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On subalgebras of $n \times n$ matrices not satisfying identities of degree $2n - 2$

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Abstract

The Amitsur–Levitzki theorem asserts that $M_n(F)$ satisfies a polynomial identity of degree $2n$. (Here, F is a field and $M_n(F)$ is the algebra of $n \times n$ matrices over F.) It is easy to give examples of subalgebras of $M_n(F)$ that do satisfy an identity of lower degree and subalgebras of $M_n(F)$ that satisfy no polynomial identity of degree $\leqslant 2n - 2$. In this paper we prove that the subalgebras of $n \times n$ matrices satisfying no nonzero polynomial of degree less than $2n$ are, up to F-algebra isomorphisms, the class of full block upper triangular matrix algebras.

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1. Introduction

This paper is concerned with $n \times n$ matrix subalgebras that do not satisfy a polynomial identity of degree $< 2n$. Our aim is to present and prove the following theorem: *Let F be a field and let A be an F-subalgebra of $M_n(F)$. If A does not satisfy the standard polynomial s_{2n-2}, then A is equivalent to a full block upper triangular matrix algebra.*

To begin, let F be a field, $M_n(F)$ the algebra of $n \times n$ matrices over F, and $F\{X\} = F\{X_1, X_2, \ldots\}$ the free associative algebra over F in countably many

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variables. A nonzero polynomial $f(X_1, \dots, X_m) \in F\{X\}$ is a *polynomial identity* for an F-algebra R (or, R satisfies f) if $f(r_1, \dots, r_m) = 0$ for all $r_1, \dots, r_m \in R$.

The standard polynomial of degree t is

$$s_t(X_1, \dots, X_t) = \sum_{\sigma \in S_t} (\mathrm{sg}\ \sigma) X_{\sigma(1)} X_{\sigma(2)} \dots X_{\sigma(t)},$$

where S_t is the symmetric group on $\{1, \dots, t\}$ and $(\mathrm{sg}\ \sigma)$ is the sign of the permutation $\sigma \in S_t$. The standard polynomial s_t is homogeneous of degree t, multilinear and alternating. If t is odd then $s_t(1, X_2, \dots, X_t) = s_{t-1}(X_2, \dots, X_t)$. Thus s_{2t} is an identity of R if and only if s_{2t+1} is an identity of R. The standard polynomial s_{q+r} is a linear combination, with coefficients being 1 or -1, of evaluations of $s_q s_r$. This can be shown as follows: We partition the set of permutations S_{q+r} by defining the equivalence relation $\tau \sim \sigma$ if the images of the interval $[1, q]$ under τ and σ are the same set. Then, we have

$$s_t(X_1, \dots, X_t) = \sum_{\bar{\sigma} \in S_t/\sim} (\mathrm{sg}\ \sigma) s_q(X_{\sigma(1)}, \dots, X_{\sigma(q)}) s_r(X_{\sigma(q+1)}, \dots, X_{\sigma(t)}). \tag{1.1}$$

The Amitsur–Levitzki theorem asserts that $M_n(F)$ satisfies any standard polynomial of degree $2n$ or higher. Moreover, if $M_n(F)$ satisfies a polynomial of degree $2n$, then it is a scalar multiple of s_{2n} (cf. [1]).

The standard polynomial s_{2n} is a minimal identity in the sense that $M_n(F)$ satisfies no polynomial identity of degree less than $2n$. More generally, if A is a subalgebra of $M_n(F)$ isomorphic to a full block upper triangular matrix algebra,

$$\begin{pmatrix} \boxed{*} & & & \\ & \boxed{*} & & * \\ & & \ddots & \\ & 0 & & \boxed{*} \end{pmatrix},$$

then A satisfies no polynomial identity of degree less than $2n$. To prove this assertion, note that every full block upper triangular matrix algebra contains the "staircase sequence" $e_{11}, e_{12}, e_{22}, e_{23}, \dots, e_{(n-1)(n-1)}, e_{(n-1)n}, e_{nn}$, and

$$s_{2n-1}\left(e_{11}, e_{12}, e_{22}, e_{23}, \dots, e_{(n-1)(n-1)}, e_{(n-1)n}, e_{nn}\right) = e_{1n}, \tag{1.2}$$

where the e_{ij} are the standard matrix units.

