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Limit theorems with asymptotic expansions for stochastic processes

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Abstract

In this paper, we consider some families of one-dimensional locally infinitely divisible Markov processes $\{\eta_t^\epsilon\}_{0\le t\le T}$ with frequent small jumps. For a smooth functional $F(x[0,T])$ on space $D[0,T]$, the following asymptotic expansions for expectations are proved: as $\epsilon\to 0$,

$$E^\epsilon F(\eta^\epsilon[0,T]) = EF(\eta^0[0,T]) + \sum_{i=1}^{s}\epsilon^{i/2}EA_iF(\eta^0[0,T]) + o(\epsilon^{s/2})$$

for some Gaussian diffusion η^0 as the weak limit of η^ϵ, suitable differential operators A_i, and a positive integer s depending on the smoothness of F.

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1. Introduction

Let $\{\eta_t^\epsilon\}_{0\le t\le T}$ on space $(\Omega^\epsilon,\mathcal{F}^\epsilon,P^\epsilon)$ be a family of stochastic processes depending on a parameter $\epsilon\ge 0$. We also write $\eta^\epsilon(t)$ instead of η_t^ϵ where it seems appropriate. Assume the trajectories of $\{\eta_t^\epsilon\}_{0\le t\le T}$ are in a metric function space $\mathbb{X}$. We write $\eta^\epsilon[0,T]$ to stress that each trajectory is an element in function space $\mathbb{X}$. Weak convergence of processes $\eta^\epsilon[0,T]$ to a process $\eta^0[0,T]$ can be formulated as follows

$$E^\epsilon F(\eta^\epsilon[0,T]) = E^0F(\eta^0[0,T]) + o(1), \tag{1}$$

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for any bounded and continuous functionals F on $\mathbb{X}$, where $E^\epsilon, \epsilon \geq 0$ denote expectations with respect to probability measures P^ϵ. Now we are interested in the exact order of $o(1)$ in (1). The inspiration behind this problem is from classical asymptotic expansions in limit theorems for sums of independent and identically distributed (i.i.d.) random variables.

It is quite clear that Berry–Esseen theorem gives a more precise result (under some restrictive conditions on moments) on convergence of distributions than central limit theorems. If more conditions are imposed on i.i.d. ξ_i, then we have an asymptotic expansion for distribution functions F_n of $(\xi_1 + \cdots + \xi_n)/\sqrt{n} : F_n(x) = F_\infty(x) + \sum_{1\leq i\leq N} P_i(x)n^{-i/2} + o(n^{-N/2})$ where F_∞ is the limiting distribution of F_n. If we write this in an expectation form for a smooth function F, then it can be proved that $EF((\xi_1+\cdots+\xi_n)/\sqrt{n}) = EF(\xi_\infty)+\sum_{1\leq i\leq N} p_i n^{-i/2}+o(n^{-N/2})$, where ξ_∞ is the random variable corresponding to distribution function F_∞ (see for example [4] for related works). This expansion leads us to expect the exact order of $o(1)$ in (1). For various classes of families of stochastic processes, results concerning the exact order of $o(1)$ were obtained in [3,5–7].

In [7] Wentzell proved

$$E^\epsilon F(\eta^\epsilon[0, T]) = E^0 F(\eta^0[0, T]) + \sum_{i=1}^{s} \epsilon^{i/2} E^0 A_i F(\eta^0[0, T]) + o(\epsilon^{s/2}) \tag{2}$$

for bounded smooth F and for some class of families of locally infinitely divisible processes η^ϵ with $s = 1$. When one tries to extend the results in [7] to general $s \geq 1$, unbounded functionals arise even if F and its derivatives are bounded, and thus several technical difficulties appear such as extending the domain of a compensating operator to include some unbounded functionals, upper estimates for a functional $f(t, x[0, t])$ which is defined in Section 3.1. Another reason to consider some unbounded functionals is from the study of the asymptotic expansions on large deviations of the form

$$E^\epsilon \left[\exp\{\epsilon^{-1} F(\xi^\epsilon)\}\right] = \exp\{\epsilon^{-1}[F(\phi_0) - S(\phi_0)]\} \left(\sum_{0\leq i\leq s/2} K_i \cdot \epsilon^i + o\left(\epsilon^{s/2}\right)\right) \tag{3}$$

for some families of locally infinitely divisible Markov processes ξ^ϵ defined through η^ϵ, where S is the normalized action functional and ϕ_0 is the unique maximizer of $F - S$. It turns out that the expansions on large deviations in (3) can be derived from the expansions on normal deviations (2) for some unbounded functionals. This idea was used by Cramér (see [1]) to derive precise large deviations for sums of independent and identically distributed random variables. But for stochastic processes, nothing has been done for precise large deviations in this direction based on normal deviations (2). This is because no such a tool exists. This paper is devoted to such a tool, namely (2).

In this paper we first show expansions (2) with $s = 1$ for certain class of unbounded functionals F (see Theorems 2.1 and 4.1), and then apply these for $s = 1$ to get expansions for general $s \geq 1$ (see Theorems 5.1 and 5.2). Those technical difficulties are tackled in Section 3 based on linear transformations of stochastic processes and suitable truncation arguments.

1.1. Locally infinitely divisible processes

Let us introduce several concepts. If $(\xi_t, P_{s,x}), t \in [s, T]$, is a Markov process (the subscript $_{s,x}$ means the process starts from x at time s), we use $P^{s,t}, 0 \leq s \leq t \leq T$, to denote the corresponding multiplicative family of linear operators acting on functions according to the formula

$$P^{s,t} f(x) = E_{s,x} f(\xi_t),$$

where $E_{s,x}$ is the expectation with respect to probability measure $P_{s,x}$. The *compensating operator* $\mathfrak{A}$ of this Markov process, taking functions $f(t,x)$ to functions of the same two arguments, is defined by

$$P^{s,t} f(t,\cdot)(x) = f(s,x) + \int_s^t P^{s,u}\mathfrak{A} f(u,\cdot)(x)du \tag{4}$$

under suitable assumptions on the measurability in (t,x) of $\mathfrak{A} f(t,x)$, where $P^{s,t} f(t,\cdot)(x)$ means that $P^{s,t}$ is applied to function $f(t,x)$ in its second argument x, and $P^{s,u}\mathfrak{A} f(u,\cdot)(x)$ means that $P^{s,u}$ is applied to function $g(u,x) := \mathfrak{A} f(u,x)$ in its second argument x. If some measurability conditions are imposed on the process $\xi_t(\omega)$, then (4) is equivalent to that

$$f(t,\xi_t) - \int_s^t \mathfrak{A} f(u,\xi_u)du$$

is a martingale with respect to the natural family of σ-algebras and every probability measure $P_{s,x}$. Of course, compensating operator $\mathfrak{A}$ is not defined uniquely. Different versions are such that $\mathfrak{A} f(u,\xi_u)$ coincide almost surely except on a set of time argument u of zero Lebesgue measure.

We say A_t is the generating operator of our process $(\xi_t, P_{s,x})$ if for $s \le t$,

$$P^{s,t} f(x) = f(x) + \int_s^t P^{s,u} A_u f(x)du$$

for suitable f. Also a generating operator has different versions. For a wide class of Markov processes, a version of the compensating operator $\mathfrak{A}$ of process ξ_t for smooth functions $f(t,x)$ is given by

$$\mathfrak{A} f(t,x) = \frac{\partial f}{\partial t}(t,x) + A_t f(t,\cdot)(x),$$

where generating operator A_t acts on functions of the spatial argument x only.

