Weighted composition operators acting between the $N_p$-space and the weighted-type space $H_\alpha^\infty$

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Abstract

In this paper, we study the boundedness and compactness of the weighted composition operator $uC_\phi$, which is induced by an analytic function $u$ on the unit disk and an analytic self-map $\phi$ of the unit disk, acting between the $N_p$-space and the weighted-type space $H_\alpha^\infty$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane, and let $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. For any $u \in H(\mathbb{D})$ and analytic self-map $\phi : \mathbb{D} \to \mathbb{D}$, the weighted composition operator $uC_\phi : H(\mathbb{D}) \to H(\mathbb{D})$ is defined by $uC_\phi f = u \cdot (f \circ \phi)$. This type of operator appears in studies on isometries of various function spaces. In fact, many isometries of function spaces are described as weighted composition operators. For details of these studies, we can refer the monograph [3]. Also many authors have studied the composition operator $C_\phi f = f \circ \phi$ and the multiplication operator $M_u f = u \cdot f$ on various analytic function spaces. In these studies, it is an interesting problem to relate operator-theoretic properties of $C_\phi$ or $M_u$ to function-theoretic properties of $\phi$ or $u$. Since $uC_\phi = M_u C_\phi$, it is natural to consider the boundedness or compactness of it on any analytic function spaces. Recently many authors have studied $uC_\phi$. 

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acting between the Bloch space, the weighted Bergman space or related analytic function spaces and estimated the norm and the essential norm of \( uC_\phi \). The purpose of this paper is to investigate the operator \( uC_\phi \) acting between the following analytic function spaces.

Let \( p \in (0, \infty) \) and \( \alpha \in (0, \infty) \). For each \( w \in \mathbb{D} \), let \( \sigma_w(z) = (w - z)/(1 - \overline{w}z) \) be a Möbius transformation which exchanges \( w \) and 0. The \( N_p \)-space consists of all \( f \in H(\mathbb{D}) \) such that

\[
\| f \|_{N_p} := \sup_{w \in \mathbb{D}} \left( \int_{\mathbb{D}} |f(z)|^2 (1 - |\sigma_w(z)|^2)^p dA(z) \right)^{\frac{1}{2}} < \infty,
\]

where \( dA \) denotes the normal area measure on \( \mathbb{D} \). The weighted-type space \( H_\alpha^\infty \) is the space of all \( f \in H(\mathbb{D}) \) such that

\[
\| f \|_{H_\alpha^\infty} := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f(z)| < \infty.
\]

And \( H_\alpha^\infty,0 \) denotes the closed subspace of \( H_\alpha^\infty \) such that \( f \in H_\alpha^\infty \) satisfies \((1 - |z|^2)^\alpha |f(z)| \to 0\) as \(|z| \to 1\). These spaces \( H_\alpha^\infty \) and \( H_\alpha^\infty,0 \) are identified with weighted Bloch spaces \( B_{\alpha+1} \) and \( B_{\alpha+1,0} \) ([8, Proposition 7]). Here \( B_{\alpha} \) and \( B_{\alpha,0} \) are defined by

\[
B_{\alpha} = \left\{ f \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty \right\},
\]

\[
B_{\alpha,0} = \left\{ f \in B_{\alpha} : \lim_{|z| \to 1} (1 - |z|^2)^\alpha |f'(z)| = 0 \right\}.
\]

When \( \alpha = 1 \), \( B_1 = B \) is the classical Bloch space.

The \( N_p \)-spaces were introduced by Palmberg [5]. One has the inclusions \( B \subset N_p \subset H_1^\infty \) if \( p \in (0, 1] \) and \( N_p = H_1^\infty \) if \( p \in (1, \infty) \) (see [5]). These situations are very similar to the relation between the \( Q_p \)-space and \( B \). That is, \( Q_p \subset B \) if \( p \in (0, 1] \) and \( Q_p = B \) if \( p \in (1, \infty) \). Here the \( Q_p \)-space consists of all analytic functions \( f \) on \( \mathbb{D} \) which satisfy the condition

\[
\sup_{w \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_w(z)|^2)^p dA(z) < \infty.
\]

We can find these facts in Xiao’s survey book [6] about \( Q_p \)-spaces.

In a recent paper [5], Palmberg has considered composition operators \( C_\phi \) on the \( N_p \)-space. Palmberg gave complete characterizations for the boundedness and compactness of \( C_\phi : N_p \to H_\alpha^\infty \). However the boundedness and compactness of the case \( C_\phi : H_\alpha^\infty \to N_p \) remain to be studied. In this paper, we will characterize the boundedness and compactness of \( uC_\phi : H_\alpha^\infty \to N_p \) or \( uC_\phi : N_p \to H_\alpha^\infty \). Our situations have not been covered by a recent progress of studies of weighted composition operators. Of course, the results in this paper will also give the characterizations for the case \( C_\phi : H_\alpha^\infty \to N_p \) and the case \( uC_\phi : N_p \to H_\alpha^\infty \) is a generalization of the result in [5]. Furthermore, by the derivative operator \( f \mapsto f' \), \( Q_p \)-spaces are closely related to \( N_p \)-spaces and Bloch-type spaces \( B_\alpha \) related to \( H_\alpha^\infty \). Hence our results also cover the corresponding results for \( C_\phi \) (with \( u = \phi' \)) acting between \( B_\alpha \) and \( Q_p \)-spaces which are presented in [6].

In Section 2, we will consider the case \( uC_\phi : H_\alpha^\infty \to N_p \). To characterize the boundedness of it, we use a \( p \)-Carleson measure characterization for \( N_p \). For the compactness, we will estimate the essential norm of \( uC_\phi \). Since the essential norm \( \| uC_\phi \|_e \) is defined to be the distance from
$uC_{\phi}$ to the space of the compact operators, $uC_{\phi}$ is compact if and only if $\|uC_{\phi}\|_c = 0$. The dual relation for $H_\alpha^\infty$ plays an important role in the proof. In Section 3, we characterize the boundedness and the compactness of $uC_{\phi} : \mathcal{N}_p \rightarrow H_\alpha^\infty$.