In Section 2 we provide the building blocks for the main theorem of this paper and its proof. This proof and some of its consequences are presented in Section 3. For polynomial identities in Ring Theory and the polynomial identities of $n \times n$ matrices, [2,4] are suggested general references.

2. Building blocks

Lemma 2.1. *Let A be a simple F-subalgebra of $M_n(F)$. Then either $A = M_n(F)$ or A satisfies the identity $s_{2n-2}(A) = 0$.*

Proof. By assumption, A is a a finite dimensional central simple algebra over its center k. Let K denote the algebraic closure of k; then $A \otimes_k K$ is a simple K-algebra in a natural way (cf. [4, §1.8]), with $\dim_K(A \otimes_k K) = \dim_k(A)$. Also, $A \otimes_k K \cong M_t(K)$ for some $t \leqslant n$. Suppose that A is a proper subalgebra of $M_n(F)$. It follows that $t < n$. Hence, by the Amitsur–Levitzki theorem, $A \otimes_k K$ satisfies s_{2n-2}, and the result follows since A is embedded as a k algebra in $A \otimes_k K$. $\square$

2.1. We now consider the case when A contains a "repetition". We will need some notation.

(i) Let $M_1, \ldots, M_t$ be matrices in A,

$$M_k = \begin{pmatrix} a_k & b_k & c_k \\ 0 & e_k & d_k \\ 0 & 0 & a_k \end{pmatrix},$$
$$a_k, c_k \in M_\ell(F), e_k \in M_m(F), b_k \in M_{\ell\times m}(F), d_k \in M_{m\times\ell}(F).$$

Given $1 \leqslant i < j \leqslant t$ and $\sigma \in S_t$, set

$$m_t^\sigma[i, j] = (\mathrm{sg}\ \sigma) a_{\sigma(1)} \ldots a_{\sigma(i-1)} b_{\sigma(i)} e_{\sigma(i+1)} \ldots e_{\sigma(j-1)} d_{\sigma(j)} a_{\sigma(j+1)} \ldots a_{\sigma(t)},$$

and denote by W the set of all matrix products

$$\{m_t^\sigma[i, j] : \sigma \in S_t \text{ and } 1 \leqslant i < j \leqslant t\}.$$

(ii) The projection ur returns the $\ell \times \ell$ upper right block of a matrix in A:

$$ur\begin{pmatrix} a & b & c \\ 0 & e & d \\ 0 & 0 & a \end{pmatrix} = c$$

(iii) Given $n \times n$ matrices $M_1, \ldots, M_t$, we say that a matrix product $M_1 \cdots M_t$ *formally contains* the factor $A_1 \cdots A_s$ if $A_1 = M_\ell, A_2 = M_{\ell+1}, \ldots, A_s = M_{\ell+s-1}$, for some $1 \leqslant \ell \leqslant t$. This notation is to distinguish to the case when $CA_1 \cdots A_s D = M_1 \cdots M_t$ as $n \times n$ matrices, for some matrices C and D. Further, if $\ell = 1$, we say that $M_1 \cdots M_t$ formally contains $A_1 \cdots A_s$ as *left factor*.

This is a good place to record a Lemma extracted from [1], which will be used later.

Lemma 2.2 (AL50, Lemma 1, 450–451). *If for an odd positive integer r we put $Y = X_{i+1} \cdots X_{i+r}$, and if s' denotes the sum of all terms of $s_m(X)$ containing the common factor Y, then*

$$s' = s_{m-r+1}(X_1, \ldots, X_i, Y, X_{i+r+1}, \ldots, X_m).$$

Lemma 2.3. *Set $t = 2(\ell + m)$, and let $M_1, \ldots, M_t$ be matrices in A such that for all $1 \leqslant k \leqslant t$,*

$$M_k = \begin{pmatrix} a_k & b_k & 0 \\ 0 & e_k & d_k \\ 0 & 0 & a_k \end{pmatrix},$$

$$\text{for } a_k \in M_\ell(F), e_k \in M_m(F), b_k \in M_{\ell\times m}(F), d_k \in M_{m\times\ell}(F).$$

Then $ur[s_t(M_1, \ldots, M_t)] = 0$.

