such that $|C_N(x)| \geqslant 3^3$. Then, there is an element $y \in N_G(N)$ such that $\langle x, x^y \rangle$ contains a four-group and $|C_N(x^y)| \geqslant 3^3$. Thus, $|C_N(\langle x, x^y \rangle)| \geqslant 3^2$ and so a four-group centralizes a group of order 3^2, a contradiction.

We have $P_1 \subseteq Z(W)$ and assume that $Z(W) \supset P_1$. Then $Z(W) \subseteq N$ and $|Z(W)| = 3^2$. But $Z(W)$ is Q-admissible and $Z(Q)$ centralizes P_1. Every subgroup of order 3 of $Z(W)$ is conjugate in $N_G(N)$ to P_1 and so $Z(Q)$ cannot centralize $Z(W)$. Hence $[Z(W), Z(Q)]$ has order 3 and is acted upon faithfully by Q, a contradiction. Hence $Z(W) = P_1$.

Assume now that N is not a characteristic subgroup of W. Then there is an automorphism σ of W such that $N^\sigma \neq N$. If $|N \cap N^\sigma| = 3^3$, then an element in $N^\sigma \backslash N$ centralizes a subgroup of order 3^3 in N, a contradiction. If $|N \cap N^\sigma| = 3^2$, then $W = N \cdot N^\sigma$ and so $N \cap N^\sigma \subseteq Z(W)$, a contradiction. It follows that N char W. We have proved:

LEMMA 16. *For every element* $x \in W \backslash N$, *we have* $|C_N(x)| \leqslant 3^2$. *Also* $Z(W) = P_1$ *and* N *is a characteristic subgroup of* W *and* W *has no Abelian subgroups of order* 3^5.

It follows that $N_G(W) \subseteq N_G(N)$ and $N_G(W) = WQ$. Hence, W is an S_3-subgroup of G and $|W| = 3^6$ and $Z(W) = P_1$.

Now, we know that P is an S_3-subgroup of $C_H(P_1)$. Then, if we use the notation in the proof of Lemma 11, we have $|M_3/P_1| = 3^4$ and $|M_3| = 3^5$; also $M_3 \lhd N_G(P_1)$ and so $W \supset M_3$. Since $W \supseteq P$, and $P \not\subseteq M_3$ we have $W = \bar{P}_1 \cdot M_3$, where $\bar{P}_1$ is conjugate in H to P_1. Since $|N| = 3^4$ and $N \supset P$, $N \subseteq W$, we get that $|N \cap M_3| = 27$. Since N contains 20 elements conjugate to s_1 in G, it follows that $N \cap M_3$ contains a subgroup $\bar{P}_2$ conjugate to P_2 in G. Since M is nilpotent, it follows from Lemma 14 that $M = M_3$. On the other hand, W has no Abelian subgroups of order 3^5, and so M is non-Abelian. Hence $M' = P_1$. Since an S_5-subgroup of $N_G(P_1)$ acts irreducibly on M/P_1, it follows that M does not possess characteristic subgroups which are strictly in between P_1 and M. *It follows that M is an extra-special group of order 3^5 and exponent* 3. Also z acts invertingly on M/P_1 and so an S_2-subgroup of $N_G(P_1)$ (which is generalized quaternion of order 16) acts fixed-point-free on M/P_1. Now $C(\bar{P}_2) \cap N(P_1) \supseteq N$ and since $\bar{P}_2$ is conjugate to P_2 in G, *all elements in $M \setminus Z(M)$ lie in a single conjugate class in $N_G(P_1)$ and are all conjugate in G to s_2*. Also $W' \subseteq M$ and $W' \subseteq N$, and so $W' \subseteq N \cap M$, where $N \cap M$ is elementary Abelian of order 27. By Lemma 16, we have that $W' \supset P_1 = Z(W)$. We know that $N_G(W) = N(W) \cap N(P_1) = WQ$, where Q is a quaternion group. Since $Z(Q)$ acts invertingly on $W'/Z(W)$, it follows that Q acts faithfully on $W'/Z(W)$ and so $|W'| = 27$. So $W' = N \cap M$ is elementary Abelian of order 27. Also $\mho^1(W) \subseteq N \cap M$ and so $W' = D(W)$ and W is of class 3 because $[W, W'] \neq 1$, $[W, W'] \supseteq Z(W)$, $[W, W'] \subseteq W'$, and Q acts faithfully on $W'/[W, W']$ which implies that $[W, W'] = Z(W) = P_1$.

Now, take an element $y \in M \setminus N$. Then, $|M : C_M(y)| = 3$ and by Lemma 16, $C_M(y)$ does not contain $N \cap M$. Thus, $M = (N \cap M)C_M(y)$ and so the group $C_M(y)$ covers $M/M \cap N$. It follows that we can have an element $y_1 \in C_M(y)$ so that $M = (M \cap N)\langle y, y_1\rangle$. But $\langle y, y_1\rangle$ is an elementary Abelian group of order 9 which complements N in W and every subgroup of order 3 in $\langle y, y_1\rangle$ is conjugate to P_2 in G. We have proved:

LEMMA 17. *The group W (of order 3^6) is an S_3-subgroup of G. Also $N_G(W) = N(W) \cap N(N) = N(W) \cap N(P_1) = WQ$, where Q is a quaternion group. The maximal normal odd order subgroup M of $N_G(P_1)$ is an extra-special group of order 3^5 and exponent 3. Every subgroup of order 3 in M which is distinct from $P_1 = Z(M)$ is conjugate to P_2 in G, and the set $M \setminus P_1$ is a single conjugate class in $N_G(P_1)$. We have $W' = D(W) = N \cap M$ which is elementary Abelian of order 27. The group N splits in W (because $N \cap M$ splits in M and $W \supset M$, $W \supset N$ and $W = MN$), and so N splits in $N_G(N)$. We have $W = NX$, $X \cap N = 1$ and every subgroup of order 3 in X is conjugate in G to P_2. We have $[W, W'] = Z(W) = P_1$ and so W is of class 3. For every $y \in M \setminus N$, we have $|C_N(y)| = 3^2$ and $C_W(y)$ covers W/N and so $C_W(y)$ (of order 3^4) is an S_3-subgroup of $C(y) \cap N(N)$ and $C_W(y)$ is non-Abelian (in fact $C_W(y) = C_M(y)$). Further, we have $C_G(s_1) = M \cdot F$, where $M \cap F = 1$ and $F/\langle z\rangle \simeq A_5$; and hence $|C_G(s_1)| = 2^3 \cdot 3^6 \cdot 5$.*

We now determine a lower bound on the order of an S_5-subgroup of G. Let R be an S_5-subgroup of H; then by Lemma 5, we have $C_H(R) = R_1\langle z\rangle$, where $R_1 = R \times P_1$ and $|P_1| = 3$. Hence $\langle z \rangle$ is an S_2-subgroup of $C_G(R)$ and so $C_G(R)$ has a normal 2-complement U. From Lemma 17, $N_U(P_1) = P_1 \times R$, and so P_1 is an S_3-subgroup of U and, hence, U has a normal 3-complement V. By Lemma 14, the group $N_G(N)$ contains an element of order 5 inverted by an involution, and since R is not inverted by any involution (see Lemma 5), R cannot be an S_5-subgroup of G. Since z acts fixed-point-free on V/R, it follows that V/R is Abelian. Let V_5 be an S_5-subgroup of V. Then $V_5 \supset R$ and since P_1 acts fixed-points-free on V_5/R, we see that $|V_5/R| \geq 5^2$ and so we have proved the following result:

LEMMA 18. *An S_5-subgroup of G has order $\geq 5^3$.*

4. CONSTRUCTION OF A SUBGROUP G_0 ISOMORPHIC TO $U_4(3)$

Let ψ as in Lemma 4, be the permutation representation of H on a subgroup of index 8. Without loss of generality, we may assume that σ_1 is an element in H such that $\psi(\sigma_1) = (123)$ and $\sigma_1{}^3 = 1$. Let a_1, u, v, and v_1 be elements of H such that $\psi(a_1) = (14)(23)$, $\psi(u) = (15)(26)(37)(48)$, $\psi(v) = (23)(67)$ and $\psi(v_1) = (67)(48)$. Now let

$$b_1 = a_1^{\sigma_1}, \qquad a_2 = a_1{}^u, \qquad \sigma_2 = \sigma_1{}^u \qquad \text{and} \qquad b_2 = a_2^{\sigma_2}.$$

Replacing a_1 by a_1^{-1} and b_1 by b_1^{-1} if necessary, we may assume that

$$b_i^{\sigma_i} = a_i b_i \qquad \text{for} \quad i = 1, 2.$$

From Lemmas 1, 3 and 4, we have the following relations:

$$\begin{aligned} &\sigma_i{}^3 = 1, \qquad a_i{}^2 = b_i{}^2 = z, \qquad a_i^{b_i} = a_i^{-1}, \qquad a_i{}^v = a_i^{-1} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 1, 2; \\ &[a_1, a_2] = [a_1, b_2] = 1, \qquad v^2 = v_1{}^2 = z, \qquad u^2 = 1, \\ &v^u = v^{-1}, \qquad v^{v_1} = v^{-1}; \qquad \sigma_i{}^v = \sigma_i^{-1} \qquad \text{for} \quad i = 1, 2, \\ &\qquad \sigma_1^{v_1} = \sigma_1 \qquad \text{and} \qquad \sigma_2^{v_1} = \sigma_2^{-1}. \end{aligned} \tag{1}$$

Replacing v_1 by v_1^{-1} and v by v^{-1} if necessary, we may assume

$$(v_1 a_2)^3 = 1 \qquad \text{and} \qquad (v_1 u)^2 = v.$$

Then $(v_1 a_1)^3 = v$ and $(v_1 a_1 a_2)^4 = z$.

Now let $L_i = \langle a_i, b_i, \sigma_i \rangle$ for $i = 1, 2$. Then we have that $L_i \simeq SL(2, 3)$, $[L_1, L_2] = 1$ and $L_1 \cap L_2 = \langle z \rangle$. The group $\langle v, u \rangle$ is a dihedral group of order 8 and normalizes the group L_1L_2. Let $H_0 = L_1L_2\langle v, u \rangle$. Then, from the relations in (1), we have that H_0 is isomorphic to the centralizer of an involution in $U_4(3)$ and $|H_0| = 2^7 \cdot 3^2$ (see Phan [7]). Since $H \simeq \hat{A}_8$ and H_0 is a maximal subgroup of H, we have $H = \langle H_0, v_1 \rangle$. From the relations in (1) and the structure of $\hat{A}_8$, we can prove the following:

LEMMA 19. *The group $H_0 = L_1L_2\langle v, u \rangle$ is a maximal subgroup of H and is isomorphic to the centralizer of an involution in $U_4(3)$. Further, the group H consists of precisely the following three distinct (H_0, H_0)-double cosets: H_0, $H_0v_1H_0$ and $H_0v_1a_1a_2v_1H_0$.*

By Lemma 2, $N_H\langle \sigma_1, \sigma_2 \rangle = \langle \sigma_1, \sigma_2 \rangle\langle v, v_1, u \rangle$, where $(vu)^8 = 1$, $(vu)^u = (vu)^3$; hence, the group $\langle v, v_1, u \rangle$ is a semidihedral group of order 16. Let $P = \langle \sigma_1, \sigma_2 \rangle$, $N_H(P) = P \cdot B$, where $B = \langle v, v_1, u \rangle$ and $N_H(\langle \sigma_1\sigma_2 \rangle) = P \cdot D$, where $D = \langle u, v \rangle$ is a dihedral group of order 8. Put $P_1 = \langle \sigma_1 \rangle$ and $P_2 = \langle \sigma_1\sigma_2 \rangle$. By Lemma 8 and the relations in (1), we see that σ_1 is not conjugate in G to $\sigma_1\sigma_2$. From Lemma 13, we have that $C_G(P) = N\langle z \rangle$, where $N \lhd N\langle z \rangle$ and N is an elementary Abelian group of order 3^4. It follows that the group B normalizes N, and, from Lemmas 14 and 17, we have $N_G(N) = N \cdot K$, $N \cap K = 1$, and there is a subgroup L of K, $L \simeq A_6$ and $[K : L] = 2$. We may also assume that $B \subset K$. From Lemma 14, there are no involutions $N_G(N) \setminus N \cdot L$ and so there are no involutions in $K \setminus L$. Hence, we must have $D \subset L$ and $v_1 \in K \setminus L$. We shall now concentrate our attention on the group $N \cdot L$.

From the structure of A_6, we must have $|N_L(\langle u, z \rangle)| = 2^3 \cdot 3$ and we can assume that

$$N_L(\langle u, z \rangle) = \langle u, v \rangle\langle \mu \rangle, \quad \text{where} \quad \mu^3 = 1, \qquad z^\mu = u, \quad u^\mu = zu, \quad \text{and} \quad \mu^{v_1} \in N \cdot L. \tag{2}$$

Then it follows that $N_{N\cdot L}(\langle u, z \rangle) = \langle u, v \rangle \cdot \langle \mu, \sigma_1\sigma_2 \rangle$. Let $P_2 = \langle \sigma_1\sigma_2 \rangle$ and $P_1 = \langle \sigma_1 \rangle$. Then, by Lemma 14, $|N_G(P_2)| = 2^3 \cdot 3^5$. Since $C_N(z) = P_2 \times P_1$, we have from (2) that $C_N(z^\mu) = C_N(u) = P_2 \times P_1^\mu$ and $C_N(uz) = P_2 \times P_1^{\mu^2}$. Hence the group $N = \langle \sigma_1\sigma_2, \sigma_1, \sigma_1^\mu, \sigma_1^{\mu^2} \rangle$. By Lemmas 15 and 17, $C_G(\sigma_1) \cap N(N) = W \cdot \langle v_1 \rangle$, where W is an S_3-subgroup of G, $|W| = 3^6$, and $N_G(W) = N(W) \cap N(N)$. Also $N(P_1) \cap N(N) = WQ$, where $Q = \langle v, v_1 \rangle$ is a quaternion group; $Z(W) = \langle \sigma_1 \rangle$, and, if $M = O_{2'}(N_G(P_1))$, M is an extra-special of exponent 3 of order 3^5. From (1), $\sigma_2 = \sigma_1^u$, and so, by Lemma 17, we have $W = M \cdot \langle \sigma_2 \rangle$. Now $M \cap N = W' = D(W)$ is an elementary Abelian group of order 27 and is normalized by $Q = \langle v, v_1 \rangle$. The group

$C_{M\cap N}(z) = \langle\sigma_1\rangle$, and so $M \cap N = \langle\sigma_1\rangle \times [M \cap N, z]$. We also know that $| [N, z] | = 9$, and obviously $[N, z] \supseteq [M \cap N, z]$. Since $| [M \cap N, z] | = 9$, we must have $[N, z] = [M \cap N, z]$. So $M \cap N = \langle\sigma_1\rangle \times [N, z]$. Now $[N, z]$ is normalized by $\langle u, z\rangle$ and without loss of generality, we can assume that $\varphi_1, \varphi_2 \in [N, z]$ such that

$$\varphi_i^z = \varphi_i^{-1} \quad \text{for} \quad i = 1, 2; \qquad \varphi_1^u = \varphi_1, \qquad \varphi_2^u = \varphi_2^{-1} \qquad \text{and}$$

the group $\langle\varphi_1, \varphi_2\rangle$ which is elementary Abelian of order 9 is normalized by $\langle v, v_1\rangle$. (3)