Throughout this paper, the notation $a \lesssim b$ means that there exists a positive constant $C$ such that $a \leq Cb$. Moreover, if both $a \lesssim b$ and $a \gtrsim b$ hold, then one says that $a \asymp b$.

2. The case $uC_{\phi} : H_\alpha^\infty \rightarrow \mathcal{N}_p$

In order to study the operator $uC_{\phi} : H_\alpha^\infty \rightarrow \mathcal{N}_p$, we need a Carleson box $S(I)$ and a $p$-Carleson measure. For each arc $I$ in the unit circle $\partial \mathbb{D}$, a Carleson box based on $I$ is the set of the form

$$S(I) = \{z \in \mathbb{D} \mid 1 - |I| \leq |z| < 1, z/|z| \in I\},$$

where $|I|$ denotes the normalized length of $I$. For each $p \in (0, \infty)$, a positive Borel measure $d\mu$ on $\mathbb{D}$ is called a $p$-Carleson measure if

$$\sup_{I \subset \partial \mathbb{D}} \frac{\mu(S(I))}{|I|^p} < \infty,$$

where the supremum is taken over all arcs $I \subset \partial \mathbb{D}$. Aulaskari et al. [1] characterized the $Q_p$-space in terms of a $p$-Carleson measure. For the $\mathcal{N}_p$-space, an analogous characterization holds.

**Lemma 1.** For $f \in H(\mathbb{D})$, $f \in \mathcal{N}_p$ if and only if $d\mu_{f,p}(z) := |f(z)|^2(1 - |z|^2)^p dA(z)$ is a $p$-Carleson measure. Furthermore it holds

$$\|f\|_{\mathcal{N}_p} \asymp \sup_{I \subset \partial \mathbb{D}} \frac{\mu_{f,p}(S(I))}{|I|^p}.$$

**Proof.** The following equality

$$1 - |\sigma_w(z)|^2 = \frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \overline{w}z|^2}$$

shows that

$$\|f\|_{\mathcal{N}_p}^2 = \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} \left(\frac{1 - |w|^2}{|1 - \overline{w}z|^2}\right)^p d\mu_{f,p}(z).$$

Combining this with [1, Lemma 2.1], we accomplish the proof. $\Box$

Xiao [7] characterized the boundedness and compactness of $C_{\phi} : \mathcal{B} \rightarrow Q_p$ by using a $p$-Carleson measure. For the case $uC_{\phi} : H_\alpha^\infty \rightarrow \mathcal{N}_p$, an argument based on a $p$-Carleson measure is also useful.

Next we consider a test function for $H_\alpha^\infty$ which will be used in the proofs of Theorems 1 and 2. For each $\alpha \in (0, \infty)$, $\theta \in [0, 2\pi)$ and $r \in (0, 1]$, we put

$$f_{\theta,r}(z) := \sum_{k=0}^{\infty} 2^{ka} (re^{i\theta})^{2^k} z^{2^k} \quad (z \in \mathbb{D}).$$

**Lemma 2.** The function $f_{\theta,r}$ belongs to $H_\alpha^\infty$ and $\|f_{\theta,r}\|_{H_\alpha^\infty} \lesssim 1$ which is independent of $\theta$ and $r$. In particular, $f_{\theta,r} \in H_\alpha^\infty_{0,0}$ if $r \in (0, 1)$.
Proof. For each $z \in \mathbb{D} \setminus \{0\}$, we have

$$|f_{\theta, r}(z)| \leq \sum_{k=0}^{\infty} 2^{k\alpha}(\sqrt{r}|z|)^{2k+1} \leq \sum_{k=0}^{\infty} \int_{k}^{k+1} 2^{\alpha x}(\sqrt{r}|z|)^{2x} dx = \int_{0}^{\infty} 2^{\alpha x}(\sqrt{r}|z|)^{2x} dx$$

$$\leq \left( \log \frac{1}{r|z|} \right)^{\alpha} \int_{\log \frac{1}{r|z|}}^{\infty} s^{\alpha-1} e^{-s} ds \leq \left( \log \frac{1}{r|z|} \right)^{\alpha}.$$  \hspace{1cm} (2)

Since $\log \frac{1}{x} \geq 1 - x$, we obtain that

$$(1 - |z|^2)^{\alpha} |f_{\theta, r}(z)| \leq 1,$$

for any $z \in \mathbb{D}$.

Furthermore it follows from (2) that

$$(1 - |z|^2)^{\alpha} |f_{\theta, r}(z)| \leq \frac{(1 - |z|^2)^{\alpha}}{\left( \log \frac{1}{r|z|} \right)^{\alpha}},$$

and so we see that $f_{\theta, r} \in H^{\infty}_{\mu, 0}$. This completes the proof.  \hspace{1cm} $\Box$

Theorem 1. Let $u \in H(\mathbb{D})$ and $\phi$ be an analytic self-map of $\mathbb{D}$. Then the following are equivalent:

(i) $uC_{\phi} : H^{\infty}_{\mu} \to N_p$ exists as a bounded operator.

(ii) $u$ and $\phi$ satisfy

$$\sup_{w \in \mathbb{D}} \int_{\mathbb{D}} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} (1 - |\sigma_{\phi}(z)|^2)^p dA(z) < \infty.$$  \hspace{1cm} (3)

(iii) $u$ and $\phi$ satisfy

$$\sup_{I \subset \partial \mathbb{D}} |I|^{-p} \int_{S(I)} \frac{|u(z)|^2 (1 - |z|^2)^p}{(1 - |\phi(z)|^2)^{2\alpha}} dA(z) < \infty.$$  \hspace{1cm} (4)

Here the supremum in (4) is taken over all arcs $I \subset \partial \mathbb{D}$.

Proof. (ii)$\Rightarrow$ (i). This implication is obvious.