(a) In this paper, we consider a class of families of locally infinitely divisible processes $(\xi_t^\epsilon, P_{0,x}^{\xi^\epsilon})$, $t \in [0,T]$ (again sometimes we also write $\xi^\epsilon(t)$ instead). For each fixed $\epsilon > 0$, let ξ^ϵ have the compensating operator (a version)

$$\mathfrak{A}^{\xi^\epsilon} f(t,x) = \frac{\partial f}{\partial t}(t,x) + A_t^{\xi^\epsilon} f(t,\cdot)(x), \tag{5}$$

where the generating operator $A_t^{\xi^\epsilon}$ is

$$\begin{aligned} A_t^{\xi^\epsilon} f(t,\cdot)(x) &= \alpha^1(t,x)\frac{\partial f}{\partial x}(t,x) + \frac{\epsilon}{2} a(t,x)\frac{\partial^2 f}{\partial x^2}(t,x) \\ &\quad + \epsilon^{-1}\int \left[f(t,x+\epsilon u) - f(t,x) - \epsilon u \frac{\partial f}{\partial x}(t,x)\right] \mu_{t,x}(du) \end{aligned} \tag{6}$$

for bounded functions $f(t,x)$ that are absolutely continuous in t for fixed x and twice continuously differentiable in x for fixed t with bounded derivatives $\partial f/\partial t$, $\partial f/\partial x$, $\partial^2 f/\partial x^2$. We impose a condition on measures $\mu_{t,x}$ in order to make sense of everything: $\int u^2 \mu_{t,x}(du) < \infty$.

The probability measures $P^{\xi^\epsilon}_{0,x}$ mean that $\xi^\epsilon_0 = x$ for some x in the real line. We will use the following symbols to denote the moments

$$\begin{aligned}\alpha^2(t,x) &= a(t,x) + \int u^2 \mu_{t,x}(du)\\ \alpha^j(t,x) &= \int u^j \mu_{t,x}(du), \qquad \beta^j(t,x) = \int |u|^j \mu_{t,x}(du), \quad j > 2.\end{aligned} \tag{7}$$

Let us set

$$\eta^\epsilon(t) = \epsilon^{-1/2}(\xi^\epsilon(t) - x_*(t)),$$

where x_* is the unique solution of differential equation $x'(t) = \alpha^1(t, x(t))$ with a prescribed initial condition. For each $\epsilon > 0$, the process $\eta^\epsilon(t)$ is also a Markov process (the generating operator $A^{\eta^\epsilon}_t$ is given in the Appendix), and it can be easily checked that for any x_0 in the real line

$$P^{\eta^\epsilon}_{0,x_0} = P^{\xi^\epsilon}_{0,x_*(0)+\epsilon^{1/2}x_0}.$$

Throughout this paper we assume that process η^ϵ starts from one same fixed point x_0 in the real line at 0 for every $\epsilon > 0$. For instance, we can set $x_*(0) = 0$ and $\xi^\epsilon(0) = \epsilon^{1/2}x_0$.

It can be proved that under some additional assumptions the process η^ϵ converges as $\epsilon \to 0$ to a process η^0 (see Theorem 1 and Lemma 6 in [7]), and η^0 is a Gaussian diffusion process on the real line with generating operator

$$A^{\eta^0}_t f(x) = \alpha^1_2(t, x_*(t)) \cdot x \cdot f'(x) + \frac{1}{2}\alpha^2(t, x_*(t)) \cdot f''(x), \tag{8}$$

where the subscript $_2$ means differentiation in second spatial argument.

(b) To formulate our main result, we introduce a class of locally infinitely divisible processes. The space $D_x[0,T]$ consists of functions defined on $[0,T]$ starting from x at 0 which are right continuous with left limits, and $D[t_1,t_2]$, $t_1 < t_2$, is the space of all functions over $[t_1,t_2]$ that are right continuous having left limits. Class **A** is the collection of locally infinitely divisible processes $\big(\xi(t), P_{t,x}\big)$ whose compensating operators are given by $\mathfrak{A}f = \partial f/\partial t + A_t f$,

$$\begin{aligned}A_t f(x) &= \alpha^1(t,x) f'(x) + \frac{1}{2}a(t,x) f''(x)\\ &\quad + \int \big[f(x+u) - f(x) - uf'(x)\big]\mu_{t,x}(du),\end{aligned}$$

and it can be approximated by pure jump processes $\big(\xi^\theta(t), P^\theta_{t,x}\big)$ with compensating operators $\mathfrak{A}^\theta f = \partial f/\partial t + A^\theta_t f$,

$$A^\theta_t f(x) = \int [f(x+u) - f(x)]\, \mu^\theta_{t,x}(du),$$

with bounded $\mu^\theta_{t,x}$ so that the distribution of $\xi^\theta[t_1,t_2]$ in space $D[t_1,t_2]$ with respect to the probability $P^\theta_{t,x}$ converges weakly in the Skorohod topology to that of $\xi[t_1,t_2]$ with respect to $P_{t,x}$, and $A^\theta_t f(x) \to A_t f(x)$ uniformly with respect to t, to x, to f changing in every class of uniformly bounded function with uniformly bounded and equicontinuous derivative f''.

(c) For each fixed $\epsilon > 0$, the process $(\eta^\epsilon(t), P^{\eta^\epsilon}_{0,x_0})$, $t \in [0,T]$ is a Markov process. We can consider its path $\eta^\epsilon[0,t]$ restricted to the interval $[0,t]$ for every $t \in [0,T]$. We can think of

$\eta^\epsilon[0,t]$ as a function of $t \in [0,T]$, and $\eta^\epsilon[0,t]$, $t \in [0,T]$, is a stochastic process taking for each $t \in [0,T]$ values in its own function space $D_{x_0}[0,t]$. Such processes have been studied by various authors (see [2,5]). We employ the term *historical processes* for such processes (see [7,2,5]).

It is proved in [5] that we can have historical process $\eta^\epsilon[0,t]$, $t \in [0,T]$, as a Markov process. That is, for every $x[0,s] \in D_{x_0}[0,s]$, the stochastic process $(\eta^\epsilon[0,t], P^{\eta^\epsilon}_{s,x[0,s]})$, $t \in [0,T]$, is a Markov process taking values for each $t \in [0,T]$ in the space $D_{x_0}[0,t]$, where $P^{\eta^\epsilon}_{s,x[0,s]}$ for some $s \in [0,T]$ denotes the probability measure under the assumption that process $\eta^\epsilon[0,t]$ starts from a function $x[0,s] \in D_{x_0}[0,s]$ at time s. For $s = 0$, the probability $P^{\eta^\epsilon}_{0,x[0,0]}$ is nothing but $P^{\eta^\epsilon}_{0,x_0}$ if $x[0,0]$ is the function defined at the single point 0 and taking at it the value x_0. Now we can also consider the multiplicative family of linear operators $P^{s,t}_{\eta^\epsilon}$ of η^ϵ acting on functionals of the form $f(x[0,t])$ according to the formula

$$P^{s,t}_{\eta^\epsilon} f(x[0,s]) = E^{\eta^\epsilon}_{s,x[0,s]} f(\eta^\epsilon[0,t])$$

with $E^{\eta^\epsilon}_{s,x[0,s]}$ denoting the expectation with respect to probability measure $P^{\eta^\epsilon}_{s,x[0,s]}$. Similarly as (4), the compensating operator $\mathfrak{A}^{\eta^\epsilon}$ of historical process $\eta^\epsilon[0,t]$, $t \in [0,T]$, acting on functionals of the form $f(t, x[0,t])$, is defined by

$$P^{s,t}_{\eta^\epsilon} f(t,\cdot)(x[0,s]) = f(s, x[0,s]) + \int_s^t P^{s,u}_{\eta^\epsilon} \mathfrak{A}^{\eta^\epsilon} f(u,\cdot)(x[0,s])du \tag{9}$$

provided that bounded $f(t, x[0,t])$ satisfies suitable smoothness assumptions (see Lemma 1 of [7] for precise assumptions).