From (3) above, we have $\varphi_1^v = \varphi_1^i\varphi_2^j$, where $i, j \in \{0, \pm 1\}$. Then

$$\varphi_1^{vu} = \varphi_1^i\varphi_2^{-j} = \varphi_1^{uv^{-1}} = \varphi_1^{vz} = (\varphi_1^i\varphi_2^j)^z = \varphi_1^{-i}\varphi_2^{-j}.$$

So we must have $i = 0$ and $\varphi_1^v = \varphi_2^{\pm 1}$. Replacing φ_2 by φ_2^{-1} if necessary, we can assume

$$\varphi_1^v = \varphi_2 \qquad \text{and} \qquad \varphi_2^v = \varphi_1^{-1}. \tag{4}$$

Now the element $a_2 \in C_G(P_1)$ and since $M \lhd N_G(P_1)$, we have

$$\langle\varphi_1^{a_2}, \varphi_2^{a_2}\rangle = \langle\varphi_3, \varphi_4\rangle \subseteq M.$$

Suppose we have $\langle\varphi_3, \varphi_4\rangle \cap \langle\sigma_1, \varphi_1, \varphi_2\rangle \neq 1$. Then, there is an element $\varphi_3^i\varphi_4^j \in \langle\sigma_1, \varphi_1, \varphi_2\rangle$ for some i and j not both zero. Since σ_2 centralizes the group $\langle\sigma_1, \varphi_1, \varphi_2\rangle$, we have

$$\begin{aligned} b_2^{-1}\varphi_1^i\varphi_2^j b_2 &= \sigma_2^{-1}a_2^{-1}\sigma_2\varphi_1^i\varphi_2^j\sigma_2^{-1}a_2\sigma_2 \\ &= \sigma_2^{-1}a_2^{-1}\varphi_1^i\varphi_2^j a_2\sigma_2 \\ &= \sigma_2^{-1}\varphi_3^i\varphi_4^j\sigma_2 \\ &= \varphi_3^i\varphi_4^j \\ &= a_2^{-1}\varphi_1^i\varphi_2^j a_2 . \end{aligned}$$

Hence, $b_2a_2^{-1} \in C_G(\varphi_1^i\varphi_2^j)$ and since $(b_2a_2^{-1})^2 = z$, we have $z \in C_G(\varphi_1^i\varphi_2^j)$. This contradicts (3). So we have that $M = \langle\sigma_1, \varphi_1, \varphi_2, \varphi_3, \varphi_4\rangle$ where

$$\varphi_1^{a_2} = \varphi_3, \qquad \varphi_2^{a_2} = \varphi_4 . \tag{5}$$

Next, we represent the group $L_2 = \langle a_2, b_2, \sigma_2\rangle$ as linear transformations on the vector space $M/\langle\sigma_1\rangle$ over the field of 3 elements. In terms of the basis

$\varphi_1\langle\sigma_1\rangle$, $\varphi_2\langle\sigma_1\rangle$, $\varphi_3\langle\sigma_1\rangle$, and $\varphi_4\langle\sigma_1\rangle$, we have the following representation of the element a_2:

$$a_2 \to \begin{bmatrix} & & 1 & 0 \\ & & 0 & 1 \\ -1 & 0 & & \\ 0 & -1 & & \end{bmatrix}.$$

Since $v^{-1}\varphi_1 v = \varphi_2$, $v^{-1}\varphi_2 v = \varphi_1^{-1}$ and $v^{-1}a_2 v = a_2^{-1}$, we have

$$\varphi_3{}^v = \varphi_4^{-1} \qquad \text{and} \qquad \varphi_4{}^v = \varphi_3 . \tag{6}$$

We have $W' = M \cap N = \langle\sigma_1, \varphi_1, \varphi_2\rangle$ and so $\varphi_3^{\sigma_2} \in \varphi_3\langle\sigma_1, \varphi_1, \varphi_2\rangle$. Similarly, $\varphi_4^{\sigma_2} \in \varphi_4\langle\sigma_1, \varphi_1, \varphi_2\rangle$. Hence we have the following representation of σ_2:

$$\sigma_2 \to \begin{bmatrix} I & 0 \\ C & I \end{bmatrix}, \qquad \text{where } C \text{ is a } 2 \times 2 \text{ matrix},$$

and I the 2×2 identity matrix. Since $(a_2\sigma_2)^3 = 1$, we must have $C = -I$ and so

$$\varphi_3^{\sigma_2} = \varphi_1^{-1}\varphi_3\sigma_1{}^{\epsilon}, \qquad \text{where} \qquad \epsilon = 0 \text{ or } \pm 1.$$

Since the group $M = \langle\sigma_1, \varphi_1, \varphi_2, \varphi_3, \varphi_4\rangle$ is extra-special,

$$\varphi_1^{\varphi_3} = \varphi_1\sigma_1{}^{\delta}, \qquad \text{where} \qquad \delta \in \{0, \pm 1\}.$$

Now $\varphi_3^{\sigma_2} = \varphi_1^{a_2\sigma_2} = \varphi_1^{-1}\varphi_3\sigma_1{}^{\epsilon}$ and so $\varphi_1^{a_2\sigma_2a_2} = \varphi_3^{-1}\varphi_1^{-1}\sigma_1{}^{\epsilon}$. But $a_2\sigma_2a_2 = \sigma_2^{-1}a_2^{-1}\sigma_2^{-1}$ and so $\varphi_1^{a_2\sigma_2a_2} = \varphi_1^{\sigma_2^{-1}a_2^{-1}\sigma_2^{-1}} = \varphi_1^{-1}\varphi_3^{-1}\sigma_1^{2\epsilon}$. But we have $\varphi_3^{-1}\varphi_1^{-1} = \varphi_1^{-1}\varphi_3^{-1}\sigma_1^{-\delta}$ and so $\sigma_1^{-\delta+\epsilon} = \sigma_1^{2\epsilon}$. Thus we must have $\delta = -\epsilon$. We have proved

$$\varphi_3^{\sigma_2} = \varphi_1^{-1}\varphi_3\sigma_1{}^{\epsilon} \qquad \text{and} \qquad \varphi_1^{\varphi_3} = \varphi_1\sigma_1^{-\epsilon} \qquad \text{where} \qquad \epsilon = 0 \text{ or } \pm 1. \tag{7}$$

Now suppose $[\varphi_1, \varphi_3] = 1$. Then conjugating by the element v, we have $[\varphi_2, \varphi_4] = 1$. Since M is extra-special,

$$\varphi_2^{\varphi_3} = \varphi_2\sigma_1{}^{\delta} \qquad \text{and} \qquad \varphi_1^{\varphi_4} = \varphi_1\sigma_1^{-\delta}, \qquad \text{where} \qquad \delta \in \{1, -1\}.$$

From (7) and the assumption that $[\varphi_1, \varphi_3] = 1$, we have $\varphi_3^{\sigma_2} = \varphi_1^{-1}\varphi_3$, i.e., $\sigma_2^{\varphi_3} = \sigma_2\varphi_1$; and $\varphi_4^{\sigma_2} = \varphi_2^{-1}\varphi_4$, i.e., $\sigma_2^{\varphi_4} = \sigma_2\varphi_2$. But the element φ_1 is conjugate in G to $\sigma_1\sigma_2$, and so by Lemma 14, $|C_G(\varphi_1)| = 2^2 \cdot 3^5$ and $C_G(\varphi_1)$ is not 3-closed. So $C_G(\varphi_1) = \langle\sigma_1, \sigma_2, \varphi_1, \varphi_2, \varphi_3, u\rangle$ and $N = \langle\sigma_1, \sigma_2, \varphi_1, \varphi_2\rangle$ is normal in $C_G(\varphi_1)$. Since $C_G(\varphi_1)/N \simeq A_4$, we have $(u\varphi_3)^3 \in \langle\sigma_1, \sigma_2, \varphi_1, \varphi_2\rangle$.