Proof. First we observe that

$$ur[M_1 \cdots M_t] = \sum_{1\leqslant i<j\leqslant t} a_{\sigma(1)} \ldots a_{\sigma(i-1)} \times b_{\sigma(i)} e_{\sigma(i+1)} \ldots e_{\sigma(j-1)} d_{\sigma(j)} a_{\sigma(j+1)} \ldots a_{\sigma(t)},$$

which implies that

$$ur[s_t(M_1, \ldots, M_t)] = \sum_{\sigma\in S_t} \sum_{1\leqslant i<j\leqslant t} m_t^\sigma[i, j]. \tag{2.3}$$

To prove that $ur[s_t(M_1, \ldots, M_t)] = 0$, we split the right hand side into two summands:

$$ur[s_t(M_1, \ldots, M_t)] = \sum_{\sigma\in S_t} \sum_{\substack{1\leqslant i<j\leqslant t \\ j-i-1\geqslant 2m}} m_t^\sigma[i, j] + \sum_{\sigma\in S_t} \sum_{\substack{1\leqslant i<j\leqslant t \\ j-i\leqslant 2m}} m_t^\sigma[i, j]. \tag{2.4}$$

Our goal is to show that each summand in (2.4) is zero. To handle the first summand we introduce the following new equivalence relation on S_t. Given fixed $1 \leqslant i < j \leqslant t$, such that $j - i - 1 \geqslant 2m$, and given $\tau, \sigma \in S_t$, say that τ is $[i, j]$-*equivalent* to σ if τ restricted to the initial and final intervals $[1, i]$ and $[j, t]$ equals the restriction of σ to the same domain. In symbols,

$$\tau \sim_{[i,j]} \sigma \iff \tau|_{[1,i]} = \sigma|_{[1,i]} \text{ and } \tau|_{[j,t]} = \sigma|_{[j,t]}.$$

For each pair (i, j), such that $1 \leqslant i < j \leqslant t$ and $j - i - 1 \geqslant 2m$, the relation $\sim_{[i,j]}$ yields a partition of S_t into disjoint subsets $P^k_{[i,j]}$, $k = 1, \ldots, \frac{t!}{(j-i-1)!}$. Then, we have

$$\sum_{\sigma\in S_t} \sum_{\substack{1\leqslant i<j\leqslant t \\ j-i-1\geqslant 2m}} m_t^\sigma[i, j] = \sum_{\substack{1\leqslant i<j\leqslant t \\ j-i-1\geqslant 2m}} \sum_k \sum_{\sigma\in P^k_{[i,j]}} m_t^\sigma[i, j]$$
$$= \sum_{\substack{1\leqslant i<j\leqslant t \\ j-i-1\geqslant 2m}} \sum_k \sum_{\sigma\in P^k_{[i,j]}} (\operatorname{sg}\sigma) a_{\sigma(1)} \cdots a_{\sigma(i-1)}$$

$$\times\, b_{\sigma(i)}e_{\sigma(i+1)}\cdots e_{\sigma(j-1)}d_{\sigma(j)}a_{\sigma(j+1)}\cdots a_{\sigma(t)}$$
$$= \sum_{\substack{1\leqslant i<j\leqslant t\\ j-i-1\geqslant 2m}}\sum_k (\mathrm{sg}\ \sigma_k)a_{\sigma_k(1)}\cdots a_{\sigma_k(i-1)}b_{\sigma_k(i)}s d_{\sigma_k(j)}a_{\sigma_k(j+1)}\cdots a_{\sigma_k(t)},$$

where $s = s_{i-j+1}(e_{\sigma_k(i+1)},\ldots,e_{\sigma_k(j-1)})$ and σ_k is a representative of the class $P^k_{[i,j]}$. Since $j-i-1\geqslant 2m$,

$$s_{i-j+1}(e_{\sigma_k(i+1)},\ldots,e_{\sigma_k(j-1)}) = 0 \quad \text{for all } k,$$

hence

$$\sum_{\sigma\in S_t}\sum_{\substack{1\leqslant i<j\leqslant t\\ j-i-1\geqslant 2m}} m_t^\sigma[i,j] = 0.$$