The first question arising here is to find the value of the compensating operator $\mathfrak{A}^{\eta^\epsilon} f(t, x[0,t])$ on some wide classes of functionals. It is natural to expect that some smoothness in t and $x[0,t]$ of $f(t, x[0,t])$ is sufficient for the existence of the compensating operator $\mathfrak{A}^{\eta^\epsilon} f(t, x[0,t])$. The partial derivative $\frac{\partial f}{\partial t}(t, x[0,t])$ could not exist since we cannot fix $x[0,t] \in D_{x_0}[0,t]$ while changing t. Wentzell introduced in [8,7] the *pseudo-partial derivative* $f_{(1)}(t, x[0,t])$ in the time argument: for each $t \in [0,T]$ and $x[0,t] \in D_{x_0}[0,t]$, the pseudo-partial derivative $f_{(1)}(t, x[0,t])$ is a functional that is measurable in $(t, x[0,t])$ and such that for $0 \le s \le t \le T$, $x[0,s] \in D_{x_0}[0,s]$,

$$f(t, x^s[0,t]) = f(s, x[0,s]) + \int_s^t f_{(1)}(u, x^s[0,u])du,$$

where the new function $x^s[0,t]$ over $[0,t]$ is defined as

$$x^s(v) = \begin{cases} x(v) & \text{for } v < s; \\ x(s) & \text{for } s \le v \le t. \end{cases}$$

The functional $f_{(1)}(t, x[0,t])$ is not defined uniquely. If one version $f_{(1)}(t, x[0,t])$ satisfies the condition

$$\lim_{t \downarrow s} f_{(1)}(t, x^s[0,t]) = f_{(1)}(s, x[0,s])$$

for all $s \in [0,T]$ and $x[0,s] \in D_{x_0}[0,s]$, then this version is

$$f_{(1)}(t, x[0,t]) = \lim_{\Delta \downarrow 0} \Delta^{-1}[f(t+\Delta, x^t[0,t+\Delta]) - f(t, x[0,t])]. \tag{10}$$

Conversely, if the limit in (10) exists for all $t \in [0, T)$ and $x[0, t] \in D_{x_0}[0, t]$, and the pre-limit quantity in (10) is bounded uniformly in all $0 < t < t + \Delta < T$ and $x[0, t] \in D_{x_0}[0, t]$, then one of the versions of the pseudo-partial derivatives is given by the limit (10).

It has been shown in [7] that $\mathfrak{A}^{\eta^\epsilon} f(t, x[0, t])$ is well-defined for bounded functionals $f(t, x[0, t])$ having pseudo-partial derivative $f_{(1)}(t, x[0, t])$, bounded second spatial derivative $f^{(2)}(t, x[0, t])(I_{\{t\}}, I_{\{t\}})$ along directions $I_{\{t\}}[0, t]$ and other suitable continuity restrictions, namely,

$$\begin{aligned}\mathfrak{A}^{\eta^\epsilon} f(t, x[0, t]) = {} & f_{(1)}(t, x[0, t]) \\ & + \epsilon^{-1/2}\left[\alpha^1(t, x_*(t) + \epsilon^{1/2}x(t)) - \alpha^1(t, x_*(t))\right] \\ & \cdot f^{(1)}(t, x[0, t])(I_{\{t\}}) \\ & + \frac{1}{2}a(t, x_*(t) + \epsilon^{1/2}x(t)) \cdot f^{(2)}(t, x[0, t])(I_{\{t\}}^{\otimes 2}) \\ & + \epsilon^{-1}\int\Big[f(t, x[0, t] + u\epsilon^{1/2}I_{\{t\}}) - f(t, x[0, t]) \\ & - u\epsilon^{1/2} \cdot f^{(1)}(t, x[0, t])(I_{\{t\}})\Big]\,\mu_{t, x_*(t)+\epsilon^{1/2}x(t)}(du). \qquad (11)\end{aligned}$$

We recall that under suitable restrictions on F and η^ϵ, expansions (2) were proved in [7] for the case $s = 1$. As explained in Section 1, unbounded functionals will arise for general $s \geq 1$. We thus first extend the domain of the compensating operator $\mathfrak{A}^{\eta^\epsilon} f(t, x[0, t])$ to include some unbounded functionals $f(t, x[0, t])$. This is done in Proposition 3.4. Sections 2 and 3 present our main result and its proof. Section 5 contains applications of our main result, which are the asymptotic expansions (2) for general s. In what follows, const_i, $i = 1, 2, \ldots$ denote generic positive constants whose values may vary in different places.

2. The main result

Before our main theorem, let us introduce the functional derivatives that will be used in this paper. We understand the differentiability of a functional $F(\phi)$ as Fréchet differentiability. As in [7], we assume that the derivatives $F^{(j)}(\phi)(\delta_1, \ldots, \delta_j)$ can be represented as integrals of the product $\delta_1(s_1) \cdot \cdots \cdot \delta_j(s_j)$ with respect to some signed measures, denoted by $F^{(j)}(\phi; \bullet)$:

$$F^{(j)}(\phi)(\delta_1, \ldots, \delta_j) = \int_{[0,T]^j} \delta_1(s_1) \cdot \cdots \cdot \delta_j(s_j) F^{(j)}(\phi; ds_1 \cdots ds_j). \qquad (12)$$

The norm of the signed measure is defined by

$$\|F^{(j)}\| := \sup_{x[0,T] \in D_0[0,T]} \left| F^{(j)}(x[0, T]; \bullet) \right| ([0, T]^j).$$

We also use $F^{(j)}(\phi)(y[0, T]^{\otimes j})$ to denote the j-th derivative $F^{(j)}(\phi)(y[0, T], \ldots, y[0, T])$ of the functional F at point $\phi[0, T]$ in directions $y[0, T]$ for short. Let us recall a differential operator A_1 which was defined in [7] for functionals F on $D[0, T]$:

$$A_1 F(x[0, T]) = \sum_{k=1}^{3} \int_{[0,T]^k} \Gamma_1^k(x[0, T]; s_1, \ldots, s_k) F^{(k)}(x[0, T]; ds_1 \cdots ds_k)$$

where

$$\Gamma_1^1(x[0,T];s_1) = \frac{1}{2}\int_0^{s_1} \alpha_{22}^1(t,x_*(t))x(t)^2 \exp\left\{\int_0^{s_1} \alpha_2^1(v,x_*(v))dv\right\} dt;$$

$$\Gamma_1^2(x[0,T];s_1,s_2) = \frac{1}{2}\int_0^{\min\{s_1,s_2\}} \alpha_2^2(t,x_*(t))x(t) \exp\left\{\sum_{i=1}^{2}\int_0^{s_i} \alpha_2^1(v,x_*(v))dv\right\} dt;$$

$$\Gamma_1^3(x[0,T];s_1,s_2,s_3) = \frac{1}{6}\int_0^{\min\{s_1,s_2,s_3\}} \alpha^3(t,x_*(t)) \exp\left\{\sum_{i=1}^{3}\int_0^{s_i} \alpha_2^1(v,x_*(v))dv\right\} dt.$$

Theorem 2.1. *Assume* $\|\alpha_{22}^1\|, \|\alpha^2\|, \|\alpha_2^2\| < \infty, \alpha_2^1(t,x) \le C_1$ *for some nonnegative constant* C_1*, and* $\|\beta^i\| < \infty$ *for* $3 \le i \le j+1$*, where* $j > 2$ *is some integer. Let every process of the family of locally infinitely divisible processes* $(\xi^\epsilon(t), P_{0,x}^{\xi^\epsilon})$ *be in class* **A**. *Processes* η^ϵ *and* x_* *are defined as above. Assume, in addition, that* $\alpha_{22}^1(t,x), \alpha_2^2(t,x)$ *and* $\alpha^3(t,x)$ *are continuous in* x *at the point* $x_*(t)$ *for all* t*, and* $|u|^{j+1}$ *is uniformly integrable with respect to* $\mu_{t,x}$.