But $\sigma_2^{(u\varphi_3)^3} = \sigma_1\varphi_1 \neq \sigma_2$ and so we have a contradiction. Hence we have proved that

$$\varphi_1^{\sigma_3} = \varphi_1\sigma_1^{-\epsilon} \quad \text{and} \quad \varphi_3^{\sigma_2} = \varphi_1^{-1}\varphi_3\sigma_1^{\epsilon}, \quad \text{where} \quad \epsilon = \pm 1.$$

Conjugating the relations above by the element v, we have (8)

$$\varphi_2^{\sigma_4} = \varphi_2\sigma_1^{-\epsilon} \quad \text{and} \quad \varphi_4^{\sigma_2} = \varphi_2^{-1}\varphi_4\sigma_1^{\epsilon}.$$

Now $\varphi_1^{\varphi_4} = \varphi_1\sigma_1^{\delta}$, where $\delta \in \{0, \pm 1\}$. If $\delta = \epsilon$, then from (8), the element $\varphi_3\varphi_4$ centralizes φ_1. Since u also centralizes φ_1, we have as above that $(u\varphi_3\varphi_4)^3 \in N$. But $\sigma_2^{(u\varphi_3\varphi_4)^3} \neq \sigma_2$, a contradiction. Similarly, we can obtain a contradiction in the case $\delta = -\epsilon$. Hence we have proved the following relations:

$$[\varphi_1, \varphi_4] = [\varphi_2, \varphi_3] = 1 \tag{9}$$

So, the elements uz and φ_3 centralize φ_2. Since $\varphi_2 \underset{G}{\sim} \sigma_1\sigma_2$, we have $(uz\varphi_3)^3 \in \langle\sigma_1, \sigma_2, \varphi_1, \varphi_2\rangle$. But from (5), we have $\varphi_3 = a_2^{-1}\varphi_1 a_2$ and using the relations in (1), we have that $uz\varphi_3 \in C_G(a_2^{-1}ua_2)$. Since there are no elements of order 9 in the group $C_G(a_2^{-1}ua_2)$, we have $(uz\varphi_3)^3 = 1$. From the relations in (1) and (6), we have

$$u\varphi_4 = v(uz\varphi_3)v^{-1} \quad \text{and so} \quad (u\varphi_4)^3 = 1.$$

Next, the elements zuv and $\varphi_3^{-1}\varphi_4^{-1}$ centralize $\varphi_1\varphi_2^{-1}$ and so $(zuv\varphi_3^{-1}\varphi_4^{-1})^3 \in \langle\varphi_1, \varphi_2, \sigma_1, \sigma_2\rangle$. But $zuv\varphi_3^{-1}\varphi_4^{-1} \in C_G(uva_1a_2)$ and since uva_1a_2 is involution, we have

$$(zuv\varphi_3^{-1}\varphi_4^{-1})^3 = 1.$$

But $v^{-1}(zuv\varphi_3^{-1}\varphi_4^{-1})v = uv\varphi_3^{-1}\varphi_4$ and so $(uv\varphi_3^{-1}\varphi_4)^3 = 1$. We have proved the following:

LEMMA 20. *The following relations hold*:

$$(a_2\sigma_2)^3 = (uz\varphi_3)^3 = (u\varphi_4)^3 = (uv\varphi_3^{-1}\varphi_4)^3 = (zuv\varphi_3^{-1}\varphi_4^{-1})^3 = 1.$$

We have from (8) that

$$\varphi_4^{\sigma_2} = \varphi_2^{-1}\varphi_4\sigma_1^{\epsilon}, \quad \text{where} \quad \epsilon = \pm 1.$$

Now suppose $\epsilon = -1$. Then $\sigma_2^{\varphi_4} = \sigma_2\varphi_2\sigma_1^{-1}$. From Lemma 20, $(u\varphi_4)^3 = 1$, and so $\sigma_1 = \sigma_1^{(u\varphi_4)^3}$. But $\sigma_1{}^u = \sigma_2$ and so $\sigma_1^{(u\varphi_4)^3} = \sigma_2$, a contradiction. Hence in (8) we must have $\epsilon = 1$. Together with the relations in (3), (4), (5), and (9) we have determined uniquely the structure of an S_3-subgroup W of G and we have proved

LEMMA 21. *The group $W = N\langle\varphi_3, \varphi_4\rangle$ is an S_3-subgroup of G and has the following structure:*

$$N = TT_1T_2$$

is an elementary abelian group of order 3^4, where

$$T = C_N(z) = \langle\sigma_1, \sigma_2\rangle,$$
$$T_1 = C_N(u) = \langle\sigma_1\sigma_2, \varphi_1\rangle,$$
$$T_2 = C_N(uz) = \langle\sigma_1\sigma_2, \varphi_2\rangle,$$

are elementary abelian groups of order 9; and

$$[\varphi_3, \sigma_1] = [\varphi_4, \sigma_1] = [\varphi_1, \varphi_4] = [\varphi_2, \varphi_3] = [\varphi_3, \varphi_4] = 1,$$

$$\varphi_1^{\varphi_3} = \varphi_1\sigma_1^{-1}, \qquad \varphi_2^{\varphi_4} = \varphi_2\sigma_1^{-1}, \qquad \varphi_3^{\sigma_2} = \varphi_1^{-1}\varphi_3\sigma_1, \qquad \varphi_4^{\sigma_2} = \varphi_2^{-1}\varphi_4\sigma_1.$$

Moreover, $v^{-1}\varphi_1 v = \varphi_2$, $v^{-1}\varphi_2 v = \varphi_1^{-1}$, $v^{-1}\varphi_3 v = \varphi_4^{-1}$ and $v^{-1}\varphi_4 v = \varphi_3$.

Now $W \subseteq C_G(\sigma_1)$ and so

$$W \cap W^{a_1a_2} \subseteq C_G(\sigma_1) \cap C_G(\sigma_1^{a_1a_2}) \subseteq C_G(\sigma_1) \cap C_G(a_1b_1) = \langle\sigma_2, a_2, b_2\rangle.$$

Since $\sigma_2 \underset{G}{\sim} \sigma_1$, we have by Lemmas 1 and 2, that an S_2-subgroup of $N_G(\langle\sigma_2\rangle)$ is a generalized quaternion group of order 16. Since the involution a_1a_2 normalizes $W \cap W^{a_1a_2}$, we have $W \cap W^{a_1a_2} = 1$. We have proved

LEMMA 22. *The group W and its conjugate $W^{a_1a_2}$ have trivial intersection.*

From the relations given in (1), we have

LEMMA 23. *The group $R = \langle v, a_2, u\rangle/\langle v\rangle$ is a dihedral group of order 8 and is generated by the involutions $\gamma_1 = a_2\langle v\rangle$ and $\gamma_2 = u\langle v\rangle$.*

Now let $B = W\langle v\rangle$, $A = \langle v, a_2, u\rangle$, $\omega(\gamma_1) = a_2$ and $\omega(\gamma_2) = u$. For any $\gamma \in R$ and $\gamma = \gamma_{i_1}, \dots, \gamma_{i_s}$, define $\omega(\gamma) = \omega(\gamma_{i_1}) \cdots \omega(\gamma_{i_s})$. We shall denote $B\gamma B$ to mean $B\omega(\gamma)B$. Then with the relations given in (1) and Lemmas 20, 21, 22, and 23, Phan [7, pp. 29–33] has proved the following:

LEMMA 24. *The set of elements $G_0 = BAB$ is a subgroup of G and if $B\gamma_1B = B\gamma_2B$, then $\gamma_1 = \gamma_2$. Further, the group G_0 has order $2^7 \cdot 3^6 \cdot 5 \cdot 7$.*

It is obvious from Lemma 17, Sylow's theorem and the fact that $B = W\langle v\rangle \subseteq G_0$, that G_0 has no subgroup of index 2 and $N_{G_0}(W) = W\langle v\rangle$. Now the group H_0 (see Lemma 19) has order $2^7 \cdot 3^2$ and is isomorphic to the centralizer of an involution in $U_4(3)$. Obviously H_0 is a subgroup of G_0. Since H_0 is a maximal subgroup of H and $v_1 \notin G_0$, we have $C_{G_0}(z) = H_0$.