This takes care of the first term in (2.4). We now turn to the second summand. For a given q, with $2\leqslant q\leqslant t$, denote by R_q the set of all q-tuples $r=(r_1,\ldots,r_q)$ of different elements from $\{1,\ldots,t\}$ and by $T_{(r_1,\ldots,r_q)}$ the set of matrix products w formally containing the common factor $b_{r_1}e_{r_2}\cdots e_{r_{q-1}}d_{r_q}$. Considering all possible q and q-tuples, the sets $T_{(r_1,\ldots,r_q)}$ form a partition of W. We are interested in the case when $q\leqslant 2m+1$. Observe that

$$\sum_{\sigma\in S_t}\sum_{\substack{1\leqslant i<j\leqslant t\\ j-i\leqslant 2m}} m_t^\sigma[i,j] = \sum_{q=2}^{2m+1}\sum_{r\in R_q}\sum_{w\in T_{(r_1,\ldots,r_q)}} w.$$

Fix q odd, a q-tuple $(r_1,\ldots,r_q)$, and the corresponding set of matrix products $T_{(r_1,\ldots,r_q)}$. Then, $\sum_{w\in T_{(r_1,\ldots,r_q)}} w$ is the sum of all matrix products formally containing the common factor $y = b_{r_1}e_{r_2}\cdots e_{r_{q-1}}d_{r_q}$. Each matrix product $w\in T_{(r_1,\ldots,r_q)}$ corresponds uniquely to a permutation $\sigma\in S_t$ and a pair (i,j), such that the q-tuple $(r_1,\ldots,r_q)$ is the image under σ of $(i,\ldots,j)$. Explicitly, the correspondence is $w = m_t^\sigma[i,j]$. We can now apply Lemma 2.2 and the alternating property of the standard polynomials. If $\sigma_0\in S_t$ is a fixed permutation such that

$$\sigma_0\colon i\to r_i,\quad 1\leqslant i\leqslant q,$$

we have

$$\sum_{w\in T_{(r_1,\ldots,r_q)}} w = (\mathrm{sg}\ \sigma_0)s_{t-q+1}\left(y, a_{\sigma_0(q+1)},\ldots,a_{\sigma_0(t)}\right)$$

where $y = b_{r_1}e_{r_2}\cdots e_{r_{q-1}}d_{r_q}$. Since $t-q+1\geqslant 2\ell$, and since all the arguments of s_{t-q+1} in the last equation are $\ell\times\ell$ matrices, it follows that

$$\sum_{w\in T_{(r_1,\dots,r_q)}} w = 0, \quad \text{when } q \text{ is odd and } (r_1,\dots,r_q) \text{ is a fixed } q\text{-tuple.} \tag{2.5}$$

Therefore

$$\sum_{\substack{q=2\\ q\,\mathrm{odd}}}^{2m+1} \sum_{r\in R_q} \sum_{w\in T_{(r_1,\dots,r_q)}} w = 0.$$

Suppose now that q is even, so $q \leqslant 2m$, and fix an arbitrary q-tuple $r = (r_1,\dots,r_q)$. We will split further the sets T_r. First consider all $w \in T_r$ formally containing in common the left factor $y = b_{r_1}e_{r_2}\cdots e_{r_{q-1}}d_{r_q}$, and call this subset L_r. Then, for each $r_0 \notin \{r_1,\dots,r_q\}$ consider the $(q+1)$-tuple (r_0, r) and the subset $G_{(r_0,r)}$ of $w \in T_r$ formally containing in common the factor $y = a_{r_0}b_{r_1}e_{r_2}\cdots e_{r_{q-1}}d_{r_q}$. The sum of all matrix products in the set T_r can be split as

$$\sum_{w\in T_r} w = \sum_{w\in L_r} w + \sum_{r_0:r_0\neq r_1,\dots,r_q}\ \sum_{w\in G_{(r_0,r)}} w.$$

