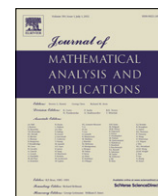


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Generalized Browder's theorem for tensor product and elementary operators

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ABSTRACT

The transfer property for the generalized Browder's theorem both of the tensor product and of the left-right multiplication operator will be characterized in terms of the B -Weyl spectrum inclusion. In addition, the isolated points of these two classes of operators will be fully characterized.

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1. Introduction

In the recent past the relationship between, on the one hand, Weyl and Browder's theorems and their generalizations and, on the other, tensor products and elementary operators has been intensively studied, see for example [1–8]. In particular, given two operators that satisfy Browder's theorem, it is proved in [6] that a necessary and sufficient condition for the tensor product operator to satisfy Browder's theorem is that the Weyl spectrum identity holds, see the latter cited article or Section 4.

The main objective of this work is to characterize when given two operators that satisfy the generalized Browder's theorem, the tensor product operator also satisfies the generalized Browder's theorem, using in particular the B -Weyl spectrum identity. Furthermore, since one inclusion always holds for operators satisfying the generalized Browder's theorem, it is enough to consider the B -Weyl spectrum inclusion, see Section 4. It is worth noticing that since Browder's and the generalized Browder's theorem are equivalent [9], the results of this work also provide a characterization for the transfer property of the Browder's theorem for the tensor product operator.

However, to prove the key characterization of Section 4, the set of isolated points of the tensor product operator need to be studied. In particular, after Section 2 where several basic definitions and facts will be recalled, the poles and the complement of the poles in the isolated points of the tensor product operator will be characterized in terms of the corresponding sets of the source operators. It is important to note that these results continue and deepen the characterization of the isolated points of the tensor product operator presented in [3, see Section 3].

Finally, since the same arguments can be applied to the left–right multiplication operator, similar characterizations will be proved for elementary operators.

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2. Preliminary definitions

From now on \mathcal{X} and \mathcal{Y} shall denote infinite dimensional complex Banach spaces and $B(\mathcal{X}, \mathcal{Y})$ the algebra of all bounded linear maps defined on \mathcal{X} and with values in \mathcal{Y} . As usual, when $\mathcal{X} = \mathcal{Y}$, $B(\mathcal{X}, \mathcal{X}) = B(\mathcal{X})$. Given $A \in B(\mathcal{X})$, $N(A)$, $R(A)$, $\sigma(A)$ and $\sigma_a(A)$ will stand for the null space, the range, the spectrum and the approximate point spectrum of A respectively. In addition, \mathcal{X}^* will denote the dual space of \mathcal{X} , and if $A \in \mathcal{X}$, then $A^* \in B(\mathcal{X}^*)$ will stand for the adjoint map of A .

Recall that $A \in B(\mathcal{X})$ is said to be a *Weyl operator*, if the dimensions both of $N(A)$ and of $\mathcal{X}/R(A)$ are finite and equal. Let $\sigma_w(A)$ be the *Weyl spectrum* of A , i.e., $\sigma_w(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Weyl}\}$, where $A - \lambda$ stands for $A - \lambda I$, I the identity map of \mathcal{X} . Note, in addition, that the concept of Weyl operator has been generalized recently. An operator $A \in B(\mathcal{X})$ will be said to be *B-Weyl*, if there exists $n \in \mathbb{N}$ for which the range of $R(A^n)$ is closed and the induced operator $A_n \in B(R(A^n))$ is Weyl [10]. It is worth noticing that if for some $n \in \mathbb{N}$, $A_n \in B(R(A^n))$ is Weyl, then $A_m \in B(R(A^m))$ is Weyl for all $m \geq n$ [11]. Naturally, from this class of operators the *B-Weyl spectrum* of $A \in B(\mathcal{X})$ can be derived in the usual way; this spectrum will be denoted by $\sigma_{BW}(A)$.

On the other hand, a Banach space operator $A \in B(\mathcal{X})$ is said to be *Drazin invertible*, if there exists a necessarily unique $B \in B(\mathcal{X})$ and some $m \in \mathbb{N}$ such that

$$A^m = A^m B A, \quad B A B = B, \quad A B = B A.$$

If $DR(B(\mathcal{X})) = \{A \in B(\mathcal{X}) : A \text{ is Drazin invertible}\}$, then the *Drazin spectrum* of $A \in B(\mathcal{X})$ is the set $\sigma_{DR}(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin DR(B(\mathcal{X}))\}$ [12,13].

The *ascent* (respectively *the descent*) of $A \in B(\mathcal{X})$ is the smallest non-negative integer $a = asc(A)$ (respectively $d = dsc(A)$) such that $N(A^a) = N(A^{a+1})$ (respectively $R(A^d) = R(A^{d+1})$); if such an integer does not exist, then $asc(A) = \infty$ (respectively $dsc(A) = \infty$). Recall that $\lambda \in \sigma(A)$ is said to be a *pole* of A , if the ascent and the descent of $A - \lambda$ are finite (hence equal). The set of poles of $A \in B(\mathcal{X})$ will be denoted by $\Pi(A)$. Note that $\Pi(A) = \sigma(A) \setminus \sigma_{DR}(A)$ [14, Theorem 4]. In particular, if $A \in B(\mathcal{X})$ is quasi-nilpotent, then according to [14, Theorem 5], necessary and sufficient for A to be nilpotent is that $\Pi(A) = \{0\}$. In addition, the set of *poles of finite rank* of A is the set $\Pi_0(A) = \{\lambda \in \Pi(A) : \alpha(A - \lambda) < \infty\}$, where $\alpha(A - \lambda) = \dim N(A - \lambda)$.

Recall that an operator $A \in B(\mathcal{X})$ is said to satisfy *Browder's theorem*, if $\sigma_w(A) = \sigma(A) \setminus \Pi_0(A)$, while A is said to satisfy the *generalized Browder's theorem*, if $\sigma_{BW}(A) = \sigma(A) \setminus \Pi(A) = \sigma_{DR}(A)$. According to [9, Theorem 2.1], the Browder's and generalized Browder's theorems are equivalent. Moreover, according to [15, Theorem 2.1(iv)], the generalized Browder's theorem is equivalent to the fact that $\text{acc } \sigma(A) \subseteq \sigma_{BW}(A)$. Here and elsewhere in this article, for $K \subseteq \mathbb{C}$, $\text{iso } K$ will stand for the set of isolated points of K and $\text{acc } K = K \setminus \text{iso } K$ for the set of limit points of K . The generalized Browder's theorem was studied in [8,9,15–17].