Hence, by a result of Phan [7], $G_0 \simeq U_4(3)$. Now, suppose G_0 is not a maximal subgroup of G. Let G_1 be a proper subgroup of G such that $G_1 \supset G_0$. Since an S_2-subgroup of G_0 is an S_2-sugroup of G, the group G_1 has no subgroup of index 2. Hence by a result of Phan [7], $C_{G_1}(z) \supset H_0$ and so $C_{G_1}(z) = H$. Also $G_1 \neq H \cdot O_{2'}(G_1)$ and so by Lemma 6, G_1 has precisely one class of involutions; and if z_1 is any involution in G_1, then $C_G(z_1) \subseteq G_1$. Also, if T is an S_2-subgroup of G_1, then T is also an S_2-subgroup of G; and since $Z(T)$ is cyclic, $N_G(T) \subseteq G_1$. Now suppose all involutions of G lie in G_1. Then the group $\Omega_1(G)$ generated by all involutions of G is contained in G_1 and is a normal subgroup of G. Obviously $\Omega_1(G) \supseteq G_0$ as G_0 is a simple group. Now $N_{G_0}(W) = W \cdot \langle v \rangle$ and so by Lemma 17 and the Frattini argument, we have $[G : \Omega_1(G)] = 2$. But this contradicts the fact that an S_2-subgroup of G has order 2^7. Hence there are involutions of G in $G \backslash G_1$. It then follows that G contains a strongly embedded subgroup in the sense of Bender [1]; and from Bender's classification of finite groups with a strongly embedded subgroup we arrive at a contradiction. Hence we have

LEMMA 25. *The group G_0 is isomorphic to $U_4(3)$ and G_0 is a maximal subgroup of G.*

5. IDENTIFICATION OF G WITH MCLAUGHLIN'S GROUP Mc

We shall now determine the action of the element v_1 on the S_3-subgroup W of G. From the relations in (3) and (4),

$$\varphi_1^{v_1} = \varphi_1{}^i\varphi_2{}^j, \qquad \varphi_2^{v_1} = \varphi_1{}^j\varphi_2^{-i}, \qquad \text{and} \qquad \varphi_1^{v_1 u} = \varphi_1{}^i\varphi_2^{-j},$$
$$\text{where} \qquad i, j \in \{0, \pm 1\}. \tag{10}$$

But from (1), we have $(v_1u)^2 = v$ and so $\varphi_1^{v_1 u} = \varphi_1^{vuv_1^{-1}} = \varphi_2^{v_1}$. Hence in (10), we must have $i = j$, i.e.,

$$\varphi_1^{v_1} = \varphi_1{}^i\varphi_2{}^j \qquad \text{and} \qquad \varphi_2^{v_1} = \varphi_1{}^i\varphi_2^{-j}, \qquad \text{where} \qquad i = \pm 1. \tag{11}$$

Again, represent the elements a_2, σ_2, v_1 as linear transformations on the vector space $M/\langle\sigma_1\rangle$ over the field of 3 elements. In terms of the basis $\varphi_1\langle\sigma_1\rangle$, $\varphi_2\langle\sigma_1\rangle$, $\varphi_3\langle\sigma_1\rangle$, and $\varphi_4\langle\sigma_1\rangle$, we have from (5), (11) and Lemma 21, the following representations of a_2, σ_2 and v_1:

$$a_2 \to \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

$$\sigma_2 \to \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix},$$

and

$$v_1 \to \begin{bmatrix} \begin{matrix} i & i \\ i & -i \end{matrix} & 0 \\ C & D \end{bmatrix}$$

where I is the 2×2 identity matrix and C, D are 2×2 matrices.

From the relations $\sigma_2^{v_1} = \sigma_2^{-1}$ and $(v_1 a_2)^3 = 1$, we have

$$v_1 \to \begin{bmatrix} \begin{matrix} i & i \\ i & -i \end{matrix} & 0 \\ -I & \begin{matrix} -i & -i \\ -i & i \end{matrix} \end{bmatrix}.$$

So with (4) and (6), we have

$$\varphi_3^{v_1} = \sigma_1{}^{\epsilon}\varphi_1^{-1}\varphi_3^{-i}\varphi_4^{-i} \quad \text{and} \quad \varphi_4^{v_1} = \sigma_1^{-\epsilon}\varphi_2^{-1}\varphi_3^{-i}\varphi_4{}^{i}, \qquad \text{where} \qquad \epsilon = 0 \text{ or } \pm 1.$$

Now, conjugating these relations by z and using Lemma 21, we find that $\epsilon = -i$ and so

$$\begin{aligned} \varphi_3^{v_1} &= \sigma_1^{-i}\varphi_1^{-1}\varphi_3^{-i}\varphi_4^{-i}, \\ \varphi_4^{v_1} &= \sigma_1{}^{i}\varphi_2^{-1}\varphi_3^{-i}\varphi_4{}^{i}, \end{aligned} \tag{12}$$

with i as in (11).

Then using the relations in (11), (12) and Lemma 21, we find that $i = 1$ and so we have proved

$$\begin{aligned} \varphi_1^{v_1} &= \varphi_1\varphi_2, & \varphi_2^{v_1} &= \varphi_1\varphi_2^{-1}, \\ \varphi_3^{v_1} &= \sigma_1^{-1}\varphi_1^{-1}\varphi_3^{-1}\varphi_4^{-1} \quad \text{and} & \varphi_4^{v_1} &= \sigma_1\varphi_2^{-1}\varphi_3^{-1}\varphi_4. \end{aligned} \tag{13}$$

The structure of the group $N_G(W) = W \cdot \langle v, v_1 \rangle$ is now completely determined. Our aim now is to show that the group G has precisely three distinct (G_0, G_0)-double cosets. We know from Lemmas 24 and 25, that the group $G_0 = BAB$, where $B = W\langle v \rangle$ and $A = \langle v, u, a_2 \rangle$ is isomorphic to $U_4(3)$; and the 8 distinct (B, B)-double cosets of G_0 are

$$\begin{aligned} &B, \quad BuB, \quad Ba_2B, \quad Ba_2uB, \quad Bua_2B, \\ &Ba_1B, \quad Ba_1a_2B \quad \text{and} \quad Ba_1a_2uB, \qquad \text{with} \qquad a_1 = ua_2u. \end{aligned} \tag{14}$$

From the structure of $U_4(3)$, $N_{G_0}(W) = W\langle v\rangle$. Since $v_1 \in N_G(W)$, we have $v_1 \notin G_0$ and hence by Lemma 25, $G = \langle G_0, v_1\rangle$. If we conjugate the 8 (B, B)-double cosets in (14) by v_1 and using the relations in (1), we have

$$\begin{aligned} B^{v_1} &= B, & (BuB)^{v_1} &= BuB,\\ (Ba_2B)^{v_1} &= Ba_2v_1a_2B, & (Ba_2uB)^{v_1} &= Ba_2v_1a_2uB,\\ (Bua_2B)^{v_1} &= Bua_2v_1a_2B, & (Ba_1B)^{v_1} &= Ba_1v_1a_1B,\\ (Ba_1a_2B)^{v_1} &= Bv_1a_1a_2v_1B, \quad \text{and} & (Ba_1a_2uB)^{v_1} &= Bv_1a_1a_2v_1uB. \end{aligned} \tag{15}$$