(i)$\Rightarrow$ (ii). For each $\theta \in [0, 2\pi)$, we set $f_{\theta} := f_{\theta, 1}$ which is defined in (1). Fix $w \in \mathbb{D}$. By Lemma 2 and Fubini’s theorem we have

$$1 \geq \int_{0}^{2\pi} \|uC_{\phi}f_{\theta}\|^2_{N_p} d\theta \geq \int_{\mathbb{D}} |u(z)|^2 (1 - |\sigma_{\phi}(z)|^2)^p \left\{ \int_{0}^{2\pi} |f_{\theta}(\phi(z))|^2 \frac{d\theta}{2\pi} \right\} dA(z).$$

Parseval’s formula gives

$$\int_{0}^{2\pi} |f_{\theta}(\phi(z))|^2 \frac{d\theta}{2\pi} = \int_{0}^{2\pi} \left| \sum_{k=0}^{\infty} 2^{k\alpha} (e^{i\theta} \phi(z))^k \right|^2 \frac{d\theta}{2\pi} = \sum_{k=0}^{\infty} (2^k)^{2\alpha} (|\phi(z)|^2)^k.$$  \hspace{1cm} (5)
When $|\phi(z)| > \frac{1}{\sqrt{2}}$, we have

$$\sum_{k=0}^{\infty} (2^k)^{2\alpha} (|\phi(z)|^2)^{2k} = 2^{-2\alpha} \sum_{k=0}^{\infty} (2^{k+1})^{2\alpha} (|\phi(z)|^2)^{2k} \geq 2^{-2\alpha} \int_{0}^{\infty} (2^x)^{2\alpha} (|\phi(z)|^2)^{2x} \, dx = \frac{2^{-2\alpha}}{\log 2} \left( \log \frac{1}{|\phi(z)|^2} \right)^{-2\alpha} \int_{\log \frac{1}{|\phi(z)|^2}}^{\infty} s^{2\alpha - 1} e^{-s} \, ds \geq \frac{1}{(1 - |\phi(z)|^2)^{2\alpha}},$$  

(6)

where the last inequality follows from $\log \frac{1}{x} \leq \log(4 - (1 - x))$ for $x \in (\frac{1}{2}, 1)$. Hence we obtain

$$\int_{|\phi(z)| > \frac{1}{\sqrt{2}}} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} (1 - |\sigma_w(z)|^2)^p \, dA(z) \lesssim 1,$$

(7)

for any $w \in \mathbb{D}$. By noting that $u \in N_p$, we have

$$\int_{|\phi(z)| \leq \frac{1}{\sqrt{2}}} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} (1 - |\sigma_w(z)|^2)^p \, dA(z) \lesssim \|u\|_{N_p}^2,$$

(8)

for any $w \in \mathbb{D}$. Inequalities (7) and (8) show that condition (3) is true.

(iii) $\Rightarrow$ (i). For every $f \in H_{\alpha}^\infty$ it follows that

$$\sup_{I \subset \partial \mathbb{D}} |I|^{-p} \int_{S(I)} |u(z)|^2 |f(\phi(z))|^2 (1 - |z|^2)^p \, dA(z) \leq \|f\|_{H_{\alpha}^\infty}^2 \cdot \sup_{I \subset \partial \mathbb{D}} |I|^{-p} \int_{S(I)} \frac{|u(z)|^2 (1 - |z|^2)^p}{(1 - |\phi(z)|^2)^{2\alpha}} \, dA(z).$$

Combining this with condition (4), we see that

$$d \mu(z) := |u(z)|^2 |f(\phi(z))|^2 (1 - |z|^2)^p \, dA(z)$$

is a $p$-Carleson measure. Thus Lemma 1 implies that $uC_{\phi} f \in N_p$ and

$$\|uC_{\phi} f\|_{N_p}^2 = \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} \left( \frac{1 - |w|^2}{1 - |\bar{w}z|^2} \right)^p |u(z)|^2 |f(\phi(z))|^2 (1 - |z|^2)^p \, dA(z) \lesssim \sup_{I \subset \partial \mathbb{D}} |I|^{-p} \int_{S(I)} |u(z)|^2 |f(\phi(z))|^2 (1 - |z|^2)^p \, dA(z) \lesssim \|f\|_{H_{\alpha}^\infty}^2,$$

and so $uC_{\phi} : H_{\alpha}^\infty \to N_p$ is bounded.

(i) $\Rightarrow$ (iii). Assume that $uC_{\phi} : H_{\alpha}^\infty \to N_p$ is bounded. Fix an arc $I \subset \partial \mathbb{D}$ and consider the test function $f_\theta \ (\theta \in [0, 2\pi))$ in the proof of (i) $\Rightarrow$ (ii). Lemmas 1 and 2, Fubini’s theorem, (5) and (6) show that

$$|I|^{-p} \int_{S(I) \cap \{|\phi(z)| > \frac{1}{\sqrt{2}}\}} \frac{|u(z)|^2 (1 - |z|^2)^p}{(1 - |\phi(z)|^2)^{2\alpha}} \, dA(z) \lesssim 1.$$

(9)
Since $u \in \mathcal{N}_p$ by the boundedness of $uC_\phi$, it follows from Lemma 1 that $|u(z)|^2(1 - |z|^2)^p dA(z)$ is a $p$-Carleson measure and

$$\sup_{I \subset \partial \mathbb{D}} |I|^{-p} \int_{S(I)} |u(z)|^2(1 - |z|^2)^p dA(z) \lesssim \|u\|_{\mathcal{N}_p}^2.$$ 

Hence we have

$$|I|^{-p} \int_{S(I) \cap \{|\phi(z)| \leq \frac{1}{2^l}\}} |u(z)|^2(1 - |z|^2)^p dA(z) \lesssim \|u\|_{\mathcal{N}_p}^2. \quad (10)$$

By (9) and (10), we obtain the condition (4).

Next we consider the compactness of $uC_\phi : H_\alpha^\infty \rightarrow \mathcal{N}_p$. To do this we need some lemmas.

Let $L_1^1(dA)$ denote the Bergman space of analytic functions $f$ on $\mathbb{D}$ such that $\int_{\partial \mathbb{D}} |f|^2 dA < \infty$. For $\beta > 0$ we consider the following integral pairing

$$(f, g)_{\beta} := \lim_{t \rightarrow 1} \int_{\partial \mathbb{D}} f(tz)g(tz)(1 - |z|^2)^{\beta-1} dA(z),$$

for $f \in L_1^1(dA)$ and $g \in B_{\beta}$. By [8, Theorem 14] we see that $(B_{\beta,0})^* = L_1^1(dA)$ and $(L_1^1(dA))^* = B_{\beta}$. Since we see that $H_\alpha^\infty = B_{\alpha+1}$ and $H_{\alpha,0} = B_{\alpha+1,0}$ for each $\alpha > 0$, we obtain the following dual relations for $H_\alpha^\infty$ (also see [8]).