Let the functional $F(x[0,T])$ *on* $D[0,T]$ *be three times differentiable with the following conditions:*

(i) *there is a constant* $B > 0$ *such that for all* $x[0,T], y[0,T] \in D[0,T]$,

$$|F(x[0,T])| \le B(1+|x(T)|^j);$$
$$\left|F^{(i)}(x[0,T])(y[0,T]^{\otimes^i})\right| \le (1+\|y\|^i)\cdot B\cdot\left(1+|x(T)|^{j-2}\right), \quad i=1,2,3;$$

(ii) $F^{(3)}(x[0,T])(I_{[t,T]}\delta, I_{[t,T]}\delta, I_{[t,T]}\delta)$ *is continuous with respect to* $x[0,T]$ *uniformly as* $x[0,T]$ *changes over an arbitrary compact subset of* $D[0,T]$, t *over* $[0,T]$*, and* $\delta[0,T]$ *over the set of Lipschitz continuous functions with constant* 1, $\|\delta\| \le 1$.

Then as $\epsilon \downarrow 0$,

$$E_{0,x_0}^{\eta^\epsilon} F(\eta^\epsilon[0,T]) = E_{0,x_0}^{\eta^0} F(\eta^0[0,T]) + \epsilon^{1/2} E_{0,x_0}^{\eta^0} A_1 F(\eta^0[0,T]) + o(\epsilon^{1/2}), \tag{13}$$

where A_1 *is a third-order differential operator defined above.*

Remark. We point out here that $|F^{(i)}(x[0,T])(y[0,T]^{\otimes^i})| \le (1+\|y\|^i)\cdot B\cdot\left(1+|x(T)|^{j-2}\right)$ in above condition (i) can be replaced by:

$$\left|F^{(i)}(x[0,T])(y_1[0,T],\ldots,y_i[0,T])\right| \le \sigma(\|y_1\|,\ldots,\|y_i\|)\cdot B\cdot\left(1+|x(T)|^{j-2}\right)$$

with $\sigma(x_1,\ldots,x_i)$ denoting any real valued function that takes a bounded set in R^i to a bounded set in R. For instance, we can choose $\sigma(x_1,\ldots,x_i) = x_1\cdots x_i$. The condition that was assumed in (i) is just for simplicity.

Some examples of functionals satisfying all the conditions in Section 2 can be produced by carefully taking into account the upper bounds. For instance, a class of examples is given by

$$F(x[0,T]) = g(x(T))$$

for some smooth function $g(x)$ having an upper bound $1+|x|^j$ together with its derivatives. We can take $g(x) = x^k$ for example where k is any positive integer. More classes of examples are given after Theorem 4.1 where a better condition on the upper bound of the functionals is assumed.

3. Proof of Theorem 2.1

3.1. Properties of $f(t, x[0,t])$

As in paper [7], for $x[0,t] \in D[0,t]$ we introduce

$$f(t, x[0,t]) := E^{\eta^0}_{t,x[0,t]} F(\eta^0[0,T]). \tag{14}$$

Let us analyze (14) in detail for $x[0,t] \in D_{x_0}[0,t]$. If $\alpha_2^1(t, x_*(t)) = 0$, then from (8) we know η^0 is a time-inhomogeneous Wiener process with local variance $\alpha^2(t, x_*(t))$. In particular, it has independent increments. So the distribution of $\eta^0[0,T]$ under probability $P^{\eta^0}_{t,x[0,t]}$ is the same as the distribution of $x^t[0,T] + \eta^0_t[0,T]$ under probability $P^{\eta^0}_{0,x_0}$, where

$$x^t(s) = x(s) \quad \text{if } s \le t, \qquad = x(t) \quad \text{if } s > t;$$
$$\eta^0_t(s) = \eta^0(s) - (\eta^0)^t(s) \quad \text{where } (\eta^0)^t(s) \text{ is defined similarly as above } x^t(s).$$

To see why these two distributions are the same (provided independent increments), we notice that for any measurable $C \subseteq D_{x_0}[0,T]$,

$$P^{\eta^0}_{t,x[0,t]}\left(\eta^0[0,T] \in C\right) = P^{\eta^0}_{t,x(t)}\left(\eta^0_t[t,T] + x(t) \in C_{x[0,t]}\right)$$

according to [5], where $C_{x[0,t]} = \{y[t,T] : x^t y[0,T] \in C\}$. By the Markov property for the process η^0 with respect to time t, and the independent increments of η^0, we have

$$P^{\eta^0}_{t,x(t)}\left(\eta^0_t[t,T] + x(t) \in C_{x[0,t]}\right) = P^{\eta^0}_{0,x_0}\left(\eta^0_t[t,T] + x(t) \in C_{x[0,t]}\right).$$

This is nothing but $P^{\eta^0}_{0,x_0}\left((x^t[0,T] + \eta^0_t[0,T]) \in C\right)$. Thus (14) becomes

$$f(t, x[0,t]) = E^{\eta^0}_{t,x[0,t]} F(\eta^0[0,T]) = E^{\eta^0}_{0,x_0} F(x^t[0,T] + \eta^0_t[0,T]). \tag{15}$$

If $\alpha_2^1(t, x_*(t)) \neq 0$, then η^0 is a general Gaussian diffusion which may not have independent increments. In this case, a linear transformation $\eta^0_*(t) = \exp\left\{-\int_0^t \alpha_2^1(s, x_*(s))ds\right\} \cdot \eta^0(t)$ is used. It is then easy to deduce that η^0_* is a time-inhomogeneous Wiener process with local variance $\alpha^2(t, x_*(t)) \cdot \exp\left\{-2\int_0^t \alpha_2^1(s, x_*(s))ds\right\}$. If we define

$$F_*(y[0,T]) = F\left(y[0,T] \cdot \exp\left\{\int_0^\bullet \alpha_2^1(s, x_*(s))ds\right\}\right),$$

then it follows from (15) that

$$\begin{aligned} f(t, x[0,t]) &= E^{\eta^0}_{t,x[0,t]} F(\eta^0[0,T]) \\ &= E^{\eta^0}_{t,x[0,t]} F\left(\eta^0_*[0,T] \cdot \exp\left\{\int_0^\bullet \alpha_2^1(s, x_*(s))ds\right\}\right) \\ &= E^{\eta^0}_{t,x[0,t]} F_*\left(\eta^0_*[0,T]\right) = E^{\eta^0}_{0,x_0} F_*(x^t[0,T] + (\eta^0_*)_t[0,T]) \\ &= E^{\eta^0}_{0,x_0} F\left(x^t[0,T] \cdot \exp\left\{\int_0^\bullet \alpha_2^1(s, x_*(s))ds\right\} + \eta^0_t[0,T]\right), \end{aligned} \tag{16}$$

that is, the functional F is taken of a function over $[0,T]$ whose value at a point $v \in [0,T]$ is

$x^t(v) \cdot \exp\{\int_0^v \alpha_2^1(s, x_*(s))ds\} + \eta_t^0(v)$. Here we use $\int_0^\bullet \alpha_2^1(s, x_*(s))ds$ to denote a real valued function defined on $[0, T]$.

3.2. Estimates

In this section, $j > 2$ is some integer as in previous section. Let us first prove an auxiliary result.