We observe from (15) that $(B\gamma B)^{v_1} \subseteq G_0 \cup G_0v_1G_0$ for all $\gamma \in A$, except when $\gamma \in a_1a_2\langle v, u\rangle$. Hence, if $v_1a_1a_2v_1 \in G_0 \cup G_0v_1G_0$, then we see from (14) and (15) that the set $G_0 \cup G_0v_1G_0$ is a group and so $G = G_0 \cup G_0v_1G_0$. Since $C_{G_0}(z) = H_0$, $v_1a_1a_2v_1 \notin G_0$ and so $v_1a_1a_2v_1 \in G_0v_1G_0$. Then from (15), we see that $G_0 \cap G_0^{v_1} = B \cup BuB$. But u normalizes the group N and so by Lemma 14, we have $\langle B, u\rangle/N \simeq A_6$. Since $\langle B, u\rangle = B \cup BuB$, we have $| G_0 \cap G_0^{v_1} | = 2^3 \cdot 3^6 \cdot 5$. Hence $| G | = | G_0 |(1 + 2^4 \cdot 7) = 2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 113$ and this contradicts Lemma 18. Hence, the three (G_0, G_0)-double cosets G_0, $G_0v_1G_0$ and $G_0v_1a_1a_2v_1G_0$ are all distinct. Our aim is to show that

$$G = G_0 \cup G_0v_1G_0 \cup G_0v_1a_1a_2v_1G_0 \tag{16}$$

We let $G_1 = G_0 \cup G_0v_1G_0 \cup G_0v_1a_1a_2v_1G_0$. To prove (16), it suffices by Lemma 25 to show that G_1 is a group. Hence, we have to verify the following three conditions:

$$v_1g_0v_1 \in G_1 \tag{17}$$

$$v_1g_0v_1a_1a_2v_1 \in G_1 \tag{18}$$

and

$$v_1a_1a_2v_1g_0v_1a_1a_2v_1 \in G_1 \tag{19}$$

where g_0 is an element in G_0.

However, (17) is true because of (14) and (15). Further, since $a_1a_2 \in G_0$, condition (19) will hold if we have verified (18). Hence it only remains to verify (18). If $g_0 \in G_0$ such that $v_1g_0v_1 \in G_0 \cup G_0v_1G_0$, then by (17) we have $v_1g_0v_1a_1a_2v_1 \in G_1$. Hence we only need to verify (18) for those $g_0 \in G_0$ such that $v_1g_0v_1 \notin G_0 \cup G_0v_1G_0$. From (15), we see that $g_0 \in Ba_1a_2B \cup Ba_1a_2uB$. But $(v_1u)^2 = v$, and, since v_1 normalizes B, we only need to verify (18) for those $g_0 \in Ba_1a_2B$. Now, using the relations in (1), (3), (5), and (13), and the fact that v_1 normalizes B, we can assume that $g_0 = a_1a_2\sigma_1{}^i\sigma_2{}^j\varphi_1{}^k\varphi_2{}^\ell$, with $i, j, k, \ell \in \{0, \pm 1\}$. Hence, to show that G_1 is a group it suffices to verify that

$$v_1a_1a_2\sigma_1{}^i\sigma_2{}^j\varphi_1{}^k\varphi_2{}^\ell v_1a_1a_2v_1 \in G_1, \qquad \text{where} \qquad i, j, k, \ell \in \{0, \pm 1\}. \tag{20}$$

We first prove the following lemmas:

LEMMA 26. *The element* $v_1 a_2 u a_2 \sigma_1 \varphi_1^{-1} \varphi_3^{-1} \varphi_4^{-1} u a_2 v_1$ *lies in the double coset* $G_0 v_1 G_0$.

Proof. We have

$$\begin{aligned} v_1 a_2 u a_2 \sigma_1 \varphi_1^{-1} \varphi_3^{-1} \varphi_4^{-1} u a_2 v_1 &= v_1 a_2 u (a_2 \sigma_1 \varphi_1^{-1} \varphi_3^{-1} \varphi_4^{-1} a_2^{-1})\, a_2 u a_2 v_1 \\ &= v_1 a_2 u \sigma_1 \varphi_3 \varphi_1^{-1} \varphi_2^{-1} a_2 u a_2 v_1 \\ &= v_1 a_2 u \varphi_1^{-1} \varphi_3 \varphi_2^{-1} a_2 u a_2 v_1 \\ &= v_1 a_2 (u \varphi_1^{-1} \varphi_2^{-1} u)\, u \varphi_3 a_2 u a_2 v_1 \\ &= v_1 a_2 \varphi_1^{-1} \varphi_2 u \varphi_3 a_2 u a_2 v_1 \\ &= v_1 (a_2 \varphi_1^{-1} \varphi_2 a_2^{-1})\, a_2 u \varphi_3 a_2 u a_2 v_1 \\ &= v_1 \varphi_3 \varphi_4^{-1} a_2 u \varphi_3 a_2 u a_2 v_1 \\ &= (v_1 \varphi_3 \varphi_4^{-1} v_1^{-1})\, v_1 a_2 u \varphi_3 a_2 u a_2 v_1 \\ &= w_0 v_1 a_2 u \varphi_3 a_2 u a_2 v_1, \qquad \text{with} \qquad w_0 \in B \\ &= w_0 v_1 a_2 u a_2 u a_2 (\varphi_3^{a_2 u a_2})\, v_1 \\ &= w_0 v_1 u a_2^{-1} u (\varphi_3^{-1})\, v_1 . \end{aligned}$$

But by (15), the element $v_1 u a_2^{-1} u \varphi_3^{-1} v_1$ lies in $B a_1 v_1 a_1 B$ and since $B \subseteq G_0$ and $a_1 \in G_0$, the lemma is proved.

Now let $w = \sigma_1^{-1} \varphi_1 \varphi_3 \varphi_4$; then from (6) and (13), we have

$$w^{-1} \sigma_2^{-1} w = \sigma_1 \sigma_2^{-1} \varphi_1^{-1} \varphi_2^{-1}, \qquad (\sigma_2^{-1})^{v_1 a_1 a_2 v_1} = \sigma_2^{a_2 v_1}$$

and

$$w^{v_1 a_1 a_2 v_1} = (\varphi_3^{-1})^{u v_1} = g_0 \in G_0 .$$

with these relations and Lemma 26, we can prove

LEMMA 27. *The element* $v_1 a_1 a_2 \sigma_1 \sigma_2^{-1} \varphi_1^{-1} \varphi_2^{-1} v_1 a_1 a_2 v_1$ *lies in the double coset* $G_0 v_1 G_0$.

Proof. From (1), we have $(v_1 a_1 a_2)^4 = z$ and so

$$\begin{aligned} v_1 a_1 a_2 \sigma_1 \sigma_2^{-1} \varphi_1^{-1} \varphi_2^{-1} v_1 a_1 a_2 v_1 &= a_1 a_2 v_1 a_1 a_2 (\sigma_1 \sigma_2^{-1} \varphi_1^{-1} \varphi_2^{-1})^{v_1 a_1 a_2 v_1} \\ &= a_1 a_2 v_1 a_1 a_2 (w^{-1} \sigma_2^{-1} w)^{v_1 a_1 a_2 v_1} \\ &= a_1 a_2 v_1 a_1 a_2 (v_1^{-1} u \varphi_3 u v_1)(v_1 a_2 \sigma_2 a_2 v_1)\, g_0 , \\ &\qquad \text{with} \quad g_0 \in G_0 . \end{aligned}$$

Now using the relations in (1) and the fact that $(a_2\sigma_2)^3 = 1$, we have

$$\begin{aligned} v_1a_1a_2\sigma_1\sigma_2^{-1}\varphi_1^{-1}\varphi_2^{-1}v_1a_1a_2v_1 &= h_0v_1a_2ua_2v_1^{-1}\varphi_3u\sigma_2^{-1}a_2v_1k_0\,, \quad \text{with} \quad h_0\,, k_0 \in G_0\,; \\ &= h_0v_1a_2ua_2v_1^{-1}\varphi_3\sigma_1^{-1}ua_2v_1k_0 \\ &= h_0v_1a_2ua_2(v_1^{-1}\varphi_3\sigma_1^{-1}v_1)\, v_1^{-1}ua_2v_1k_0 \\ &= h_0v_1a_2ua_2\sigma_1\varphi_1^{-1}\varphi_3^{-1}\varphi_4^{-1}v_1^{-1}ua_2v_1k_0\,. \end{aligned}$$

But $(v_1u)^2 = v$ and $(v_1a_2)^3 = 1$ and so

$$v_1a_1a_2\sigma_1\sigma_2^{-1}\varphi_1^{-1}\varphi_2^{-1}v_1a_1a_2v_1 = h_0'(v_1a_2ua_2\sigma_1\varphi_1^{-1}\varphi_3^{-1}\varphi_4^{-1}ua_2v_1)\, k_0',$$

with h_0', $k_0' \in G_0$. But by lemma 26, the element in brackets lies in $G_0v_1G_0$ and so the lemma is proved.