**Lemma 3 ([8]).** Using the integral pairing given by

$$(f, g)_\alpha := \lim_{t \rightarrow 1} \int_{\partial \mathbb{D}} f(tz)g(tz)(1 - |z|^2)^{\alpha} dA(z),$$

we have

$$(H_\alpha^\infty)^* \cong L_1^1(dA) \quad \text{and} \quad (L_1^1(dA))^* \cong H_\alpha^\infty.$$ 

For each $n \in \mathbb{N}$ we consider the function $g^n_{\theta,r}(z) := z^n f_{\theta,r}(z)$. Here $f_{\theta,r}$ is the test function defined in (1). Since $z^n H_{\alpha,0}^\infty \subset H_{\alpha,0}^\infty$, it follows from Lemma 2 that $\{g^n_{\theta,r}\}_{n \in \mathbb{N}} \subset H_{\alpha,0}^\infty$ and the norm $\|g^n_{\theta,r}\|_{H_{\alpha,0}^\infty}$ is uniformly bounded on $\theta, r$ and $n$. The following Lemma 4 shows that $\{g^n_{\theta,r}\}_{n \in \mathbb{N}}$ converges to 0 weakly in $H_\alpha^\infty$.

**Lemma 4.** For every $A \in (H_\alpha^\infty)^*$ we have $\sup_{\theta, r} |A(g^n_{\theta,r})| \rightarrow 0$ as $n \rightarrow \infty$.

**Proof.** This lemma is proved by a modification of [4, Lemma 5]. For the sake of the reader, however, we describe the proof. It is enough to prove that $\sup_{\theta, r} |A(g^n_{\theta,r})| \rightarrow 0$ as $n \rightarrow \infty$ for any $A \in (H_{\alpha,0}^\infty)^*$. Lemma 3 shows that there exists an $h \in L_1^1(dA)$ such that $A(f) = (h, f)_\alpha$ for $f \in H_{\alpha,0}^\infty$. By using the estimate in the proof of Lemma 2, we have

$$\sup_{\theta, r} |A(g^n_{\theta,r})| \leq \sup_{\theta, r} \lim_{t \rightarrow 1} \int_{\partial \mathbb{D}} |tz|^n |f_{\theta,r}(tz)| |h(tz)|(1 - |z|^2)^{\alpha} dA(z)$$

$$\lesssim \lim_{t \rightarrow 1} \int_{\partial \mathbb{D}} |tz|^n |h(tz)| dA(z)$$

$$\leq \lim_{t \rightarrow 1} \frac{1}{t^2} \int_{\partial \mathbb{D}} |z|^n |h(z)| dA(z)$$

$$= \int_{\partial \mathbb{D}} |z|^n |h(z)| dA(z).$$
Since $|z|^n |h(z)| \to 0$ \((n \to \infty)\), \(|z|^n |h(z)| \leq |h(z)| \) and \(|h| \in L^1(dA)\), the dominated convergence theorem shows that $\lim_{n \to \infty} \int_{D} |z|^n |h(z)| dA(z) = 0$. We obtain the desired result. \hfill \Box

By the above Lemma 4, we see that $T g^n_{\theta, r} \to 0$ in $\mathcal{N}_p$ as $n \to \infty$ for any compact operator $T : H^\infty_{\alpha} \to \mathcal{N}_p$, $\theta \in [0, 2\pi)$ and $r \in (0, 1)$. To prove Theorem 2, however, we will need the following strong result.

Lemma 5. For any compact operator $T : H^\infty_{\alpha} \to \mathcal{N}_p$, it holds that

$$\lim_{n \to \infty} \sup_{\theta, r} \| T g^n_{\theta, r} \|_{\mathcal{N}_p} = 0,$$

where the supremum is taken over all $\theta \in [0, 2\pi)$ and $r \in (0, 1)$.

Proof. This lemma is verified by the complete continuity of compact operators and Lemma 4. So we omit the detail of the proof. \hfill \Box

Lemma 6. Let $X, Y \in \{ H^\infty_{\alpha}, \mathcal{N}_p \}$. Suppose that $uC_\phi(X) \subset Y$. Then $uC_\phi : X \to Y$ is compact if and only if for every bounded sequence $\{ f_j \}$ in $X$ which converges to 0 uniformly on compact subsets of $D$, we have $\lim_{j \to \infty} \| uC_\phi f_j \|_Y = 0$.

Proof. This is an extension of a well-known result on the compactness of the composition operator on the Hardy spaces (see [2, Proposition 3.11]). We see that any bounded sequence in $H^\infty_{\alpha}$ forms a normal family. Also the relation $\| f \|_{H^\infty_{\alpha}} \lesssim \| f \|_{\mathcal{N}_p}$ and the growth estimate for $f \in H^\infty_{\alpha}$ imply that any bounded sequence in $\mathcal{N}_p$ forms a normal family. Hence a similar argument by using Montel theorem also proves this lemma, and so we omit its proof. \hfill \Box

Theorem 2. Let $u \in H(D)$ and $\phi$ be an analytic self-map of $D$. Suppose that $uC_\phi : H^\infty_{\alpha} \to \mathcal{N}_p$ is bounded. Then it holds that

\begin{equation}
\| uC_\phi \|_e^2 \leq \limsup_{r \to 1} \sup_{x \in D} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^2} (1 - |\sigma_w(z)|^2)^p dA(z) \tag{11}
\end{equation}

\begin{equation}
\leq \limsup_{r \to 1} \sup_{I \subset \partial D} \frac{|I|^{-p}}{\int_{I \cap \{|\phi(z)| > r\}} \frac{|u(z)|^2 (1 - |\sigma_w(z)|^2)^p}{(1 - |\phi(z)|^2)^2} dA(z) \tag{12}.
\end{equation}

Here the supremum in (12) is taken over all arcs $I \subset \partial D$.