Lemma 3.1. *Under conditions* $\|\beta^i\| < \infty$ *for* $3 \le i \le j+1$, $\|\alpha^2\| < \infty$ *and condition* $\alpha_2^1(t, x) \le C_1$ *for some nonnegative constant* C_1, *then for any* $\epsilon_0 > 0$,

$$\sup_{t\in[0,T]} \sup_{\epsilon\in[0,\epsilon_0)} E_{0,x_0}^{\eta^\epsilon} |\eta^\epsilon(t)|^{j+1} < (|x_0|^{j+1} + 1) \cdot \text{const.}(T, j) < \infty.$$

Proof. Without loss of generality we may assume j is an odd number. First we can find generating operators $A_t^{\eta^\epsilon}$ of processes η^ϵ from generating operators $A_t^{\xi^\epsilon}$ of processes ξ^ϵ (see the Appendix for details). Let us consider a sequence of functions $f_n(x) = \frac{x^k}{1+(x/n)^k}$ for an even positive integer k. Then, according to the formula for $A_t^{\eta^\epsilon}$ in the Appendix, the generating operator $A_t^{\eta^\epsilon}$ applying to f_n gives

$$\begin{aligned} A_t^{\eta^\epsilon} f_n(x) &= \epsilon^{-1/2} f_n'(x) \left[\alpha^1(t, x\epsilon^{1/2} + x_*(t)) - \alpha^1(t, x_*(t))\right] \\ &\quad + \frac{1}{2} a(t, x\epsilon^{1/2} + x_*(t)) f_n''(x) \\ &\quad + \epsilon^{-1} \int \left[f_n(x + \epsilon^{1/2}u) - f_n(x) - \epsilon^{1/2}u \cdot f_n'(x)\right] \mu_{t, x\epsilon^{1/2}+x_*(t)}(du) \\ &= x \cdot f_n'(x) \cdot \alpha_2^1(t, \theta_1 x\epsilon^{1/2} + x_*(t)) + \frac{1}{2} a(t, x\epsilon^{1/2} + x_*(t)) f_n''(x) \\ &\quad + \frac{f_n''(x)}{2} \int u^2 \mu_{t, x\epsilon^{1/2}+x_*(t)}(du) \\ &\quad + \cdots + \frac{1}{k!} \int \epsilon^{\frac{k-2}{2}} u^k f_n^{(k)}(x + \theta_2 \epsilon^{1/2} u) \mu_{t, x\epsilon^{1/2}+x_*(t)}(du) \end{aligned}$$

which is less than or equal to

$$\le \text{const}_1 + \text{const}_2 \times f_n(x),$$

since $f_n^{(k)}$ is bounded, and $x \cdot f_n'(x), f_n'', f_n''', \ldots, f_n^{(k-1)}$ are bounded by $\text{const}_3 + \text{const}_4 \times f_n(x)$. So

$$\begin{aligned} E_{0,x_0}^{\eta^\epsilon}\left[f_n\left(\eta_t^\epsilon\right)\right] &\le f_n(x_0) + \int_0^t \left(\text{const}_1 + \text{const}_2 \times E_{0,x_0}^{\eta^\epsilon}\left[f_n\left(\eta_s^\epsilon\right)\right]\right) ds \\ &\le C + \text{const}_2 \int_0^t E_{0,x_0}^{\eta^\epsilon}\left[f_n\left(\eta_s^\epsilon\right)\right] ds, \end{aligned}$$

where constant $C = C(x_0, k, T) = |x_0|^k + T \cdot \text{const}_1$. Applying Gronwall's lemma with such nonnegative $E^{\eta^\epsilon}\left[f_n\left(\eta_t^\epsilon\right)\right]$, we obtain

$$E_{0,x_0}^{\eta^\epsilon}\left[f_n\left(\eta_t^\epsilon\right)\right] \le C \cdot \exp\{\text{const}_2 \cdot t\},$$

then the proof is done by sending n to infinity and choosing $k = j + 1$. □

Lemma 3.2. *Assume* $\|\beta^i\| < \infty$ *for* $3 \le i \le j+1$, $\|\alpha^2\| < \infty$ *and condition* $\alpha_2^1(t,x) \le C_1$ *for some nonnegative constant* C_1*, and condition* (i) *in Theorem 2.1 holds, then there is a constant* $B' = B'(x_0, T, j)$ *such that*

$$|f(t, x[0,t])| \le B'(1 + |x(t)|^j),$$

for all $x[0,t] \in D_{x_0}[0,t]$.

Proof. According to (16), we have

$$\begin{aligned}
&|f(t, x[0,t])| \\
&= \left| E^{\eta^0}_{0,x_0} F\left(x^t[0,T] \cdot \exp\left\{ \int_0^{\bullet} \alpha_2^1(s, x_*(s))ds \right\} + \eta_t^0[0,T] \right) \right| \\
&\le B \cdot E^{\eta^0}_{0,x_0} \left[1 + \left(|x(t)| \cdot \exp\left\{ \int_0^T \alpha_2^1(u, x_*(u))du \right\} + |\eta^0(T) - x(t)| \right)^j \right] \\
&\le B(1 + |x(t)|^j \cdot \text{const}_1 + \text{const}_2 \cdot E^{\eta^0}_{0,x_0} |\eta^0(T)|^j) \\
&\le B'(1 + |x(t)|^j), \quad \text{for some } B' \text{ from Lemma 3.1}. \quad \square
\end{aligned}$$

Lemma 3.3. *Assume* $\|\beta^i\| < \infty$ *for* $3 \le i \le j+1$, $\|\alpha^2\| < \infty$ *and condition* $\alpha_2^1(t,x) \le C_1$ *for some nonnegative constant* C_1*, and condition (i) in Theorem 2.1 holds, then there is a constant* $B'' = B''(x_0, T, j)$ *such that*

$$\left| f^{(i)}(t, x[0,t])(I_{\{t\}}^{\otimes i}) \right| \le B''(1 + |x(t)|^{j-2}), \quad i = 1, 2, 3,$$

for all $x[0,t] \in D_{x_0}[0,t]$.

Proof. From Lemma 4 in [7], with $\psi_{[t,T]}(s) = I_{[t,T]}(s) \exp\left\{ \int_t^s \alpha_2^1(v, x_*(v)dv) \right\}$,

$$f^{(i)}(t, x[0,t])(I_{\{t\}}^{\otimes i}) = E^{\eta^0}_{t,x[0,t]} F^{(i)}(\eta^0[0,T])(\psi_{[t,T]}^{\otimes i}).$$

It follows from (16) that

$$\begin{aligned}
f^{(i)}(t, x[0,t])(I_{\{t\}}^{\otimes i}) = {} & E^{\eta^0}_{0,x_0} F^{(i)} \left(x^t[0,T] \cdot \exp\left\{ \int_0^{\bullet} \alpha_2^1(s, x_*(s))ds \right\} + \eta_t^0[0,T] \right) \\
& \times (\psi_{[t,T]}^{\otimes i}).
\end{aligned} \tag{17}$$

Thus

$$\begin{aligned}
&\left| f^{(i)}(t, x[0,t])(I_{\{t\}}^{\otimes i}) \right| \\
&= \left| E^{\eta^0}_{0,x_0} F^{(i)} \left(x^t[0,T] \cdot \exp\left\{ \int_0^{\bullet} \alpha_2^1(s, x_*(s))ds \right\} + \eta_t^0[0,T] \right) (\psi_{[t,T]}^{\otimes i}) \right| \\
&\le B \cdot E^{\eta^0}_{t,x_0} \Bigg[(1 + \|\psi_{[t,T]}\|^i) \Bigg(1 + |x(t)| \cdot \exp\left\{ \int_0^T \alpha_2^1(u, x_*(u))du \right\} \\
&\quad + |\eta^0(T) - x(t)| \Bigg)^{j-2} \Bigg] \\
&\le B''(1 + |x(t)|^{j-2}), \quad \text{for some } B'' \text{ depending on } x_0, T \text{ and } j. \quad \square
\end{aligned}$$

According to Lemma 3.1, we can select $B' = \text{const}_5(1 + |x_0|^j)$ and $B'' = \text{const}_6(1 + |x_0|^{j-2})$.

3.3. Proof of Theorem 2.1

For each fixed $\epsilon > 0$, the compensating operator $\mathfrak{A}^{\eta^\epsilon}$ of $\eta^\epsilon[0,T]$ is well defined for nice bounded functionals $f(t, x[0,T])$ (see (11) and Lemma 1 in [7]). As explained in the introduction, we need to extend the domain of the compensating operator $\mathfrak{A}^{\eta^\epsilon}$ to include some unbounded functionals.