We shall now proceed to verify (20):

(I) All elements of the form $v_1a_1a_2\sigma_1{}^i\sigma_2{}^jv_1a_1a_2v_1$, with i, $j \in \{0, \pm 1\}$ lie in H and so by Lemma 19, these elements lie in G_1.

(II) Now consider elements of the form $v_1a_1a_2\sigma_1{}^i\varphi_1{}^j\varphi_2{}^kv_1a_1a_2v_1$, where $i, j, k \in \{0, \pm 1\}$. Using the relations in (1) and (5), we have

$$\begin{aligned} v_1a_1a_2\sigma_1{}^i\varphi_1{}^j\varphi_2{}^kv_1a_1a_2v_1 &= v_1a_1(a_2\sigma_1{}^i\varphi_1{}^j\varphi_2{}^ka_2^{-1})\, a_2v_1a_1a_2v_1 \\ &= v_1a_1\sigma_1{}^i\varphi_3^{-j}\varphi_4^{-k}a_2v_1a_2a_1v_1 \\ &= v_1a_1\sigma_1{}^i\varphi_3^{-j}\varphi_4^{-k}v_1a_2a_1v_1va_1 \\ &= v_1a_1v_1(v_1^{-1}\sigma_1{}^i\varphi_3^{-j}\varphi_4^{-k}v_1)\, a_2a_1v_1va_1 \\ &= v^{-1}a_1v_1a_1(v_1^{-1}\sigma_1{}^i\varphi_3^{-j}\varphi_4^{-k}v_1)\, a_2a_1v_1va_1\,. \end{aligned}$$

But the element $v_1^{-1}\sigma_1{}^i\varphi_3^{-j}\varphi_4^{-k}v_1 \in G_0$ and so by (14) and (15),

$$v_1a_1a_2\sigma_1{}^i\varphi_1{}^j\varphi_2{}^kv_1a_1a_2v_1 \in G_1\,.$$

(III) From the relations in (1) and (3), we have

$$\begin{aligned} u(v_1a_1a_2\sigma_2{}^i\varphi_1{}^j\varphi_2{}^kv_1a_1a_2v_1)u &= vv_1a_1a_2\sigma_1{}^i\varphi_1{}^j\varphi_2^{-k}v_1v^{-1}a_1a_2v_1^{-1}v \\ &= vv_1a_1a_2\sigma_1{}^i\varphi_1{}^j\varphi_2^{-k}v_1a_1a_2v_1\,. \end{aligned}$$

Hence by (II) above, elements of the form $v_1a_1a_2\sigma_2{}^i\varphi_1{}^j\varphi_2{}^kv_1a_1a_2v_1 \in G_1$, where $i, j, k \in \{0, \pm 1\}$.

(IV) Now by Lemma 27, the element $w = v_1a_1a_2\sigma_1\sigma_2^{-1}\varphi_1^{-1}\varphi_2^{-1}v_1a_1a_2v_1$ lies in $G_0v_1G_0$ and so the element

$$w^* = v_1a_1a_2\sigma_1\sigma_2^{-1}\varphi_1\varphi_2v_1a_1a_2v_1 = zwz \in G_0v_1G_0\,.$$

But

$$v_1a_1a_2\sigma_1\sigma_2\varphi_1\varphi_2v_1a_1a_2v_1 = w^*v_1a_2\sigma_2a_2v_1$$
$$= w^*v_1\sigma_2^{-1}a_2^{-1}\sigma_2^{-1}v_1$$
$$= w^*\sigma_2v_1a_2^{-1}v_1\sigma_2$$
$$= w^*\sigma_2a_2v_1a_2\sigma_2 .$$

Since $w^* \in G_0v_1G_0$, we have by (14) and (15) that the element

$$v_1a_1a_2\sigma_1\sigma_2\varphi_1\varphi_2v_1a_1a_2v_1 \in G_1 .$$

Now $u(v_1a_1a_2\sigma_1\sigma_2\varphi_1\varphi_2v_1a_1a_2v_1)u = vv_1a_1a_2\sigma_1\sigma_2\varphi_1\varphi_2^{-1}v_1a_1a_2v_1$ and so the element $v_1a_1a_2\sigma_1\sigma_2\varphi_1\varphi_2^{-1}v_1a_1a_2v_1 \in G_1$. Similarly, we can show that all the elements of the form

$$v_1a_1a_2\sigma_1\sigma_2\varphi_1{}^i\varphi_2{}^jv_1a_1a_2v_1 \in G_1 , \qquad \text{where} \quad i, j \in \{1, -1\}.$$

Next, consider the element $\ell = v_1a_1a_2\sigma_1\sigma_2\varphi_1{}^iv_1a_1a_2v_1$, where $i \in \{1, -1\}$. The element u centralizes $a_1a_2\sigma_1\sigma_2\varphi_1{}^i$ and $v_1a_1a_2v_1$ and so the element

$$(a_1a_2\sigma_1\sigma_2\varphi_1{}^i)^{v_1a_1a_2v_1} \in C_G(u).$$

But by (2), there exists an element $\mu \in L \subseteq G_0$ such that $\mu^{v_1} \in N \cdot L \subseteq G_0$; and $z^\mu = u$, $u^\mu = uz$. Hence, $v_1a_1a_2\ell \in C_G(z^\mu)$. But by Lemma 19, $C_G(z) = H_0 \cup H_0v_1H_0 \cup H_0v_1a_1a_2v_1H_0$ and so $v_1a_1a_2\ell$ lies in one of the following sets: $\mu^{-1}H_0\mu$, $\mu^{-1}H_0v_1H_0\mu$, or $\mu^{-1}H_0v_1a_1a_2v_1H_0\mu$. If $v_1a_1a_2\ell$ lies in $\mu^{-1}H_0\mu$ or $\mu^{-1}H_0v_1H_0\mu$, then it follows easily that ℓ will then lie in G_1. If $v_1a_1a_2\ell = \mu^{-1}h_0v_1a_1a_2v_1k_0\mu$, where h_0, $k_0 \in H_0$, then the element

$$\ell = a_1a_2v_1^{-1}\mu^{-1}h_0v_1a_1a_2v_1k_0\mu = a_1a_2(v_1^{-1}\mu^{-1}v_1)(v_1^{-1}h_0v_1a_1a_2v_1)\, k_0\mu.$$

But $v_1^{-1}\mu v_1 \in G_0$ and $v_1^{-1}h_0v_1a_1a_2v_1 \in C_G(z)$ and so $\ell \in G_1$, i.e.,

$$v_1a_1a_2\sigma_1\sigma_2\varphi_1{}^iv_1a_1a_2v_1 \in G_1 \qquad \text{where} \qquad i \in \{1, -1\}.$$

Similarly, we can prove that the elements of the form $v_1a_1a_2\sigma_1\sigma_2\varphi_2{}^iv_1a_1a_2v_1$, where $i \in \{1, -1\}$ lie in G. (In this case we use the element uz instead of u.) Hence we have proved that all elements of the form

$$v_1a_1a_2\sigma_1\sigma_2\varphi_1{}^i\varphi_2{}^jv_1a_1a_2v_1 \in G_1 , \qquad \text{where} \quad i, j \in \{0, \pm 1\}. \tag{21}$$