In what follows, given Banach spaces \mathcal{X} and \mathcal{Y} , $\mathcal{X} \overline{\otimes} \mathcal{Y}$ will stand for the completion, endowed with a reasonable uniform cross-norm, of the algebraic tensor product $\mathcal{X} \otimes \mathcal{Y}$ of \mathcal{X} and \mathcal{Y} . In addition, if $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$, then $A \otimes B \in B(\mathcal{X} \overline{\otimes} \mathcal{Y})$ will denote the tensor product operator defined by A and B .

On the other hand, $\tau_{AB} \in B(B(\mathcal{Y}, \mathcal{X}))$ will denote the multiplication operator defined by $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$, i.e., $\tau_{AB}(U) = AUB$, where $U \in B(\mathcal{Y}, \mathcal{X})$ and \mathcal{X} and \mathcal{Y} are two Banach spaces. Note that $\tau_{AB} = L_A R_B$, where $L_A \in B(B(\mathcal{Y}, \mathcal{X}))$ and $R_B \in B(B(\mathcal{Y}, \mathcal{X}))$ are the left and right multiplication operators defined by A and B respectively, i.e., $L_A(U) = AU$ and $R_B(U) = UB$, $U \in B(\mathcal{Y}, \mathcal{X})$.

3. The isolated points

In this section the isolated points both of the tensor product and of the left–right multiplication operator will be studied. To this end, some preparation is needed.

Remark 3.1. Let \mathcal{X} be a Banach space, consider $A \in B(\mathcal{X})$ and set $I(A) = \text{iso } \sigma(A) \setminus \Pi(A)$.

- (i) Necessary and sufficient for $\lambda \in \sigma(A)$ to belong to $I(A)$ is that there exist M and N , two closed and complemented subspaces of \mathcal{X} invariant for A , such that if $A_1 = A|_M$ and $A_2 = A|_N$, then $A_1 - \lambda$ is quasi-nilpotent but not nilpotent and $A_2 - \lambda$ is invertible. Note that $\sigma(A) = I(A) = \{\lambda\}$ if and only if $N = 0$.
- (ii) Let $\lambda \in \sigma(A)$. The complex number λ belongs to $\Pi(A)$ if and only if there are M' and N' two closed and complemented subspaces of \mathcal{X} invariant for A , such that if $A' = A|_{M'}$ and $A'' = A|_{N'}$, then $A' - \lambda$ is nilpotent and $A'' - \lambda$ is invertible. As in statement (i), $\sigma(A) = \Pi(A) = \{\lambda\}$ is equivalent to the fact that $N' = 0$. Statements (i)–(ii) are well known and they can be easily deduced from [13, Theorem 12] and [14, Theorem 5]. Now let \mathcal{Y} be a Banach space and consider $B \in B(\mathcal{Y})$.
- (iii) Since $\sigma(A \otimes B) = \sigma(A)\sigma(B) = \sigma(\tau_{AB})$ [18, Theorem 2.1] and [19, Corollary 3.4], according to [3, Theorem 6],

$$(\text{iso } \sigma(A \otimes B)) \setminus \{0\} = (\text{iso } (\tau_{AB})) \setminus \{0\} = (\text{iso } \sigma(A) \setminus \{0\})(\text{iso } \sigma(B) \setminus \{0\}).$$

(iv) Set

$$\mathbb{L} = (I(A) \setminus \{0\})(I(B) \setminus \{0\}) \cup (I(A) \setminus \{0\})(\Pi(B) \setminus \{0\}) \cup (\Pi(A) \setminus \{0\})(I(B) \setminus \{0\}).$$

Then clearly, $(\text{iso } \sigma(A \otimes B)) \setminus \{0\} = (\text{iso } (\tau_{AB})) \setminus \{0\} = \mathbb{L} \cup (\Pi(A) \setminus \{0\})(\Pi(B) \setminus \{0\})$.

- (v) Let $\lambda \in (\text{iso } \sigma(A \otimes B)) \setminus \{0\} = (\text{iso } (\tau_{AB})) \setminus \{0\}$. Then, it is not difficult to prove that there exist finite sequences $\{\mu_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$ of points $\mu_i \in \text{iso } \sigma(A) \setminus \{0\}$ and $v_i \in \text{iso } \sigma(B) \setminus \{0\}$ such that $\lambda = \mu_i v_i$ for all $i = 1, \dots, n$.
- (vi) Note that if $0 \in \text{iso } \sigma(A \otimes B) = \text{iso } \sigma(\tau_{AB})$, then one of the following possibilities holds:
- (a) $\sigma(A) = \{0\}$ or $\sigma(B) = \{0\}$;
- (b) $(\sigma(A) \neq \{0\} \text{ and } \sigma(B) \neq \{0\}) \implies 0 \in \text{iso } \sigma(A) \text{ and } 0 \notin \sigma(B)$ or $0 \notin \sigma(A) \text{ and } 0 \in \text{iso } \sigma(B)$;
- (c) $(\sigma(A) \neq \{0\}, \sigma(B) \neq \{0\}, 0 \in \sigma(A) \cap \sigma(B)) \implies 0 \in \text{iso } \sigma(A) \cap \text{iso } \sigma(B)$.

In the next theorem the position of $0 \in \mathbb{C}$ in the isolated points will be characterized. To this end, if \mathcal{X} and \mathcal{Y} are two Banach spaces, then I_1 and I_2 will denote the identity map on \mathcal{X} and \mathcal{Y} respectively. Moreover, given $x \in \mathcal{X}$ and $f \in \mathcal{Y}^*$, $U_{x,f} \in B(\mathcal{Y}, \mathcal{X})$ is the map defined as follows: $U_{x,f}(y) = xf(y)$, $y \in \mathcal{Y}$.