Proof. In order to prove upper estimates in (11) and (12), we put $C_k f(z) = f \left( \frac{k}{k+1} z \right)$ for each positive integer $k$ and $f \in H(D)$. Then we see that every $C_k$ is bounded on $H^\infty_{\alpha}$. By applying Lemma 6 to the case $u \equiv 1$ and $\phi(z) = \frac{k}{k+1} z$, we see that $C_k$ is compact on $H^\infty_{\alpha}$. So we have that

$$\| uC_\phi \|_e \leq \liminf_{k \to \infty} \| uC_\phi - uC_\phi C_k \| = \liminf_{k \to \infty} \sup_{\| f \|_{H^\infty_{\alpha}} \leq 1} \| uC_\phi (Id - C_k) f \|_{\mathcal{N}_p}. \tag{13}$$

Here $Id$ denotes the identity operator on $H^\infty_{\alpha}$. Fix a positive integer $k$ and an $f \in H^\infty_{\alpha}$ with $\| f \|_{H^\infty_{\alpha}} \leq 1$. The term $\| uC_\phi (Id - C_k) f \|_{\mathcal{N}_p}^2$ is less than or equal to

\begin{equation}
\sup_{z \in D} \int_{\{|\phi(z)| \leq r\}} |u(z)|^2 \left| f(\phi(z)) - f \left( \frac{k}{k+1} \phi(z) \right) \right|^2 (1 - |\sigma_w(z)|^2)^p dA(z)
\end{equation}

\begin{equation}
+ \sup_{z \in D} \int_{\{|\phi(z)| > r\}} |u(z)|^2 \left| f(\phi(z)) - f \left( \frac{k}{k+1} \phi(z) \right) \right|^2 (1 - |\sigma_w(z)|^2)^p dA(z), \tag{14}
\end{equation}
for any \( r \in (0, 1) \). By the growth estimate for \( f \in H_\alpha^\infty \), we have that
\[
\left| f(\phi(z)) - f\left(\frac{k}{k + 1}\phi(z)\right) \right| \leq \frac{2\|f\|_{H_\alpha^\infty}}{(1 - |\phi(z)|^2)^\alpha}.
\] (15)

This implies that
\[
\sup_{w \in \mathbb{D}} \int_{|\phi(z)| > r} \left| u(z) \right|^2 \left| f(\phi(z)) - f\left(\frac{k}{k + 1}\phi(z)\right) \right|^2 (1 - |\sigma_w(z)|^2)^p dA(z)
\]
\[
\leq \sup_{w \in \mathbb{D}} \int_{|\phi(z)| > r} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^2\alpha} (1 - |\sigma_w(z)|^2)^p dA(z),
\] (16)

for any \( r \in (0, 1) \). Note that this estimate does not depend on \( k \).

Now let us prove that
\[
\sup_{\|f\|_{H_\alpha^\infty} \leq 1} \sup_{w \in \mathbb{D}} \int_{|\phi(z)| \leq r} \left| u(z) \right|^2 \left| f(\phi(z)) - f\left(\frac{k}{k + 1}\phi(z)\right) \right|^2 (1 - |\sigma_w(z)|^2)^p dA(z)
\]
\[
\times (1 - |\sigma_w(z)|^2)^p dA(z) \to 0,
\] (17)
as \( k \to \infty \). We put \( v = \phi(z) \) and denote the radial segment by \( [\frac{k}{k+1} v, v] \). By integrating \( f' \) along \( [\frac{k}{k+1} v, v] \), we obtain that
\[
\left| f(v) - f\left(\frac{k}{k + 1} v\right) \right| \leq \frac{1}{k + 1} |v| |f'(\xi(v))|,
\] (18)

for some \( \xi(v) \in [\frac{k}{k+1} v, v] \). An application of Cauchy’s estimate to \( f' \) on the circle with center at \( \xi(v) \) and radius \( R \in (0, 1 - r) \) shows that
\[
|f'(\xi(v))| \leq \frac{1}{R} \max_{|\xi|=R+r} |f(\xi)|.
\] (19)

Combining this with (18), we obtain that
\[
\sup_{\|f\|_{H_\alpha^\infty} \leq 1} \sup_{w \in \mathbb{D}} \int_{|\phi(z)| \leq r} \left| u(z) \right|^2 \left| f(\phi(z)) - f\left(\frac{k}{k + 1}\phi(z)\right) \right|^2 (1 - |\sigma_w(z)|^2)^p dA(z)
\]
\[
\leq \frac{r^2}{(k + 1)^2} \frac{1}{\{1 - (R + r)^2\}^{2\alpha}} \|u\|_{\mathcal{N}_p}^2.
\]

Since \( u \in \mathcal{N}_p \) by the boundedness of \( uC_\phi : H_\alpha^\infty \to \mathcal{N}_p \), we obtain (17). By (13), (14), (16) and (17), we have that
\[
\|uC_\phi\|_{\mathcal{N}_p}^2 \leq \sup_{w \in \mathbb{D}} \int_{|\phi(z)| > r} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^2\alpha} (1 - |\sigma_w(z)|^2)^p dA(z),
\]
for any \( r \in (0, 1) \), and so we obtain the upper estimate in (11).