Proposition 3.4. *Assume $\|\beta^i\| < \infty$ for $3 \le i \le j+1$, $\|\alpha^2\| < \infty$, $\|\alpha_2^2\| < \infty$, $\|\alpha_{22}^1\| < \infty$, $\alpha_2^1(t,x) \le C_1$ for some nonnegative constant C_1, and condition* (i) *in Theorem 2.1 holds, then the compensating operator $\mathfrak{A}^{\eta^\epsilon}$ of historical process $\eta^\epsilon[0,t]$, $t \in [0,T]$, can be applied to functional $f(t, x[0,t])$ on $D[0,t]$ given by (14) with F in Theorem 2.1.*

Proof. For $x[0,t] \in D[0,t]$, let us define

$$f_n(t, x[0,t]) = f(t, x[0,t]) \cdot h\left(\frac{x(t)}{n}\right) \cdot h\left(\frac{x(0)}{n}\right),$$

where h is a non-negative smooth function with $\sup_x h(x) \le 1$, and is equal to 1 in $(-1, 1)$ and to 0 outside $(-2, 2)$.

Claim. *For each positive integer n,*

$$E_{0,x_0}^{\eta^\epsilon} f_n(T, \eta^\epsilon[0,T]) = f_n(0, x_0) + \int_0^T E_{0,x_0}^{\eta^\epsilon} \mathfrak{A}^{\eta^\epsilon} f_n(t, \eta^\epsilon[0,t]) dt. \tag{18}$$

As explained at the beginning of Section 3.3, equality (18) holds for nice bounded functionals with suitable continuity conditions. We will show the following:

(a) *$f_n^{(1)}(t, x[0,t])(I_{\{t\}})$ is continuous in $x[0,t]$ with respect to Skorohod topology for every t;*
(b) *$f_n^{(2)}(t, x[0,t])(I_{\{t\}}, I_{\{t\}})$ is bounded and uniformly continuous in $x[0,t]$ in uniform topology;*
(c) *$f_n(t, x[0,t])$ and the pseudo-partial derivative $(f_n)_{(1)}(t, x[0,t])$ are bounded and continuous in $x[0,t]$ in uniform topology for every t.*

Proof of (a) and (b): Let us compute the first Gâteaux derivative of $f_n(t, x[0,t])$ in its second argument along direction $y[0,t]$ as follows

$$\begin{aligned}
&\lim_{\delta \to 0} \delta^{-1} [f_n(t, x[0,t] + \delta y[0,t]) - f_n(t, x[0,t])] \\
&\quad = \lim_{\delta \to 0} \delta^{-1} \left[f(t, x[0,t] + \delta y[0,t]) \cdot h\left(\frac{x(t) + \delta y(t)}{n}\right) \cdot h\left(\frac{x(0) + \delta y(0)}{n}\right) \right. \\
&\qquad \left. - f(t, x[0,t]) \cdot h\left(\frac{x(t)}{n}\right) \cdot h\left(\frac{x(0)}{n}\right) \right] \\
&\quad = f^{(1)}(t, x[0,t])(y[0,t]) \cdot h\left(\frac{x(t)}{n}\right) \cdot h\left(\frac{x(0)}{n}\right) \\
&\qquad + f(t, x[0,t]) \left[h'\left(\frac{x(t)}{n}\right) \cdot \frac{y(t)}{n} \cdot h\left(\frac{x(0)}{n}\right) \right. \\
&\qquad \left. + h\left(\frac{x(t)}{n}\right) \cdot h'\left(\frac{x(0)}{n}\right) \cdot \frac{y(0)}{n} \right].
\end{aligned}$$

We note that the first Gâteaux derivative at $x[0,t]$ as a functional on space $D[0,t]$ is linear and continuous in $y[0,t]$, and it is continuous in $x[0,t]$ as a mapping from $D[0,t]$ to $L(D[0,t], R)$

which is the space of all linear and continuous functionals over $D[0,t]$, so $f_n(t, x[0,t])$ is first Fréchet differentiable in the second argument $x[0,t]$ and for $t > 0$,

$$f_n^{(1)}(t, x[0,t])(I_{\{t\}}) = \left[f^{(1)}(t, x[0,t])(I_{\{t\}}) \cdot h\left(\frac{x(t)}{n}\right) + f(t, x[0,t]) \cdot h'\left(\frac{x(t)}{n}\right) \cdot \frac{1}{n} \right] \cdot h\left(\frac{x(0)}{n}\right).$$

Similarly $f_n(t, x[0,t])$ is twice Fréchet differentiable and for $t > 0$,

$$f_n^{(2)}(t, x[0,t])(I_{\{t\}}^{\otimes 2}) = \left[f^{(2)}(t, x[0,t])(I_{\{t\}}^{\otimes 2}) \cdot h\left(\frac{x(t)}{n}\right) + f^{(1)}(t, x[0,t])(I_{\{t\}}) \cdot h'\left(\frac{x(t)}{n}\right) \cdot \frac{2}{n} + f(t, x[0,t]) \cdot h''\left(\frac{x(t)}{n}\right) \cdot \frac{1}{n^2} \right] \cdot h\left(\frac{x(0)}{n}\right).$$

From definition of h, we get

$$\begin{aligned} \sup_{x[0,t]\in D[0,t]} |f_n(t, x[0,t])| &= \sup_{x[0,t]\in D[0,t]} |f(t, x[0,t])| \cdot h\left(\frac{x(t)}{n}\right) \cdot h\left(\frac{x(0)}{n}\right) \\ &\le B' \cdot \sup_{x[0,t]\in D[0,t]} \left(1 + |x(t)|^j\right) \cdot h\left(\frac{x(t)}{n}\right) \cdot h\left(\frac{x(0)}{n}\right) \\ &\le \text{const}_5 (1 + (2n)^j)^2. \end{aligned}$$

And from Lemma 3.3, it follows

$$\sup_{x[0,t]\in D[0,t]} \left| f_n^{(1)}(t, x[0,t])(I_{\{t\}}) \right| < \infty; \qquad \sup_{x[0,t]\in D[0,t]} \left| f_n^{(2)}(t, x[0,t])(I_{\{t\}}^{\otimes 2}) \right| < \infty.$$

Note that f_n, $f_n^{(1)}$ and $f_n^{(2)}$ are continuous in $x[0,t]$. Lemmas in Section 3.2 tell us we can consider f_n, $f_n^{(1)}$ and $f_n^{(2)}$ just for $x(t)$ in $[-2n, 2n]$ because outside this interval h, h' and h'' are zeros, from which uniform continuity follows. For instance, the proof of uniform continuity for $f_n^{(2)}(t, x[0,t])(I_{\{t\}}^{\otimes 2})$ can be done as follows. For $x[0,t]$, $y[0,t] \in D[0,t]$ with $\|y - x\| \le 1$, we estimate the following

$$\begin{aligned} &\left| f_n^{(2)}(t, y[0,t])(I_{\{t\}}^{\otimes 2}) - f_n^{(2)}(t, x[0,t])(I_{\{t\}}^{\otimes 2}) \right| \\ &\quad = \left| f^{(2)}(t, y[0,t])(I_{\{t\}}^{\otimes 2}) h\left(\frac{y(t)}{n}\right) h\left(\frac{y(0)}{n}\right) - f^{(2)}(t, x[0,t])(I_{\{t\}}^{\otimes 2}) h\left(\frac{x(t)}{n}\right) h\left(\frac{x(0)}{n}\right) + \text{other terms} \right| \\ &\quad \le I_1 + I_2 + |\text{other terms}|, \end{aligned}$$

where

$$I_1 = \left| f^{(2)}(t, y[0,t])(I_{\{t\}}^{\otimes 2}) h\left(\frac{y(t)}{n}\right) h\left(\frac{y(0)}{n}\right) - f^{(2)}(t, y[0,t])(I_{\{t\}}^{\otimes 2}) h\left(\frac{x(t)}{n}\right) h\left(\frac{x(0)}{n}\right) \right|$$

$$\leq \left| f^{(2)}(t, y[0,t])(I_{\{t\}}^{\otimes 2}) \right| \cdot \left(\left| h'\left(\frac{\theta x(t) + (1-\theta) y(t)}{n}\right) \right| \frac{1}{n} \cdot |y(t) - x(t)| \right.$$
$$\left. + \left| h'\left(\frac{\theta' x(0) + (1-\theta') y(0)}{n}\right) \right| \frac{1}{n} \cdot |y(0) - x(0)| \right)$$
$$\leq \text{const}_6 (1 + (2n+1)^{j-2})^2 \cdot \frac{2}{n} \cdot \sup_x |h'(x)| \cdot \|y - x\|.$$