Theorem 3.2. *Let \mathcal{X} and \mathcal{Y} be two Banach spaces and consider $A \in B(\mathcal{X})$, $B \in B(\mathcal{Y})$, $A \otimes B \in B(\mathcal{X} \overline{\otimes} \mathcal{Y})$ and $\tau_{AB} \in B(B(\mathcal{Y}, \mathcal{X}))$. Suppose that $0 \in \text{iso } \sigma(A \otimes B) = \text{iso } \sigma(\tau_{AB})$.*

- (i) *If $\sigma(A) = \Pi(A) = \{0\}$ or $\sigma(B) = \Pi(B) = \{0\}$, then $\sigma(A \otimes B) = \Pi(A \otimes B) = \{0\} = \Pi(\tau_{AB}) = \sigma(\tau_{AB})$.*
- (ii) *If $\sigma(A) = I(A) = \{0\}$ and B is not nilpotent or $\sigma(B) = I(B) = \{0\}$ and A is not nilpotent, then $\sigma(A \otimes B) = I(A \otimes B) = \{0\} = I(\tau_{AB}) = \sigma(\tau_{AB})$.*
- (iii) *If $0 \in \Pi(A)$ and $0 \notin \sigma(B)$ or $0 \notin \sigma(A)$ and $0 \in \Pi(B)$, then $0 \in \Pi(A \otimes B) \cap \Pi(\tau_{AB})$.*
- (iv) *If $0 \in I(A)$ ($\sigma(A) \neq \{0\}$) and $0 \notin \sigma(B)$ or $0 \notin \sigma(A)$ and $0 \in I(B)$ ($\sigma(B) \neq \{0\}$), then $0 \in I(A \otimes B) \cap I(\tau_{AB})$.*
- (v) *If $0 \in \Pi(A) \cap \Pi(B)$, then $0 \in \Pi(A \otimes B) \cap \Pi(\tau_{AB})$.*
- (vi) *If $0 \in I(A) \cap \Pi(B)$ and B is not nilpotent, $0 \in \Pi(A) \cap I(B)$ and A is not nilpotent, or $0 \in I(A) \cap I(B)$, then $0 \in I(A \otimes B) \cap I(\tau_{AB})$.*

Proof. (i) According to Remark 3.1(ii), A or B is nilpotent, which implies that $A \otimes B$ is nilpotent.

On the other hand, since $L_A \in B(B(\mathcal{X}))$ or $R_B \in B(B(\mathcal{Y}))$ is nilpotent, τ_{AB} is nilpotent.

(ii) Suppose that $\sigma(A) = I(A) = \{0\}$ and B is not nilpotent. Clearly, $\sigma(A \otimes B) = \{0\}$. In addition, according to Remark 3.1(i), A is not nilpotent. In particular, for each $k \in \mathbb{N}$ there exist $x_k \in \mathcal{X}$ and $y_k \in \mathcal{Y}$ such that $\|A^k(x_k)\| = 1$ and $\|B^k(y_k)\| = 1$. Therefore, since $\mathcal{X} \overline{\otimes} \mathcal{Y}$ is endowed with a reasonable uniform cross norm, $\|(A \otimes B)^k(x_k \otimes y_k)\| = 1$, for each $k \in \mathbb{N}$. As a result, $A \otimes B$ is not nilpotent, equivalently $I(A \otimes B) = \{0\}$.

On the other hand, it is clear that $\sigma(\tau_{AB}) = \{0\}$. Moreover, since B is not nilpotent, $B^* \in B(\mathcal{Y}^*)$ is not nilpotent. In particular, for each $k \in \mathbb{N}$ there exist $x_k \in \mathcal{X}$ and $f_k \in \mathcal{Y}^*$ such that $\|A^k(x_k)\| = 1$ and $\|(B^*)^k(f_k)\| = 1$. Consider $U_{x_k, f_k} \in B(\mathcal{Y}, \mathcal{X})$. Then, $\|\tau_{AB}^k(U_{x_k, f_k})\| = 1$. Consequently, τ_{AB} is not nilpotent and $I(\tau_{AB}) = \{0\}$.

The remaining case can be proved in a similar way.

(iii) If $0 \in \Pi(A)$ and $0 \notin \sigma(B)$ or $0 \notin \sigma(A)$ and $0 \in \Pi(B)$, then it is not difficult to prove that $A \otimes B$ and τ_{AB} are Drazin invertible, equivalently $0 \in \Pi(A \otimes B) \cap \Pi(\tau_{AB})$.

(iv) If $0 \in I(A)$, then, according to Remark 3.1(i), there exist M_1 and M_2 two closed and complemented subspaces of \mathcal{X} invariant for A such that $A_1 \in B(M_1)$ is quasi-nilpotent but not nilpotent and $A_2 \in B(M_2)$ is invertible, where $A_1 = A|_{M_1}$ and $A_2 = A|_{M_2}$. Now, clearly $\mathcal{X} \overline{\otimes} \mathcal{Y} = M_1 \overline{\otimes} \mathcal{Y} \oplus M_2 \overline{\otimes} \mathcal{Y}$, $A_1 \otimes B$ is quasi-nilpotent and, since $M_2 \neq 0$ ($\sigma(A) \neq \{0\}$), $A_2 \otimes B$ is invertible. However, using an argument similar to the one in the proof of statement (ii), $A_1 \otimes B$ is not nilpotent. Consequently, according to Remark 3.1(i), $0 \in I(A \otimes B)$.

To prove that $0 \in I(\tau_{AB})$, consider the decompositions of \mathcal{X} and A recalled in the previous paragraph. Note that $B(\mathcal{Y}, \mathcal{X}) = B(\mathcal{Y}, M_1) \oplus B(\mathcal{Y}, M_2)$ and then, decomposing τ_{AB} as a block operator, τ_{AB} is a diagonal operator with entries $\tau_{A_1 B} \in B(B(\mathcal{Y}, M_1))$ and $\tau_{A_2 B} \in B(B(\mathcal{Y}, M_2))$. Clearly, $\tau_{A_1 B}$ is quasi-nilpotent and $\tau_{A_2 B}$ is invertible. However, using an argument similar to the one in the proof of statement (ii), $\tau_{A_1 B}$ is not nilpotent. In particular, $0 \in I(\tau_{AB})$.

The remaining case can be proved in a similar way.

(v) If $0 \in \Pi(A) \cap \Pi(B)$, then A and B are Drazin invertible, which implies that $A \otimes I_2$ and $I_1 \otimes B$ are Drazin invertible. Since $A \otimes I_2$ and $I_1 \otimes B$ commute, according to [12, Proposition 2.6], $A \otimes B$ is Drazin invertible, equivalently $0 \in \Pi(A \otimes B)$.

On the other hand, it is not difficult to prove that L_A and R_B are Drazin invertible. Moreover, since L_A and R_B commute, τ_{AB} is Drazin invertible, in particular $0 \in \Pi(\tau_{AB})$.