On the other hand, by Lemma 1, we have that
\[
\|uC_\phi(Id - C_k)f\|_{\mathcal{N}_p}^2 \leq \sup_{I \subset \partial \mathbb{D}} |I|^{-p} \int_{S(I) \cap \{|\phi(z)| \leq r\}} |u(z)|^2 |f(\phi(z)) - f\left(\frac{k}{k + 1}\phi(z)\right)|^2 (1 - |z|^2)^p dA(z)
\]
\[
\leq \frac{r^2}{(k + 1)^2} \frac{1}{\{1 - (R + r)^2\}^{2\alpha}} \|u\|_{\mathcal{N}_p}^2.
\]
for any \( r \in (0, 1) \). Inequality (15) gives that

\[
\sup_{I \subset \partial \mathbb{D}} |I|^{-p} \int_{I} |u(z)|^2 \left| f(\phi(z)) - f \left( \frac{k}{k+1} \phi(z) \right) \right|^2 (1 - |z|^2)^p dA(z) \\
\leq \sup_{I \subset \partial \mathbb{D}} |I|^{-p} \int_{I} |u(z)|^2 (1 - |z|^2)^p dA(z) + f \left( \frac{k}{k+1} \phi(z) \right)^2 (1 - |z|^2)^p dA(z),
\]

for any \( r \in (0, 1) \) and any \( f \in H_1^\infty \) with \( \|f\|_{H_1^\infty} \leq 1 \). By inequalities (18), (19) and Lemma 1, we obtain that

\[
\sup_{I \subset \partial \mathbb{D}} |I|^{-p} \int_{I} |u(z)|^2 \left| f(\phi(z)) - f \left( \frac{k}{k+1} \phi(z) \right) \right|^2 (1 - |z|^2)^p dA(z) \\
\leq \frac{r^2}{R^2(k + 1)^2 (1 - (R + r)^2)^{2\alpha}} \|u\|_{\mathcal{N}_p}^2 \to 0,
\]

as \( k \to \infty \) for any \( f \in H_1^\infty \) with \( \|f\|_{H_1^\infty} \leq 1 \). Note that the above (22) holds uniformly on the unit ball of \( H_1^\infty \). By (13), (20)–(22) and letting \( r \to 1 \), we obtain that

\[
\|uC_\phi\|_{\mathcal{V}_p}^2 \leq \limsup_{r \to 1} \sup_{I \subset \partial \mathbb{D}} |I|^{-p} \int_{I} |u(z)|^2 \left( 1 - |z|^2 \right)^p dA(z).
\]

This is the upper estimate in (12).

Next we will prove lower estimates in (11) and (12). We consider the functions \( \{g^n_{\theta, t}\}_{n \in \mathbb{N}} \) defined in Lemma 4. Note that the norm \( \|g^n_{\theta, t}\|_{H_1^\infty} \) is uniformly bounded on \( \theta, t \) and \( n \) by Lemma 2. For any compact operators \( \mathcal{K} : H_1^\infty \to \mathcal{N}_p \), we have that

\[
\|uC_\phi - \mathcal{K}\| \geq \|uC_\phi - \mathcal{K}g^n_{\theta, t}\|_{\mathcal{N}_p} = \|uC_\phi g^n_{\theta, t}\|_{\mathcal{N}_p} - \|\mathcal{K}g^n_{\theta, t}\|_{\mathcal{N}_p},
\]

for any \( \theta, t \) and \( n \). By Fatou’s lemma we have that

\[
\sup_{\theta, t} \|uC_\phi g^n_{\theta, t}\|_{\mathcal{N}_p}^2 \geq \liminf_{t \to 1} \int_{\mathbb{D}} |u(z)|^2 |g^n_{\theta, t}(\phi(z))|^2 (1 - |\sigma_w(z)|^2)^p dA(z) \\
\geq \int_{|\phi(z)| > r} |u(z)|^2 |\phi(z)|^2 |f_\theta(\phi(z))|^2 (1 - |\sigma_w(z)|^2)^p dA(z),
\]

for any \( r \in (0, 1) \). Here \( f_\theta(w) \) denotes the function \( f_{\theta, 1}(w) \). By integrating these inequalities with respect to \( \theta \) from 0 to \( 2\pi \) and Fubini’s theorem, we obtain that

\[
\sup_{\theta, t} \|uC_\phi g^n_{\theta, t}\|_{\mathcal{N}_p}^2 \geq \int_{|\phi(z)| > r} |u(z)|^2 |\phi(z)|^2 |f_\theta(\phi(z))|^2 (1 - |\sigma_w(z)|^2)^p \\
\times \left\{ \int_0^{2\pi} |f_\theta(\phi(z))|^2 d\theta \right\} dA(z).
\]
By (5) and (6), it follows that
\[
\int_0^{2\pi} |f_\theta(\phi(z))|^2 \frac{d\theta}{2\pi} \gtrsim \frac{1}{(1 - |\phi(z)|^2)^{2\alpha}},
\]
for any \( z \in \mathbb{D} \) with \( |\phi(z)| > \frac{1}{\sqrt{2}} \). Combining (24) with (25), we obtain
\[
sup_{\theta, r} \|uC_{\phi} g_{\theta, r}^n\|_{N_p}^2 \gtrsim \sup_{w \in \mathbb{D}} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} (1 - |\sigma_w(z)|^2)^p dA(z),
\]
for any \( r \in (1/\sqrt{2}, 1) \). Letting \( r \to 1 \), then this gives that
\[
sup_{\theta, t} \|uC_{\phi} g_{\theta, t}^n\|_{N_p}^2 \gtrsim \lim_{r \to 1} \sup_{w \in \mathbb{D}} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} (1 - |\sigma_w(z)|^2)^p dA(z).
\]
Note that this estimate does not depend on \( n \). Since \( \sup_{\theta, I} \|K g_{\theta, I}^n\|_{N_p} \to 0 \) as \( n \to \infty \) for any compact operators \( K : H_\alpha^\infty \to N_p \) by Lemma 5, (23) and (26) imply that
\[
\|uC_{\phi}\|_C^2 \gtrsim \lim_{r \to 1} \sup_{w \in \mathbb{D}} \int_{\{|\phi(z)| > r\}} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} (1 - |\sigma_w(z)|^2)^p dA(z).
\]
Furthermore, by Lemma 1, we obtain that
\[
sup_{\theta, I} \|uC_{\phi} g_{\theta, I}^n\|_{N_p}^2 \gtrsim |I|^{-p} \int_{S(I)} |u(z)|^2 |g_{\theta, I}(\phi(z))|^2 (1 - |z|^2)^p dA(z),
\]
for all arcs \( I \). A similar argument in the proof of (27) shows that
\[
\|uC_{\phi}\|_C^2 \gtrsim \lim_{r \to 1} \sup_{I \subset \partial \mathbb{D}} |I|^{-p} \int_{S(I) \cap \{|\phi(z)| > r\}} \frac{|u(z)|^2 (1 - |z|^2)^p}{(1 - |\phi(z)|^2)^{2\alpha}} dA(z) = 0.
\]
We accomplish the proof. \( \square \)

**Corollary 1.** Under the same assumptions in Theorem 1, the following are equivalent:

(i) \( uC_{\phi} : H_\alpha^\infty \to N_p \) exists as a compact operator.