Term I_2 can be estimated as follows,

$$I_2 = \left| f^{(2)}(t, y[0,t])(I_{\{t\}}^{\otimes 2}) - f^{(2)}(t, x[0,t])(I_{\{t\}}^{\otimes 2}) \right| h\left(\frac{x(t)}{n}\right) h\left(\frac{x(0)}{n}\right)$$
$$= \left| f^{(3)}(t, x[0,t] + \theta(t, x[0,t], y[0,t]) \right.$$
$$\left. \cdot (y[0,t] - x[0,t]))(I_{\{t\}}, I_{\{t\}}, y[0,t] - x[0,t]) \right| \cdot h\left(\frac{x(t)}{n}\right) h\left(\frac{x(0)}{n}\right)$$
$$\leq \text{const}_6 (1 + (2n+1)^{j-2})^2 \cdot \|y - x\|,$$

where $\theta(t, x[0,t], y[0,t])$ is between 0 and 1 depending on t, $x[0,t]$ and $y[0,t]$. The other terms are

$$|\text{other terms}|$$
$$= \left| f^{(1)}(t, y[0,t])(I_{\{t\}}) \cdot h'\left(\frac{y(t)}{n}\right) \frac{2}{n} - f^{(1)}(t, x[0,t])(I_{\{t\}}) \cdot h'\left(\frac{x(t)}{n}\right) \frac{2}{n} \right.$$
$$\left. + f(t, y[0,t]) \cdot h''\left(\frac{y(t)}{n}\right) \cdot \frac{1}{n^2} - f(t, x[0,t]) \cdot h''\left(\frac{x(t)}{n}\right) \cdot \frac{1}{n^2} \right| h\left(\frac{x(0)}{n}\right).$$

Similar estimates can be made for the other terms. The uniform continuity of the second derivative $f_n^{(2)}(t, x[0,t])(I_{\{t\}}^{\otimes 2})$ in $x[0,t]$ then follows from these estimates.

Proof of (c): First we show the functional $f(t, x[0,t]) = E_{t,x[0,t]}^{\eta^0} F(\eta^0[0,T])$ given by (14) has a pseudo-partial derivative $f_{(1)}(t, x[0,t])$ with F in Theorem 2.1. If functional F were bounded together with its derivatives, then pseudo-partial derivative $f_{(1)}(t, x[0,t])$ exists and even an explicit formula can be found (see Lemma 4 of [7]). Here F in Theorem 2.1 may not be bounded, so we set, for $x[0,t] \in D[0,t]$ and h defined above,

$$\hat{f}_m(t, x[0,t]) = E_{t,x[0,t]}^{\eta^0} \left[F(\eta^0[0,T]) h\left(\frac{\eta^0(T)}{m}\right) \right].$$

It is clear that $\lim_{m\to\infty} \hat{f}_m(t, x[0,t]) = f(t, x[0,t])$. Now $F(\eta^0[0,T]) h\left(\frac{\eta^0(T)}{m}\right)$ is bounded together with its derivatives, then it follows that $\hat{f}_m(t, x[0,t])$ has a pseudo-partial derivative $(\hat{f}_m)_{(1)}(t, x[0,t])$, that is, for any $0 \leq s \leq t \leq T$, $x[0,t] \in D[0,t]$,

$$\hat{f}_m(t, x^s[0,t]) = \hat{f}_m(s, x[0,s]) + \int_s^t (\hat{f}_m)_{(1)}(u, x^s[0,u]) du. \tag{19}$$

What is more,

$$(\hat{f}_m)_{(1)}(t, x[0,t]) = -\alpha_2^1(t, x_*(t)) x(t) (\hat{f}_m)^{(1)}(t, B^{-1}x[0,t])(I_{\{t\}})$$
$$- \frac{1}{2} \alpha^2(t, x_*(t)) (\hat{f}_m)^{(2)}(t, B^{-1}x[0,t])(I_{\{t\}}^{\otimes 2}),$$

where $Bx(t) = \exp\{\int_0^t \alpha_2^1(s, x_*(s))ds\} \cdot x(t)$ is a one-to-one mapping from $D[0, T]$ onto itself. By sending m to infinity in (19) we get

$$f(t, x^s[0,t]) = f(s, x[0,s]) + \int_s^t \Bigg(-\alpha_2^1(u, x_*(u))x(t) f^{(1)}(u, B^{-1}x^s[0,u])(I_{\{u\}}) \\ - \frac{1}{2}\alpha^2(u, x_*(u)) f^{(2)}(u, B^{-1}x^s[0,u])(I_{\{u\}}^{\otimes 2}) \Bigg) du.$$

This suggests that a version $f_{(1)}(t, x[0,t])$ of the pseudo-partial derivatives is given by

$$f_{(1)}(t, x[0,t]) = -\alpha_2^1(t, x_*(t))x(t) f^{(1)}(t, B^{-1}x[0,t])(I_{\{t\}}) \\ - \frac{1}{2}\alpha^2(t, x_*(t)) f^{(2)}(t, B^{-1}x[0,t])(I_{\{t\}}^{\otimes 2}).$$

From (15) we know

$$\lim_{t \downarrow s} f_{(1)}(t, x^s[0,t]) = f_{(1)}(s, x[0,s])$$

for all $s \in [0, T]$ and $x[0,s] \in D[0,s]$, so this version satisfies

$$f_{(1)}(t, x[0,t]) = \lim_{\Delta \downarrow 0} \Delta^{-1}[f(t+\Delta, x^t[0,t+\Delta]) - f(t, x[0,t])].$$

We thus can compute the pseudo-partial derivative of f_n as follows:

$$\begin{aligned}(f_n)_{(1)}(t, x[0,t]) &= \lim_{\Delta \downarrow 0} \Delta^{-1}\left[f_n(t+\Delta, x^t[0,t+\Delta]) - f_n(t, x[0,t])\right] \\ &= \lim_{\Delta \downarrow 0} \Delta^{-1}\left[f(t+\Delta, x^t[0,t+\Delta]) - f(t, x[0,t])\right] \\ &\quad \cdot h\left(\frac{x(t)}{n}\right) \cdot h\left(\frac{x(0)}{n}\right) \\ &= f_{(1)}(t, x[0,t]) \cdot h\left(\frac{x(t)}{n}\right) \cdot h\left(\frac{x(0)}{n}\right).\end{aligned}$$

From Lemma 3.3 we have

$$\sup_{x[0,t] \in D[0,t]} \left|(f_n)_{(1)}(t, x[0,t])\right| < \infty.$$

The continuity in $x[0,t]$ of $(f_n)_{(1)}(t, x[0,t])$ is proved in the same way as in (a) and (b). The claim is thus proved.