(vi) If $0 \in I(A) \cap \Pi(B)$, then, according to Remark 3.1(i)–(ii), there exist M_1 and M_2 (respectively N_1 and N_2) two closed and complemented subspaces of \mathcal{X} (respectively \mathcal{Y}) invariant for A (respectively B) such that A_1 is quasi-nilpotent but not nilpotent and A_2 is invertible (respectively B_1 is nilpotent and B_2 is invertible), where $A_1 = A|_{M_1}$ and $A_2 = A|_{M_2}$ (respectively $B_1 = B|_{N_1}$ and $B_2 = B|_{N_2}$). Now, it is clear that $\mathcal{X} \overline{\otimes} \mathcal{Y} = M_1 \overline{\otimes} N_1 \oplus M_2 \overline{\otimes} N_1 \oplus M_1 \overline{\otimes} N_2 \oplus M_2 \overline{\otimes} N_2$, $A_1 \otimes B_1 \in B(M_1 \overline{\otimes} N_1)$ and $A_2 \otimes B_1 \in B(M_2 \overline{\otimes} N_1)$ are nilpotent, $A_1 \otimes B_2 \in B(M_1 \overline{\otimes} N_2)$ is quasi-nilpotent and $A_2 \otimes B_2 \in B(M_2 \overline{\otimes} N_2)$ is invertible. As a result, to prove that $0 \in I(A \otimes B)$, it is enough to prove that $A_1 \otimes B_2 \in B(M_1 \overline{\otimes} N_2)$ is not nilpotent. However, since B is not nilpotent, $N_2 \neq 0$, and then, using the argument in the proof of statement (ii), $A_1 \otimes B_2$ is not nilpotent.

On the other hand, according to the decomposition of \mathcal{X} and \mathcal{Y} recalled in the previous paragraph, $\tau_{AB} \in B(B(\mathcal{Y}, \mathcal{X}))$ can be considered as a diagonal operator with diagonal entries $(\tau_{AB})_{11} \in B(B(N_1, M_1))$, $(\tau_{AB})_{22} \in B(B(N_2, M_1))$, $(\tau_{AB})_{33} \in B(B(N_1, M_2))$ and $(\tau_{AB})_{44} \in B(B(N_2, M_2))$. Clearly, $(\tau_{AB})_{11}$ and $(\tau_{AB})_{33}$ are nilpotent, $(\tau_{AB})_{44}$ is invertible and $(\tau_{AB})_{22}$ is quasi-nilpotent. Thus, to prove that $0 \in I(\tau_{AB})$, it is enough to prove that $(\tau_{AB})_{22}$ is not nilpotent. However, since $N_2 \neq 0$ and B_2 is invertible, using the argument in the proof of statement (ii), $(\tau_{AB})_{22} \in B(B(N_2, M_1))$ is not nilpotent.

Similar arguments prove the remaining cases both for $A \otimes B$ and for τ_{AB} . \square

The following proposition will be useful to study the isolated non-null points.

Proposition 3.3. *Let \mathcal{X} and \mathcal{Y} be two Banach spaces and suppose that $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$ are such that $\sigma(A) = \{\mu\}$, $\sigma(B) = \{\nu\}$, $\mu\nu \neq 0$. Consider $A \otimes B \in B(\mathcal{X} \otimes \mathcal{Y})$ and $\tau_{AB} \in B(B(\mathcal{Y}, \mathcal{X}))$. Then, $\sigma(A \otimes B) = \sigma(\tau_{AB}) = \{\mu\nu\}$ and the following statements hold.*

- (i) *If $A - \mu$ and $B - \nu$ are nilpotent, then $A \otimes B - \mu\nu$ and $\tau_{AB} - \mu\nu$ are nilpotent.*
- (ii) *If either $A - \mu$ or $B - \nu$ is quasi-nilpotent but not nilpotent, then $A \otimes B - \mu\nu$ and $\tau_{AB} - \mu\nu$ are not nilpotent.*

Proof. Clearly $\sigma(A \otimes B) = \sigma(\tau_{AB}) = \sigma(A)\sigma(B) = \{\mu\nu\}$.

(i) Note that $A \otimes B - \mu\nu = (A - \mu) \otimes B + \mu \otimes (B - \nu)$. Since $(A - \mu) \otimes B$ and $\mu \otimes (B - \nu)$ are nilpotent and commute, an easy calculation proves that $A \otimes B - \mu\nu$ is nilpotent.

On the other hand, since $\tau_{AB} - \mu\nu = L_{(A-\mu)}R_B + \mu R_{(B-\nu)}$, a similar argument proves that $\tau_{AB} - \mu\nu$ is nilpotent.

(ii) Since $A \otimes B - \mu\nu = (A - \mu) \otimes B + \mu \otimes (B - \nu)$, it is not difficult to prove that

$$(A \otimes B - \mu\nu)I_1 \otimes B^{-1} = I_1 \otimes B^{-1}(A \otimes B - \mu\nu) = (A - \mu) \otimes I_2 - \mu\nu \otimes (B^{-1} - \nu^{-1}).$$

Moreover, since $A \otimes B - \mu\nu$ and $I_1 \otimes B^{-1}$ commute, $A \otimes B - \mu\nu$ is nilpotent if and only if $(A \otimes B - \mu\nu)I_1 \otimes B^{-1}$ is nilpotent.

Suppose that $(B - \nu) \in B(\mathcal{Y})$ is quasi-nilpotent but not nilpotent. Then, $(B^{-1} - \nu^{-1}) \in B(\mathcal{Y})$ is quasi-nilpotent but not nilpotent. In fact, it is clear that $\sigma(B^{-1}) = \{\nu^{-1}\}$. In addition, if $B^{-1} - \nu^{-1}$ were nilpotent, then a straightforward calculation proves that B must be algebraic. However, since $B - \nu$ is quasi-nilpotent, $B - \nu$ must be nilpotent, which is impossible.

Next note that since $\sigma_{\bar{a}}(A) = \sigma(A) = \{\mu\}$, there exists $(x_n)_{n \in \mathbb{N}} \subset \mathcal{X}$ such that $\|x_n\| = 1$, $n \in \mathbb{N}$, and $((A - \mu)(x_n))_{n \in \mathbb{N}}$ converges to $0 \in \mathcal{X}$. Then, given $k \in \mathbb{N}$, $c_{k,j} = \frac{k!}{(k-j)!j!}$ and $y_k \in \mathcal{Y}$ such that $|\mu\nu|^k \|(B^{-1} - \nu^{-1})^k(y_k)\| = 2$, there exist $n_k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_k$, $\|\sum_{j=1}^k c_{k,j}(-\mu\nu)^{k-j}(A - \mu)^j(x_n) \otimes (B^{-1} - \nu^{-1})^{k-j}(y_k)\| < 1$. As a result, for $n \geq n_k$,

$$\|((A - \mu) \otimes I_2 - \mu\nu \otimes (B^{-1} - \nu^{-1}))^k(x_n \otimes y_k)\| > 1.$$

Therefore, $A \otimes B - \mu\nu$ is not nilpotent. A similar argument, using $A \otimes B - \mu\nu = A \otimes (B - \nu) + (A - \mu) \otimes \nu$, proves the case $A - \mu$ quasi-nilpotent but not nilpotent for the tensor product operator.