(V) Now let $x = v_1a_1a_2\sigma_1^{-1}\sigma_2\varphi_1{}^i\varphi_2{}^jv_1a_1a_2v_1$, with $i, j \in \{0, \pm 1\}$. Then $x^{-1} = v_1a_1a_2v_1\sigma_1\sigma_2^{-1}\varphi_1^{-i}\varphi_2^{-j}a_1a_2v_1z = v_1a_1a_2\sigma_1\sigma_2\varphi_1{}^k\varphi_2{}^\ell v_1a_1a_2v_1z$, where k, $\ell \in \{0, \pm 1\}$. Hence by (21) the element $x^{-1} \in G_1$ and so $x \in G_1$. From the relations in (1) and Lemma 21, we can prove in a similar way that elements of the form

$$v_1a_1a_2\sigma_1^{-1}\sigma_2^{-1}\varphi_1{}^i\varphi_2{}^jv_1a_1a_2v_1 \qquad \text{or} \qquad v_1a_1a_2\sigma_1\sigma_2^{-1}\varphi_1{}^i\varphi_2{}^jv_1a_1a_2v_1$$

all lie in G_1. With this, we have verified (20) and so we have proved

LEMMA 28. *The elements in $G_1 = G_0 \cup G_0v_1G_0 \cup G_0v_1a_1a_2v_1G_0$ form a group and so $G = G_1$.*

Now $G_0^{v_1} \cap G_0 = \langle B, u\rangle$ and by Lemma 14 we have $|G_0^{v_1} \cap G_0| = 2^3 \cdot 3^6 \cdot 5$. Hence $[G_0 : G_0^{v_1} \cap G_0] = 112$. Next, the group

$$G_0^{v_1a_1a_2v_1} \cap G_0 \supseteq \langle H_0^{v_1a_1a_2v_1} \cap H_0, w\rangle,$$

where $w = \sigma_1^{-1}\varphi_1\varphi_3\varphi_4$ and since $|H_0^{v_1a_1a_2v_1} \cap H_0| = 2^6$ (see Lemma 19), we have $2^6 \cdot 3 \mid |G_0^{v_1a_1a_2v_1} \cap G_0|$. By Lemma 18, the order of G is divisible by 5^3 and so $n = [G_0 : G_0^{v_1a_1a_2v_1} \cap G_0] \equiv 2 \pmod 5$. Hence $n \mid 2 \cdot 3^5 \cdot 7$ and so $n = 2 \cdot 3 \cdot 7$, $2 \cdot 3^5 \cdot 7$ or $2 \cdot 3^4$. However, if $n = 2 \cdot 3 \cdot 7$ or $2 \cdot 3^5 \cdot 7$ then $|G|$ is not divisible by 5^3, contradicting Lemma 18; and so we must have $n = 2 \cdot 3^4 = 162$. Hence

$$|G| = 2^7 \cdot 3^6 \cdot 5 \cdot 7(1 + 112 + 162) = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$$

and we have

LEMMA 29. *The group G is a primitive transitive rank 3 permutation group on 275 points, for which the stabilizer of a point G_a is isomorphic to $U_4(3)$ with subdegrees 1, 112 and 162.*

We shall now consider the graph associated with the rank 3 permutation group G on the 275 conjugates in G of the subgroup G_0 which is isomorphic to $U_4(3)$. (for a description of the construction of the graph, see Wales [11]). The group G is then a subgroup of the full automorphism group of the graph. Our aim is to show that this graph is isomorphic to the graph in McLaughlin [6]. We shall use the notation in [11] and set Ω to be the set of all conjugates in G of G_0. So by Lemma 29, we have $|\Omega| = 275$ and for $\{a\}$ in Ω, $G_a \simeq U_4(3)$. The orbits of G_a are $\{a\}$, $\Delta(a)$, and $\Gamma(a)$, where $k = |\Delta(a)| = 112$ and $\ell = |\Gamma(a)| = 162$. From the relation $\mu\ell = k(k-\lambda-1)$, we have $\mu = 56$ and $\lambda = 30$. Now let $b \in \Delta(a)$ and $c \in \Gamma(a)$. Then $|G_{ab}| = 2^3 \cdot 3^6 \cdot 5$ and $|G_{ac}| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$. Todd [10] has determined the character table of $U_4(3)$ and he has also determined all possible subgroups of index $\leqslant 200$ in $U_4(3)$. From his work, we see that all subgroups of the order $2^3 \cdot 3^6 \cdot 5 = |G_{ab}|$ are conjugate in $U_4(3)$ and all subgroups of order

$2^6 \cdot 3^2 \cdot 5 \cdot 7 = |G_{ac}|$ are isomorphic to $L_3(4)$. There are precisely 2 conjugate classes of subgroups in $U_4(3)$ which are isomorphic to $L_3(4)$ but there are conjugate in the automorphism group of $U_4(3)$. From this it follows that the permutation representations of G_a on $\Delta(a)$ and G_a on $\Gamma(a)$ are unique. Hence it follows that G_a is a primitive rank 3 permutation group on the 112 points in $\Delta(a)$ with subdegrees 1, 30 and 81; and G_a has subdegrees 1, 56, and 105 on $\Gamma(a)$. The point b is joined to $\lambda = 30$ points in $\Delta(a)$ which is a union of orbits of G_{ab} on $\Delta(a)$, and since there is precisely one orbit of length 30 of G_{ab} on $\Delta(a)$, this union is unique. Similarly, the union of orbits of G_{ac} on $\Gamma(a)$ is unique. From the work of Todd [10], we see that G_{ab} has precisely 2 orbits on $\Gamma(a)$, each of length 81. Hence at this stage, we have 2 possible graphs associated with the rank 3 permutation group G. However, it can be shown that these 2 graphs are isomorphic in the automorphism group of G_a. Hence, the graph associated with the rank 3 permutation group G is unique and is isomorphic to the graph defined by McLaughlin [6]. Since McLaughlin has shown that the group Mc is of index 2 in the full automorphism group of this graph and $|G| = |\text{Mc}|$, we must have that G is isomorphic to the McLaughlin's simple group Mc of order 898,128,000. The theorem is proved.

Finally, we shall prove the two remarks mentioned in the introduction: From (1), we have $(v_1a_1a_2)^4 = z$ and so $(v_1a_1a_2)^{a_1a_2} = (v_1a_1a_2)^3$. Hence $\langle v_1a_1a_2, a_1a_2\rangle$ is a semidihedral group of order 16 and so the group $G = G_0\langle v_1, a_1a_2\rangle G_0$, where $G_0 \simeq U_4(3)$ and $\langle v_1, a_1a_2\rangle = \langle v_1a_1a_2, a_1a_2\rangle$ is a semidihedral group of order 16. Next, since $[G : H] = 3^4 \cdot 5^2 \cdot 11$ and G has precisely one class of involutions, there are precisely $3^4 \cdot 5^2 \cdot 11$ involution in G. Since $[G_0 : H_0] = 3^4 \cdot 5 \cdot 7$ and G_0 has 1 class of involutions, there are $3^4 \cdot 5 \cdot 7$ involutions in G_0. It can be shown that $|C_{G_0}(v_1a_1a_2v_1)| = 2^3 \cdot 3 \cdot 7$ and so there are $[G_0 : C_{G_0}(v_1a_1a_2v_1)] = 2^4 \cdot 3^5 \cdot 5$ involutions in the double coset $G_0v_1a_1a_2v_1G_0$. But $3^4 \cdot 5 \cdot 7 + 2^4 \cdot 3^5 \cdot 5 = 3^4 \cdot 5^2 \cdot 11$ and so there are no involution in the double coset $G_0v_1G_0$.

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