(ii) \( u \) and \( \phi \) satisfy
\[
\lim_{r \to 1} \sup_{w \in \mathbb{D}} \int_{\{|\phi(z)| > r\}} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} (1 - |\sigma_w(z)|^2)^p dA(z) = 0.
\]

(iii) \( u \) and \( \phi \) satisfy
\[
\lim_{r \to 1} \sup_{I \subset \partial \mathbb{D}} |I|^{-p} \int_{S(I) \cap \{|\phi(z)| > r\}} \frac{|u(z)|^2 (1 - |z|^2)^p}{(1 - |\phi(z)|^2)^{2\alpha}} dA(z) = 0.
\]

**Here the supremum in (12) is taken over all arcs \( I \subset \partial \mathbb{D} \).**

3. **The case \( uC_{\phi} : N_p \to H_\alpha^\infty \)**

In this section, we will consider the operator \( uC_{\phi} : N_p \to H_\alpha^\infty \). The case \( u \equiv 1 \) can be found in the work [5] by Palmberg.
Theorem 2. Let \( u \in H(\mathbb{D}) \) and \( \phi \) be an analytic self-map of \( \mathbb{D} \). Then \( uC_\phi : \mathcal{N}_p \rightarrow H_\alpha^\infty \) exists as a bounded operator if and only if \( u \) and \( \phi \) satisfy
\[
\sup_{z \in \mathbb{D}} \frac{|u(z)|(1 - |z|^2)^\alpha}{1 - |\phi(z)|^2} < \infty.
\] (28)

Proof. Note that \( \mathcal{N}_p \subset H_1^\infty \) and \( \|f\|_{H_1^\infty} \lesssim \|f\|_{\mathcal{N}_p} \) (see [5, Proposition 3.1]). First assume that the condition (28) is true. Then for each \( z \in \mathbb{D} \) and \( f \in \mathcal{N}_p \) we have
\[
(1 - |z|^2)^\alpha |uC_\phi f(z)| \leq \frac{|u(z)|(1 - |z|^2)^\alpha}{1 - |\phi(z)|^2} \|f\|_{H_1^\infty} \lesssim \frac{|u(z)|(1 - |z|^2)^\alpha}{1 - |\phi(z)|^2} \|f\|_{\mathcal{N}_p}.
\]
This implies that \( \|uC_\phi f\|_{H_\alpha^\infty} \lesssim \|f\|_{\mathcal{N}_p} \), namely \( uC_\phi \) is bounded from \( \mathcal{N}_p \) into \( H_\alpha^\infty \).

To prove the necessity of (28), we consider the following function
\[
k_w(z) = \frac{1 - |w|^2}{1 - w\overline{z}} \quad (z \in \mathbb{D}),
\] (29)
with \( w = \phi(z_0) \) for fixed \( z_0 \in \mathbb{D} \). Since \( k_w \in \mathcal{N}_p \) and \( \|k_w\|_{\mathcal{N}_p} \leq 1 \), the boundedness of \( uC_\phi \) gives
\[
1 \gtrsim \|uC_\phi k_w\|_{H_\alpha^\infty} \geq (1 - |z_0|^2)^\alpha |u(z_0)| \|k_w(\phi(z_0))\| = \frac{|u(z_0)|(1 - |z_0|^2)^\alpha}{1 - |\phi(z_0)|^2}.
\]
This completes the proof. \( \square \)

Theorem 3. Let \( u \in H(\mathbb{D}) \) and \( \phi \) be an analytic self-map of \( \mathbb{D} \). Then \( uC_\phi : \mathcal{N}_p \rightarrow H_\alpha^\infty \) is bounded. Then it holds that
\[
\|uC_\phi\|_e \asymp \limsup_{|\phi(z)| \to 1} \frac{|u(z)|(1 - |z|^2)^\alpha}{1 - |\phi(z)|^2}.
\] (30)

Proof. To prove the upper estimate in (30), we consider the operator \( C_k \) as in the proof of Theorem 2. For each positive integer \( k \), we have that
\[
\|C_k f\|_{\mathcal{N}_p}^2 = \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} \left| f\left( \frac{k}{k+1} z \right) \right|^2 (1 - |\sigma_w(z)|^2)^p dA(z)
\leq \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} \left( 1 - \frac{k}{k+1} |z|^2 \right)^2 (1 - |\sigma_w(z)|^2)^p dA(z)
\lesssim \frac{(k+1)^4}{(2k+1)^2} \|f\|_{\mathcal{N}_p}^2.
\]
Hence \( C_k \) is bounded on \( \mathcal{N}_p \). By an application of Lemma 6, we see that every \( C_k \) is compact on \( \mathcal{N}_p \), and so \( uC_\phi C_k \) is compact from \( \mathcal{N}_p \rightarrow H_\alpha^\infty \). Thus we have that
\[
\|uC_\phi\|_e \leq \|uC_\phi - uC_\phi C_k\| = \sup_{\|f\|_{\mathcal{N}_p} \leq 1} \|uC_\phi (Id - C_k) f\|_{H_\alpha^\infty},
\] (31)
where \( Id \) denotes the identity operator on \( \mathcal{N}_p \). Fix a positive integer \( k \) and an \( f \in \mathcal{N}_p \) with \( \| f \|_{\mathcal{N}_p} \leq 1 \). For any \( r \in (0, 1) \) we have that
\[
\| uC_\phi(Id - C_k)f \|_{H^\alpha_a} \leq \sup_{|\phi(z)| > r} (1 - |z|^2)\alpha|u(z)| \left| f(\phi(z)) - f\left(\frac{k}{k+1}\phi(z)\right)\right| + \sup_{|\phi(z)| \leq r} (1 - |z|^2)\alpha|u(z)| \left| f(\phi(z)) - f\left(\frac{k}{k+1}\phi(z)\right)\right|.
\]
Since it holds that
\[
\left| f(\phi(z)) - f\left(\frac{k}{k+1}\phi(z)\right)\right| \leq \frac{2\| f \|_{\mathcal{N}_p}}{1 - |\phi(z)|^2}.
\]
we have that
\[
\sup_{|\phi(z)| > r} (1 - |z|^2)\alpha|u(z)| \left| f(\phi(z)) - f\left(\frac{k}{k+1}\phi(z)\right)\right| \lesssim \sup_{|\phi(z)| > r} \frac{|u(z)|(1 - |z|^2)\alpha}{1 - |\phi(z)|^2},
\]
for any \( r \in (0, 1) \).