On the other hand, since $\tau_{AB} - \mu\nu = L_{(A-\mu)}R_B + \mu R_{(B-\nu)}$, adapting the argument used before it is not difficult to prove that $\tau_{AB} - \mu\nu$ is not nilpotent if and only if $L_{(A-\mu)} - \mu\nu R_{(B^{-1}-\nu^{-1})}$ is not nilpotent. To prove this latter fact, consider the same sequence $(x_n)_{n \in \mathbb{N}} \subset \mathcal{X}$ of the tensor product operator case. In addition, since $(B^{-1} - \nu^{-1})^* \in B(\mathcal{Y}^*)$ is not nilpotent, for each $k \in \mathbb{N}$ there exists $f_k \in \mathcal{Y}^*$ such that $|\mu\nu|^k \|((B^{-1} - \nu^{-1})^*)^k(f_k)\| = 2$. However, an argument similar to the one used in the tensor product operator case proves that there is $n \in \mathbb{N}$ such that

$$\|(L_{(A-\mu)} - \mu\nu R_{(B^{-1}-\nu^{-1})})^k(U_{x_n, f_k})\| > 1.$$

Therefore, $\tau_{AB} - \mu\nu$ is not nilpotent. A similar argument, using $\tau_{AB} - \mu\nu = L_A R_{(B-\nu)} + \nu L_{(A-\mu)}$, proves the case $A - \mu$ quasi-nilpotent but not nilpotent for the multiplication operator. \square

Given \mathcal{X} and \mathcal{Y} two Banach spaces and $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$, in [3, Theorem 6] the limit and the isolated points both of the tensor product operator $A \otimes B \in B(\mathcal{X} \otimes \mathcal{Y})$ and of the elementary operator $\tau_{AB} \in B(B(\mathcal{Y}, \mathcal{X}))$ were studied. In the following theorem $I(A \otimes B) \setminus \{0\}$, $I(\tau_{AB}) \setminus \{0\}$, $\Pi(A \otimes B) \setminus \{0\}$ and $\Pi(\tau_{AB}) \setminus \{0\}$ will be characterized in terms of the corresponding sets of A and B .

Theorem 3.4. *Let \mathcal{X} and \mathcal{Y} be two Banach spaces and consider $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$. Then, the following statements hold.*

- (i) $\mathbb{L} = I(A \otimes B) \setminus \{0\} = I(\tau_{AB}) \setminus \{0\}$.
- (ii) $\Pi(A \otimes B) \setminus \{0\} = \Pi(\tau_{AB}) \setminus \{0\} = (\Pi(A) - \{0\})(\Pi(B) - \{0\}) \setminus \mathbb{L}$.

Proof. In the first place, note that according to Remark 3.1(iv), statement (i) implies statement (ii).

To prove statement (i), let $\lambda \in \text{iso } \sigma(A \otimes B) \setminus \{0\}$. Then, according to Remark 3.1(v), there exist $n \in \mathbb{N}$ and finite spectral sets $\{\mu\} = \{\mu_1, \dots, \mu_n\} \subseteq \text{iso } \sigma(A)$ and $\{\nu\} = \{\nu_1, \dots, \nu_n\} \subseteq \text{iso } \sigma(B)$ such that $\lambda = \mu_i \nu_i$ for all $1 \leq i \leq n$. Corresponding to these spectral sets there are closed subspaces M_1, M_2 and $(M_{1i})_{i=1}^n$ of \mathcal{X} invariant for A and closed subspaces N_1, N_2 and $(N_{1i})_{i=1}^n$ of \mathcal{Y} invariant for B such that $\mathcal{X} = M_1 \oplus M_2$, $M_1 = \bigoplus_{i=1}^n M_{1i}$, $\mathcal{Y} = N_1 \oplus N_2$, $N_1 = \bigoplus_{i=1}^n N_{1i}$, $\sigma(A_1) = \{\mu\}$, $\sigma(A_2) = \sigma(A) \setminus \{\mu\}$, $\sigma(A_{1i}) = \{\mu_i\}$, $\sigma(B_1) = \{\nu\}$, $\sigma(B_2) = \sigma(B) \setminus \{\nu\}$ and $\sigma(B_{1i}) = \{\nu_i\}$, where $A_1 = A|_{M_1}$, $A_2 = A|_{M_2}$, $A_{1i} = A|_{M_{1i}}$, $B_1 = B|_{N_1}$, $B_2 = B|_{N_2}$ and $B_{1i} = B|_{N_{1i}}$. Note that $A \otimes B - \lambda$ is invertible on the closed invariant subspaces $M_1 \otimes \overline{N_2}, M_2 \otimes \overline{N_1}, M_2 \otimes \overline{N_2}$ and $M_{1j} \otimes \overline{N_{1k}}$, $1 \leq j \neq k \leq n$. Moreover, $\mathcal{X} \otimes \mathcal{Y}$ is the direct sum of these subspaces and $M_{1i} \otimes \overline{N_{1i}}$, $1 \leq i \leq n$.

Suppose that $\lambda \in \mathbb{L}$. Then, there exist $\mu \in \text{iso } \sigma(A) \setminus \{0\}$ and $\nu \in \text{iso } \sigma(B) \setminus \{0\}$ such that $\lambda = \mu\nu$ and either $\mu \in I(A) \setminus \{0\}$ or $\nu \in I(B) \setminus \{0\}$. Applying what has been done in the previous paragraph to $\lambda \in \mathbb{L}$, there exist an $n = n(\lambda) \in \mathbb{N}$ and an i , $1 \leq i \leq n$, such that $\mu = \mu_i$ and $\nu = \nu_i$. Therefore, according to Proposition 3.3(ii) and Remark 3.1(i), $\lambda \in I(A \otimes B) \setminus \{0\}$.

On the other hand, consider $\lambda \in I(A \otimes B) \setminus \{0\}$. As before, there exist an $n = n(\lambda) \in \mathbb{N}$ and $\mu_i \in \text{iso } \sigma(A) \setminus \{0\}$ and $\nu_i \in \text{iso } \sigma(B) \setminus \{0\}$ such that $\lambda = \mu_i \nu_i$, $i = 1, \dots, n$. Now, if $\lambda \notin \mathbb{L}$, then for each $i = 1, \dots, n$, $\mu_i \in (\Pi(A) \setminus \{0\})$ and $\nu_i \in (\Pi(B) \setminus \{0\})$. However, according to Proposition 3.3(i) and Remark 3.1(ii), $\lambda \in (\Pi(A \otimes B) \setminus \{0\})$, which is impossible.