For the terms in L_r we have

$$\sum_{w\in L_{(r_1,\dots,r_q)}} w = (\mathrm{sg}\ \sigma_0)\, y s_{t-q}\left(a_{\sigma_0(q+1)},\dots,a_{\sigma_0(t)}\right), \tag{2.6}$$

where $y = b_{r_1}e_{r_2}\cdots e_{r_{q-1}}d_{r_q}$, and where $\sigma_0 \in S_t$ is a fixed permutation such that

$$\sigma_0 : i \to r_i, \quad 1 \leqslant i \leqslant q.$$

Since $t - q \geqslant 2\ell$, we obtain

$$\sum_{w\in L_r} w = 0. \tag{2.7}$$

Finally, for a suitable fixed r_0, the sequence (r_0, r) has odd length, so we can argue as in (2.5) to obtain

$$\sum_{w\in G_{(r_0,r)}} w = (\mathrm{sg}\,\sigma_0)s_{t-q+1}\left(y, a_{\sigma_0(q+2)},\dots,a_{\sigma_0(t)}\right) = 0,$$

where $y = a_{r_0}b_{r_1}e_{r_2}\cdots e_{r_{q-1}}d_{r_q}$, and where $\sigma_0 \in S_t$ is a fixed permutation such that

$$\sigma_0 = \begin{cases} 1 \to r_0, & \\ i \to r_{i-1}, & \text{for} \quad 2 \leqslant i \leqslant q+1. \end{cases}$$

This finishes the proof of Lemma 2.3. $\square$

Proposition 2.1. *Let*

$$A=\left\{\begin{pmatrix}a & b & c\\ 0 & e & d\\ 0 & 0 & a\end{pmatrix}: a, c\in M_\ell(F), e\in M_m(F), b\in M_{\ell\times m}(F), d\in M_{m\times \ell}(F)\right\}.$$

Then, A satisfies $s_{2(\ell+m)}$.

Proof. For any t and matrices $M_k\in A$, $k=1\dots t$, set

$$M_k=\begin{pmatrix}a_k & b_k & c_k\\ 0 & e_k & d_k\\ 0 & 0 & a_k\end{pmatrix},$$
$$a_k, c_k\in M_\ell(F), e_k\in M_m(F), b_k\in M_{\ell\times m}(F), d_k\in M_{m\times\ell}(F).$$

By direct calculations, we obtain

$$ur[s_t(M_1,\dots,M_t)]=\sum_{i=1}^{t}s_t(a_1,\dots,a_{i-1},c_i,a_{i+1},\dots,a_t)$$
$$+\sum_{\sigma\in S_t}\sum_{1\leqslant i<j\leqslant t}m_t^\sigma[i,j].$$

Now set $t=2(\ell+m)$. It follows from (2.3) that

$$\sum_{\sigma\in S_t}\sum_{1\leqslant i<j\leqslant t}m_t^\sigma[i,j]=ur[s_t(M_1',\dots,M_t')]=0,$$

where M_k' is the matrix in A obtained by replacing the upper right corner c_k of M_k by $0\in M_\ell(F)$. Suitable applications of the Amitsur–Levitzki identity give us

$$ur[s_t(M_1,\dots,M_t)]=0,$$

$$s_t\left(\begin{pmatrix}a_1 & b_1\\ 0 & e_1\end{pmatrix},\dots,\begin{pmatrix}a_t & b_t\\ 0 & e_t\end{pmatrix}\right)=0,$$

and

$$s_t\left(\begin{pmatrix}e_1 & d_1\\ 0 & a_1\end{pmatrix},\dots,\begin{pmatrix}e_t & d_t\\ 0 & a_t\end{pmatrix}\right)=0.$$

Combining these three equations, it follows that $s_t(M_1,\dots,M_t)=0$. $\square$

3. Main theorem

In this section we prove that if an F-subalgebra of $M_n(F)$ does not satisfy the standard polynomial s_{2n-2}, then it is isomorphic as F-algebra to a full block upper triangular matrix algebra.