On the other hand, by the same argument in the proof of Theorem 2, inequalities (18) and (19) show that for an \( R \in (0, 1 - r) \)
\[
\sup_{|\phi(z)| \leq r} (1 - |z|^2)\alpha|u(z)| \left| f(\phi(z)) - f\left(\frac{k}{k+1}\phi(z)\right)\right| \lesssim \frac{\| u \|_{H^\alpha_a}}{R(k+1)\{1 - (R + r)^2\}} \to 0,
\]
as \( k \to \infty \). By (31)–(34) we obtain that
\[
\| uC_\phi \|_e \lesssim \sup_{|\phi(z)| > r} \frac{|u(z)|(1 - |z|^2)\alpha}{1 - |\phi(z)|^2},
\]
for any \( r \in (0, 1) \). Letting \( r \to 1 \) in the above inequality, then we have the desired estimate in (30).

Now we will prove the lower estimate in (30). Take a sequence \( \{z_j\} \) in \( \mathbb{D} \) with \( |\phi(z_j)| \to 1 \) as \( j \to \infty \), arbitrarily. Put \( w_j = \phi(z_j) \) and \( f_j(z) = k_{w_j}(z) \) where \( k_{w_j} \) is the function defined by (29) in the proof of Theorem 3. Then \( \{f_j\} \) forms a bounded sequence in \( \mathcal{N}_p \) and converges to 0 uniformly on compact subsets of \( \mathbb{D} \). More precisely, inequalities \( \| f_j \|_{H_1^{\infty}} \lesssim \| f_j \|_{\mathcal{N}_p} \leq 1 \) show that \( \{f_j\} \) is a bounded sequence in \( H_1^{\infty} \). Now we prove that \( f_j \to 0 \) weakly in \( \mathcal{N}_p \). For any \( \Lambda \in (\mathcal{N}_p)^* \), by the Hahn–Banach theorem, we can choose a \( \Gamma \in (H_1^{\infty})^* \) such that \( \Gamma = \Lambda \) on \( \mathcal{N}_p \). By the inclusion \( (H_1^{\infty})^* \subset (H_1^{\infty,0})^* \) and Lemma 3 there exists a \( g \in L^1_a(dA) \) such that \( \Gamma(f_j) = \langle g, f_j \rangle_1 \). So we have that
\[
|\Lambda(f_j)| = |\Gamma(f_j)| = |\langle g, f_j \rangle_1|
\]
\[
\leq \liminf_{j \to 1} \int_{\mathbb{D}} |g(tz)| |f_j(tz)|(1 - |z|^2)dA(z)
\]
\[
\leq \int_{\mathbb{D}} |g(z)| |f_j(z)|(1 - |z|^2)dA(z).
\]
Since \( |g(z)| |f_j(z)|(1 - |z|^2) \to 0 \) as \( j \to \infty \), \( |g(z)| |f_j(z)|(1 - |z|^2) \lesssim |g(z)| \) for all \( z \in \mathbb{D} \) and \( |g| \in L^1(dA) \), the dominated convergence theorem shows that \( \Lambda(f_j) \to 0 \) as \( j \to \infty \).
for any \( A \in (\mathcal{N}_p)^* \). Thus we see that \( \|Kf_j\|_{H_\alpha^\infty} \to 0 \) as \( j \to \infty \) for any compact operator \( K : \mathcal{N}_p \to H_\alpha^\infty \).

On the other hand, it holds that

\[
\|uC^\phi f_j\|_{H_\alpha^\infty} \geq \left( 1 - |z_j|^2 \right)^\alpha |u(z_j)\|_{H_\alpha^\infty} \geq |u(z_j)|(1 - |z_j|^2)^\alpha (1 - |\phi(z_j)|^2) \over (1 - |\phi(z_j)|^2)^2 ,
\]

and so we have that

\[
\|uC^\phi f_j\|_{H_\alpha^\infty} \geq \frac{|u(z_j)|(1 - |z_j|^2)^\alpha}{1 - |\phi(z_j)|^2} ,
\]

for any positive integer \( j \). Combining these arguments with the following inequalities

\[
\|uC^\phi - K\| \geq \|uC^\phi f_j\|_{H_\alpha^\infty} \geq \|uC^\phi f_j\|_{H_\alpha^\infty} - \|Kf_j\|_{H_\alpha^\infty} ,
\]

for any compact operator \( K : \mathcal{N}_p \to H_\alpha^\infty \), we obtain that

\[
\|uC^\phi\| \geq \limsup_{j \to \infty} \frac{|u(z_j)|(1 - |z_j|^2)^\alpha}{1 - |\phi(z_j)|^2} .
\]

Since \( \{z_j\} \subset \mathbb{D} \) with \( |\phi(z_j)| \to 1 \) as \( j \to \infty \) is arbitrary, this implies that

\[
\|uC^\phi\| \geq \limsup_{|\phi(z)| \to 1} \frac{|u(z)|(1 - |z|^2)^\alpha}{1 - |\phi(z)|^2} ,
\]

which completes the proof.

**Corollary 2.** Under the same assumptions in Theorem 3, \( uC^\phi : \mathcal{N}_p \to H_\alpha^\infty \) exists as a compact operator if and only if \( u \) and \( \phi \) satisfy

\[
\lim_{|\phi(z)| \to 1} \frac{|u(z)|(1 - |z|^2)^\alpha}{1 - |\phi(z)|^2} = 0 .
\]

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**References**