To prove that $\mathbb{L} = I(\tau_{AB}) \setminus \{0\}$, as in the tensor product operator case, consider the decompositions of \mathcal{X} and \mathcal{Y} into closed complemented invariant subspaces for A and B respectively and, as in [Theorem 3.2](#), decompose τ_{AB} as a diagonal operator. Then, to conclude the proof, adapt the argument developed to prove that $\mathbb{L} = I(A \otimes B) \setminus \{0\}$ to the case under consideration. \square

Applying the main results of this section, it is not difficult to prove that the Drazin spectra of the tensor product and of the elementary operator coincide. Note that since the spectra of these operators are equal, both the set of limit points and the one of isolated points of the aforementioned operators are identical.

Corollary 3.5. *Let \mathcal{X} and \mathcal{Y} be two Banach spaces and consider $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$. Then, the following statements hold.*

- (i) $\Pi(A \otimes B) = \Pi(\tau_{AB})$.
- (ii) $I(A \otimes B) = I(\tau_{AB})$.
- (iii) $\sigma_{DR}(A \otimes B) = \sigma_{DR}(\tau_{AB})$.

Proof. Statements (i)–(ii) can be derived from [Theorems 3.2](#) and [3.4](#). To prove statement (iii), apply [[13](#), [Theorem 12](#)]. \square

4. The B -Weyl spectrum inclusion

Recall that given $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$ two operators satisfying Browder's theorem, the *Weyl spectrum equality* for $A \otimes B$, i.e., the identity

$$\sigma_w(A \otimes B) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B),$$

is equivalent to the fact that $A \otimes B$ satisfies Browder's theorem [[6](#), [Theorem 3](#)]. Note that the inclusion

$$\sigma_w(A \otimes B) \subseteq \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B)$$

always holds, so that the relevant inclusion is the reverse inclusion “ \supseteq ”.

Similarly, under the same conditions for A and B , the *Weyl spectrum equality* for τ_{AB} , i.e., the identity

$$\sigma_w(\tau_{AB}) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B),$$

is equivalent to the fact that τ_{AB} satisfies Browder's theorem [[8](#), [Theorem 4.5](#)]. As in the tensor product operator case, the following inclusion always holds:

$$\sigma_w(\tau_{AB}) \subseteq \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B).$$

Given $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$ two operators that satisfy the generalized Browder's theorem, the *B -Weyl spectrum inclusion* for $A \otimes B$ (respectively for τ_{AB}) will be said to hold, if

$$\begin{aligned} \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) &\subseteq \sigma_{BW}(A \otimes B) \\ (\text{respectively if } \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) &\subseteq \sigma_{BW}(\tau_{AB})). \end{aligned}$$

In this section the B -Weyl spectrum inclusion will be studied in relation to the *transfer property for the generalized Browder's theorem*, i.e., the conditions under which given $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$ two operators that satisfy the generalized Browder's theorem, $A \otimes B \in B(\mathcal{X} \otimes \mathcal{Y})$ and $\tau_{AB} \in B(B(\mathcal{Y}, \mathcal{X}))$ also satisfy the generalized Browder's theorem. Note that since the Browder's and generalized Browder's theorems are equivalent [[9](#), [Theorem 2.1](#)], the results of this section also provide a characterization of the transfer property for the Browder's theorem both for the tensor product and the left–right multiplication operator.

In the first place the B -Weyl spectrum inclusion will be proved to be an equality, when it holds. However, since for the main results of this article the relevant condition is an inclusion, the B -Weyl spectrum inclusion will be focused on.

Lemma 4.1. *Let \mathcal{X} and \mathcal{Y} be two Banach spaces and consider $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$ two operators that satisfy the generalized Browder's theorem. Then,*

$$(\sigma_{BW}(A \otimes B) \cup \sigma_{BW}(\tau_{AB})) \subseteq \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B).$$

Proof. Suppose that $0 \in \sigma_{BW}(A \otimes B)$. Then, according to [[10](#), [Theorem 2.3](#)], $0 \in \text{acc } \sigma(A \otimes B)$ or $0 \in I(A \otimes B)$. Since A and B satisfy the generalized Browder's theorem, if $0 \in \text{acc } \sigma(A \otimes B)$, then $0 \in \text{acc } \sigma(A) \subseteq \sigma_{BW}(A)$ or $0 \in \text{acc } \sigma(B) \subseteq \sigma_{BW}(B)$. On the other hand, if $0 \in I(A \otimes B)$, then according to [Theorem 3.2](#), A and B are not nilpotent and $0 \in I(A) \subseteq \sigma_{BW}(A)$ or $0 \in I(B) \subseteq \sigma_{BW}(B)$. However, in all these cases $0 \in \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B)$.

Next consider $\lambda \neq 0$, $\lambda \in \sigma(A \otimes B) \setminus (\sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B))$. In particular, for each $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$ such that $\lambda = \mu\nu$, $\mu \in \sigma(A) \setminus \sigma_{BW}(A)$ and $\nu \in \sigma(B) \setminus \sigma_{BW}(B)$. However, since A and B satisfy the generalized Browder's theorem, $\mu \in \Pi(A)$ and $\nu \in \Pi(B)$. Consequently, according to [Theorem 3.4](#), $\lambda \in \Pi(A \otimes B)$. Therefore, $\lambda \in \sigma(A \otimes B) \setminus \sigma_{BW}(A \otimes B)$ [[10](#), [Theorem 2.3](#)].

A similar argument proves the inclusion for τ_{AB} . \square

In what follows the transfer property for the generalized Browder's theorem will be studied.

Theorem 4.2. *Let \mathcal{X} and \mathcal{Y} be two Banach spaces and consider $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$ two operators that satisfy the generalized Browder's theorem. If the B -Weyl spectrum inclusion for $A \otimes B$ (respectively for τ_{AB}) holds, then $A \otimes B$ (respectively τ_{AB}) satisfies the generalized Browder's theorem.*

Proof. According to [15, Theorem 2.1(iv)], $\text{acc } \sigma(A) \subseteq \sigma_{BW}(A)$ and $\text{acc } \sigma(B) \subseteq \sigma_{BW}(B)$. Now, since the B -Weyl spectrum inclusion for $A \otimes B$ holds, according to [3, Theorem 6],

$$\text{acc } \sigma(A \otimes B) \subseteq \sigma(A)(\text{acc } \sigma(B)) \cup (\text{acc } \sigma(A))\sigma(B) \subseteq \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) \subseteq \sigma_{BW}(A \otimes B).$$

Therefore, $A \otimes B$ satisfies the generalized Browder's theorem. Since $\sigma(\tau_{AB}) = \sigma(A \otimes B) = \sigma(A)\sigma(B)$, the same argument proves the statement concerning the operator τ_{AB} . \square