3.1. We first introduce our notation and review some necessary background (cf. [3]).

(i) Let t be a positive integer, let $\ell_1, \ell_2, \dots, \ell_t$ be positive integers summing up to n, and set

$$E_{(\ell_1,\ell_2,\dots,\ell_t)}(F) = \begin{pmatrix} M_{\ell_1}(F) & M_{\ell_1\times\ell_2}(F) & \cdots & M_{\ell_1\times\ell_{t-1}}(F) & M_{\ell_1\times\ell_t}(F) \\ 0 & M_{\ell_2}(F) & \cdots & M_{\ell_2\times\ell_{t-1}}(F) & M_{\ell_2\times\ell_t}(F) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & M_{\ell_{t-1}}(F) & M_{\ell_{t-1}\times\ell_t}(F) \\ 0 & 0 & \cdots & 0 & M_{\ell_t}(F) \end{pmatrix},$$

a full block upper triangular matrix subalgebra of $M_n(F)$.

(ii) Recall that every F-algebra automorphism τ of $M_n(F)$ is inner (i.e., there exists an invertible Q in $M_n(F)$ such that $\tau(a) = QaQ^{-1}$ for all $a \in M_n(F)$). We will say that two F-subalgebras A, A' of $M_n(F)$ are *equivalent* provided there exists an automorphism τ of $M_n(F)$ such that $\tau(A) = A'$.

(iii) We will say that a subalgebra Λ of $E_{(\ell_1,\ell_2,\dots,\ell_t)}(F)$ is an $(\ell_1, \ell_2, \dots, \ell_t)$-*extension of simple blocks* if the projections $\pi_i : \Lambda \to M_{\ell_i}(F)$, for $1 \leqslant i \leqslant t$, are all irreducible representations (when F is algebraically closed, of course, the representation π_i is irreducible if and only if $\pi_i(\Lambda) = M_{\ell_i}$). Note that, every F-subalgebra A of $M_n(F)$ is equivalent to an $(\ell_1, \ell_2, \dots, \ell_t)$-extension of simple blocks Λ for some suitable $(\ell_1, \ell_2, \dots, \ell_t)$.

Theorem 3.1. *Let F be a field and let A be an F-subalgebra of $M_n(F)$. If A does not satisfy the standard polynomial s_{2n-2}, then A is equivalent to a full block upper triangular matrix algebra.*

Proof. Without loss of generality we assume that A is an extension of simple blocks. We proceed by induction on t, the number of diagonal blocks in A. If $t = 1$, A is a simple algebra and therefore, in view of Lemma 2.1, A is a full matrix algebra. Now suppose that there are t diagonal blocks of sizes $\ell_1, \ell_2, \dots, \ell_t$, with $n = \ell_1 + \cdots + \ell_t$. The desired conclusion is: If A does not satisfy s_{2n-2}, then it is full block upper triangular. Assume that each matrix $B \in A$ has the form

$$B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1(t-1)} & B_{1t} \\ 0 & B_{22} & \cdots & B_{2(t-1)} & B_{2t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & B_{(t-1)(t-1)} & B_{(t-1)t} \\ 0 & 0 & \cdots & 0 & B_{tt} \end{pmatrix}.$$

If A does not satisfy s_{2n-2}, then it does not satisfy $s_{2(\ell_1+\ell_2+\cdots\ell_{t-1})-2}s_{2\ell_t}$. This implies that the image of A gotten by deleting the last row and the last column does not satisfy $s_{2(\ell_1+\ell_2+\cdots\ell_{t-1})-2}$, and by induction it is full block upper triangular. Similarly, the image of A gotten by deleting the first row and the first column is full block upper triangular. Since A can not be semisimple, A is full block upper triangular unless the projections $B \to B_{11}$ and $B \to B_{tt}$ are equivalent representations of A, which means that there is a fixed matrix T such that $TB_{11}T^{-1} = B_{tt}$ for all $B \in A$. But then Proposition 2.1 implies that A satisfies s_{2n-2}. □

Corollary 3.1. *The standard polynomial s_{2n-2} is an identity for any proper subalgebra of $U_n(F)$, the algebra of upper triangular matrices over the field F.*

Proof. Immediate from Theorem 3.1. □

Remark. The standard polynomial of degree $2n-2$ is not necessarily an identity for any proper subalgebra of $U_n(C)$ when C is a commutative ring: Let I be a nonzero ideal of C, and consider the C-subalgebra B of $U_n(C)$ defined by the property that the $(1, 2)$-entry of matrices in B lie in I. A staircase argument shows that $s_{2n-2}(B) \neq 0$.

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