Remark 4.3. (i) Note that the converse of Theorem 4.2 does not in general hold. In fact, let \mathcal{X} and \mathcal{Y} be two Banach spaces and consider $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$ two operators such that A is nilpotent and B satisfies the generalized Browder's theorem. As a result, $A \otimes B \in B(\mathcal{X} \otimes \mathcal{Y})$ is nilpotent, what is more, A and $A \otimes B$ satisfy the generalized Browder's theorem (the sets of limit points of these two operators are empty). On the other hand, since A and $A \otimes B$ are nilpotent, according to [10, Theorem 2.3], $\sigma_{BW}(A) = \emptyset = \sigma_{BW}(A \otimes B)$. In particular, necessary and sufficient for $\sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) = \emptyset$ is that $\sigma_{BW}(B) = \emptyset$ (observe, however, that the operators A , B and $A \otimes B$ satisfy the equality $\sigma_w(A \otimes B) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B)$). Naturally, the same can be said for the operator τ_{AB} .

(ii) Let \mathcal{X} be a Banach space and consider $A \in B(\mathcal{X})$ an operator that satisfies the generalized Browder's theorem. According to [13, Theorem 3], [20, Theorem 1.5] and [21, Theorem 2.7], $\sigma_{BW}(A) = \emptyset$ if and only if A is algebraic, i.e., there exists a non-constant polynomial $P \in \mathbb{C}[X]$ such that $P(A) = 0$. Clearly, since the spectrum of an algebraic operator is a finite set (actually in this case $\sigma(A) = \Pi(A)$ [20, Theorem 1.5]), algebraic operators satisfy the generalized Browder's theorem. Moreover, if $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$ are algebraic operators, then $\sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) = \emptyset$ and the B -Weyl spectrum inclusion both for $A \otimes B \in B(\mathcal{X} \otimes \mathcal{Y})$ and for $\tau_{AB} \in B(B(\mathcal{Y}, \mathcal{X}))$ holds. Furthermore, it is not difficult to prove, using in particular Theorems 3.2 and 3.4, that $A \otimes B$ and τ_{AB} satisfy the generalized Browder's theorem and $\sigma_{BW}(A \otimes B) = \emptyset = \sigma_{BW}(\tau_{AB})$. Therefore, to characterize when the transfer property implies the B -Weyl spectrum inclusion, it is enough to consider two cases: first, when only one operator is algebraic (observe that according to (i) the algebraic operator must not be nilpotent); second, when both operators are not algebraic.

Before going on, to study the converse of Theorem 4.2 set

$$\mathbb{S} = \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B).$$

Theorem 4.4. *Let \mathcal{X} and \mathcal{Y} be two Banach spaces and consider $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$ such that A is an algebraic but not nilpotent operator and B is a non-algebraic operator that satisfies the generalized Browder's theorem. Then, if $A \otimes B \in B(\mathcal{X} \otimes \mathcal{Y})$ (respectively if $\tau_{AB} \in B(B(\mathcal{Y}, \mathcal{X}))$) satisfies the generalized Browder's theorem, the following statements are equivalent.*

- (i) *The B -Weyl spectrum inclusion for $A \otimes B$ (respectively for τ_{AB}) holds;*
- (ii) *B is not Drazin invertible.*

Furthermore, if one of the equivalent statements holds, then $\mathbb{S} = \sigma_{BW}(A \otimes B)$ (respectively $\mathbb{S} = \sigma_{BW}(\tau_{AB})$), while if this is not the case, then $\mathbb{S} = \sigma_{BW}(A \otimes B) \cup \{0\}$ (respectively $\mathbb{S} = \sigma_{BW}(\tau_{AB}) \cup \{0\}$).

Proof. Note that since according to Remark 4.3(ii) $\sigma(A) = \Pi(A)$, $\mathbb{S} = \Pi(A)\sigma_{BW}(B)$. On the other hand, recall that $\sigma_{BW}(B) = I(B) \cup \text{acc } \sigma(B)$ and $\sigma_{BW}(A \otimes B) = I(A \otimes B) \cup \text{acc } \sigma(A \otimes B)$ [13, Theorem 12]. In particular, since $I(A \otimes B) \setminus \{0\} = (\Pi(A) \setminus \{0\})(I(B) \setminus \{0\})$ (Theorem 3.4) and $\text{acc } \sigma(A \otimes B) \setminus \{0\} = (\Pi(A) \setminus \{0\})(\text{acc } \sigma(B) \setminus \{0\})$ [3, Theorem 6], $\mathbb{S} \setminus \{0\} = \sigma_{BW}(A \otimes B) \setminus \{0\}$. Therefore, according to Lemma 4.1, statement (i) is equivalent to the following implication: $0 \in \mathbb{S} \Rightarrow 0 \in \sigma_{BW}(A \otimes B)$. Now, $0 \in \mathbb{S}$ if and only if $0 \in \Pi(A)$ or $0 \in \sigma_{BW}(B)$. If $0 \in \sigma_{BW}(B)$, then using in particular statements (ii), (iv) and (vi) of Theorem 3.2 and the fact that A is not nilpotent, it is not difficult to prove that $0 \in \sigma_{BW}(A \otimes B)$. On the other hand, if $0 \in \Pi(A)$, according to what has been proved, if $0 \in \sigma_{BW}(B)$, then $0 \in \sigma_{BW}(A \otimes B)$, while if $0 \in \Pi(B)$ or $0 \notin \sigma(B)$, equivalently if B is Drazin invertible, then according to statements (iii) and (v) of Theorem 3.2, $0 \in \Pi(A \otimes B) = \sigma(A \otimes B) \setminus \sigma_{BW}(A \otimes B)$. Consequently, since all the possible cases have been considered, the B -Weyl spectrum inclusion for $A \otimes B$ holds if and only if B is not Drazin invertible.

The last statement is clear.

The statements concerning the operator $\tau_{AB} \in B(B(\mathcal{Y}, \mathcal{X}))$ can be proved in a similar way. \square

Naturally, under the same conditions as Theorem 4.4, if the properties of A and B are interchanged, similar statements can be proved. Next follows the remaining case, i.e., when both operators are not algebraic.

Theorem 4.5. *Let \mathcal{X} and \mathcal{Y} be two Banach spaces and consider $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$ two non-algebraic operators that satisfy the generalized Browder's theorem. Then, if $A \otimes B \in B(\mathcal{X} \otimes \mathcal{Y})$ (respectively if $\tau_{AB} \in B(B(\mathcal{Y}, \mathcal{X}))$) satisfies the generalized Browder's theorem, the following statements are equivalent.*

- (i) The B -Weyl spectrum inclusion for $A \otimes B$ (respectively for τ_{AB}) holds;
(ii) $0 \notin \Pi(A \otimes B)$ ($=\Pi(\tau_{AB})$);
(iii) $A \otimes B$ (respectively τ_{AB}) is invertible or $A \otimes B$ (respectively τ_{AB}) is not Drazin invertible.
Furthermore, if one of the equivalent statements holds, then $\mathbb{S} = \sigma_{BW}(A \otimes B)$ (respectively $\mathbb{S} = \sigma_{BW}(\tau_{AB})$), while if this is not the case, then $\mathbb{S} = \sigma_{BW}(A \otimes B) \cup \{0\}$ (respectively $\mathbb{S} = \sigma_{BW}(\tau_{AB}) \cup \{0\}$).

Proof. Consider the operator $A \otimes B \in B(\mathcal{X} \overline{\otimes} \mathcal{Y})$. Recall that since A , B and $A \otimes B$ satisfy the generalized Browder's theorem, according to [13, Theorem 12], $\sigma_{BW}(A) = I(A) \cup \text{acc } \sigma(A)$, $\sigma_{BW}(B) = I(B) \cup \text{acc } \sigma(B)$ and $\sigma_{BW}(A \otimes B) = I(A \otimes B) \cup \text{acc } \sigma(A \otimes B)$. Now set $\mathbb{A} = (\text{acc } \sigma(A))\sigma(B) \cup \sigma(A)(\text{acc } \sigma(B))$ and $\mathbb{B} = I(A)I(B) \cup I(A)\Pi(B) \cup \Pi(A)I(B)$. Note that $\mathbb{B} \setminus \{0\} = \mathbb{L} = I(A \otimes B) \setminus \{0\}$ (Theorem 3.4) and $\mathbb{S} = \mathbb{A} \cup \mathbb{B}$.

(i) \Rightarrow (ii). Suppose that $0 \in \Pi(A \otimes B) \subseteq \sigma(A \otimes B) = \sigma(A)\sigma(B)$. Then, since neither A nor B is algebraic, $0 \in \mathbb{S} \subseteq \sigma_{BW}(A \otimes B)$, which is impossible for $\Pi(A \otimes B) = \sigma(A \otimes B) \setminus \sigma_{BW}(A \otimes B)$.

(ii) \Rightarrow (iii). Apply [13, Theorem 12].

(iii) \Rightarrow (i). Note that three cases must be considered: $0 \notin \sigma(A \otimes B)$; $0 \in \text{acc } \sigma(A \otimes B)$; $0 \in I(A \otimes B)$. Suppose that $0 \notin \sigma(A \otimes B)$ or $0 \in \text{acc } \sigma(A \otimes B)$. Then, according to [3, Theorem 6], $\text{acc } \sigma(A \otimes B) = \mathbb{A}$. If $0 \notin \sigma(A \otimes B)$, then $\mathbb{B} = I(A \otimes B)$, in particular, $\mathbb{S} = \sigma_{BW}(A \otimes B)$, while if $0 \in \text{acc } \sigma(A \otimes B) = \mathbb{A}$, since $\mathbb{B} \setminus \{0\} = I(A \otimes B)$, $\mathbb{S} = \sigma_{BW}(A \otimes B)$.

Next suppose that $0 \in I(A \otimes B) \subseteq \sigma_{BW}(A \otimes B) \subseteq \sigma(A \otimes B)$. Since neither A nor B is algebraic, $0 \in \mathbb{S}$. Moreover, since $\mathbb{B} \setminus \{0\} = I(A \otimes B) \setminus \{0\}$ and $\text{acc } \sigma(A \otimes B) = \mathbb{A} \setminus \{0\}$ [3, Theorem 6], $\mathbb{S} = \sigma_{BW}(A \otimes B)$.

Concerning the last statement, according to Lemma 4.1, it is enough to consider the case $\sigma_{BW}(A \otimes B) \subsetneq \mathbb{S}$. Suppose that $0 \in \Pi(A \otimes B)$. Then, since neither A nor B is algebraic, $0 \in \mathbb{S} \setminus \sigma_{BW}(A \otimes B)$. However, since $\text{acc } \sigma(A \otimes B) = \mathbb{A} \setminus \{0\}$ [3, Theorem 6] and $I(A \otimes B) \setminus \{0\} = \mathbb{B} \setminus \{0\}$, $\mathbb{S} = \sigma_{BW}(A \otimes B) \cup \{0\}$.

Finally, a similar argument proves the statements concerning the left–right multiplication operator. \square

Remark 4.6. Note that under the same hypotheses as Theorems 4.4 and 4.5, if $\sigma_{BW}(A \otimes B) \subsetneq \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B)$, then $\sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) = \sigma_{BW}(A \otimes B) \cup \{0\}$. Moreover, a similar observation holds for τ_{AB} .

In the following theorem the transfer property for the generalized Browder's theorem will be characterized. Note that if an operator is not Drazin invertible, then it is not algebraic. Recall that according to [9, Theorem 2.1], Browder's theorem and the generalized Browder's theorem are equivalent. Moreover, recall that Browder's theorem both for the tensor product operator and for the elementary operator is equivalent to the respective Weyl spectrum equality, see [6, Theorem 3] and [8, Theorem 4.5] respectively.

Theorem 4.7. Let \mathcal{X} and \mathcal{Y} be two Banach spaces and consider $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$ two operators that satisfy the generalized Browder's theorem. Suppose either that A and B are not algebraic and $0 \notin \Pi(A \otimes B)$ ($=\Pi(\tau_{AB})$) or that only one of them, say A , is algebraic but not nilpotent and the other, say B , is not Drazin invertible.

- (a) The following statements are equivalent.
(i) The (generalized) Browder's theorem for $A \otimes B \in B(\mathcal{X} \overline{\otimes} \mathcal{Y})$ holds;
(ii) $\sigma_{BW}(A \otimes B) = \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B)$.
(iii) $\sigma_w(A \otimes B) = \sigma_w(A)\sigma(B) \cup \sigma(A)\sigma_w(B)$.
(b) The following statements are equivalent.
(i) The (generalized) Browder's theorem for $\tau_{AB} \in B(B(\mathcal{Y}, \mathcal{X}))$ holds;
(ii) $\sigma_{BW}(\tau_{AB}) = \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B)$.
(iii) $\sigma_w(\tau_{AB}) = \sigma_w(A)\sigma(B) \cup \sigma(A)\sigma_w(B)$.

Proof. Apply Theorems 4.2, 4.4 and 4.5. \square

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