Now we apply $\lim_{n\to\infty}$ to both sides of (18). For $\lim_{n\to\infty} E_{0,x_0}^{\eta^\epsilon} f_n(T, \eta^\epsilon[0,T])$, we can find a dominating function as follows

$$\left|f_n(T, \eta^\epsilon[0,T])\right| \le \left|f(T, \eta^\epsilon[0,T])\right| \le B'(1 + |\eta^\epsilon(T)|^j).$$

Noticing that $B'(1 + |\eta^\epsilon(T)|^j)$ is integrable by Lemma 3.1, we get from Lebesgue dominated convergence theorem that $\lim_{n\to\infty} E_{0,x_0}^{\eta^\epsilon} f_n(T, \eta^\epsilon[0,T]) = E_{0,x_0}^{\eta^\epsilon} f(T, \eta^\epsilon[0,T])$. Now for the limit on the right hand side $\lim_{n\to\infty} \int_0^T E_{0,x_0}^{\eta^\epsilon} \mathfrak{A}^{\eta^\epsilon} f_n(t, \eta^\epsilon[0,t])dt$, we use Lebesgue dominated convergence theorem twice to first interchange $\lim_{n\to\infty}$ and $\int_0^T$ and then interchange $\lim_{n\to\infty}$

and $E^{\eta^\epsilon}_{0,x_0}$. More precisely, the dominating function can be chosen as follows:

$$\left|\mathfrak{A}^{\eta^\epsilon} f_n(t,\eta^\epsilon[0,t])\right| \le \left(\text{const}_1 + \text{const}_2 \cdot |\eta^\epsilon(t)|^j\right)\left(1+|x_0|^j\right)$$

where finite constants const_1 and const_2 are independent of n (but they depend on $\|\alpha_2^2\|$ and $\|\alpha_{22}^1\|$, and this is why we assume these two are finite). At the end, we get

$$E^{\eta^\epsilon}_{0,x_0} f(T,\eta^\epsilon[0,T]) = f(0,x_0) + \int_0^T E^{\eta^\epsilon}_{0,x_0}\mathfrak{A}^{\eta^\epsilon} f(t,\eta^\epsilon[0,t])dt,$$

which means $\mathfrak{A}^{\eta^\epsilon} f(t,x[0,t])$ is well defined. □

Proof of Theorem 2.1. Since we have Proposition 3.4, we rewrite $\mathfrak{A}^{\eta^\epsilon} f(t,x[0,t])$ as follows,

$$\begin{aligned}\mathfrak{A}^{\eta^\epsilon} f(t,x[0,t]) = \epsilon^{1/2}\Bigg\{&\frac{1}{2}\alpha_{22}^1(t,x_*(t)+\epsilon^{1/2}\theta_1^\epsilon)\cdot x(t)^2\cdot f^{(1)}(t,x[0,t])(I_{\{t\}})\\ &+\frac{1}{2}\alpha_2^2(t,x_*(t)+\epsilon^{1/2}\theta_2^\epsilon)\cdot x(t)\cdot f^{(2)}(t,x[0,t])(I_{\{t\}}^{\otimes 2})\\ &+\int\frac{1}{6}u^3 f^{(3)}(t,x[0,t]+\epsilon^{1/2}\theta_3^\epsilon\\ &\cdot u\cdot I_{\{t\}})(I_{\{t\}}^{\otimes 3})\mu_{t,x_*(t)+\epsilon^{1/2}x(t)}(du)\Bigg\},\end{aligned} \tag{20}$$

where θ_i^ϵ, $i=1,2,3$, are between 0 and 1 depending on ϵ (and some other elements).

We want to show $\epsilon^{-1/2}\mathfrak{A}^{\eta^\epsilon} f(t,x[0,t])$ converges to

$$\begin{aligned}Bf(t,x[0,t]) := \Bigg\{&\frac{1}{2}\alpha_{22}^1(t,x_*(t))\cdot x(t)^2\cdot f^{(1)}(t,x[0,t])(I_{\{t\}})\\ &+\frac{1}{2}\alpha_2^2(t,x_*(t))\cdot x(t)\cdot f^{(2)}(t,x[0,t])(I_{\{t\}}^{\otimes 2})\\ &+\frac{1}{6}\alpha^3(t,x_*(t))f^{(3)}(t,x[0,t])(I_{\{t\}}^{\otimes 3})\Bigg\},\end{aligned} \tag{21}$$

uniformly as $x[0,t]$ varies over every compact subset of $D_{x_0}[0,t]$. We prove such uniform convergence in several steps (note that uniform convergence can be easily derived except for the last integral term in (20)).

Step 1: As $\epsilon\to 0$,

$$f^{(3)}(t,x[0,t]+\epsilon^{1/2}\theta_3^\epsilon\cdot u\cdot I_{\{t\}})(I_{\{t\}}^{\otimes 3}) \to f^{(3)}(t,x[0,t])(I_{\{t\}}^{\otimes 3})$$

uniformly for $|u|\le$ const. and for $x[0,t]$ changing over every compact subset of $D_{x_0}[0,t]$. This is from (ii) of Theorem 2.1 and (17). To see this, we notice that (17) gives that for $i=3$ and for any $y[0,t]\in D_{x_0}[0,t]$,

$$\begin{aligned}f^{(3)}(t,y[0,t])(I_{\{t\}}^{\otimes 3}) = E^{\eta^0}_{0,x_0}F^{(3)}\Bigg(&y^t[0,T]\cdot\exp\left\{\int_0^\bullet\alpha_2^1(s,x_*(s))ds\right\}\\ &+\eta_t^0[0,T]\Bigg)(\psi_{[t,T]}^{\otimes 3})\end{aligned}$$

$$= \int_{D_0[0,T]} F^{(3)}\left(y^t[0,T]\cdot\exp\left\{\int_0^{\bullet}\alpha_2^1(s,x_*(s))ds\right\}\right.$$
$$\left. +\ z[0,T]\right)(\psi_{[t,T]}^{\otimes^3})\mu_{\eta_t^0[0,T]}(dz[0,T]), \tag{22}$$

where $\mu_{\eta_t^0[0,T]}$ represents the distribution of $\eta_t^0[0,T]$ on space $D_0[0,T]$. We now replace $y[0,t]$ in (22) by $y_\epsilon[0,t] = x[0,t] + \epsilon^{1/2}\theta_3^\epsilon \cdot u \cdot I_{\{t\}}[0,t]$ for $x[0,t] \in D_{x_0}[0,t]$ (the initial point $y_\epsilon(0) = x(0) = x_0$ for $t > 0$), then from (ii) we know the integrand converges to $F^{(3)}(x^t[0,T]\cdot\exp\{\int_0^\bullet \alpha_2^1(s,x_*(s))ds\} + z[0,T])(\psi_{[t,T]}^{\otimes^3})$ uniformly for $|u| \le$ const. and for $x[0,t]$, $z[0,T]$ changing over every compact subset of $D_{x_0}[0,t]$, $D_0[0,t]$ respectively. What is more, integrand $F^{(3)}(y_\epsilon^t[0,T]\cdot\exp\{\int_0^\bullet \alpha_2^1(s,x_*(s))ds\} + z[0,T])(\psi_{[t,T]}^{\otimes^3})$ is bounded on every compact set (for $z[0,T]$) and uniformly integrable with respect to probability measure $\mu_{\eta_t^0[0,T]}$. These facts complete the proof of Step 1.

Step 2:

$$\left|\int u^3 f^{(3)}(t,x[0,t]+\epsilon^{1/2}\theta_3^\epsilon u I_{\{t\}})(I_{\{t\}}^{\otimes^3})\mu_{t,x_*(t)+\epsilon^{1/2}x(t)}(du)\right.$$
$$\left. -\alpha^3(t,x_*(t))f^{(3)}(t,x[0,t])(I_{\{t\}}^{\otimes^3})\right|$$
$$\le \int |u|^3\left|f^{(3)}(t,x[0,t]+\epsilon^{1/2}\theta_3^\epsilon\cdot u\cdot I_{\{t\}})(I_{\{t\}}^{\otimes^3})\right.$$
$$\left. - f^{(3)}(t,x[0,t])(I_{\{t\}}^{\otimes^3})\right|\mu_{t,x_*(t)+\epsilon^{1/2}x(t)}(du)$$
$$+ \left|f^{(3)}(t,x[0,t])(I_{\{t\}}^{\otimes^3})\right|\cdot\left|\int u^3\mu_{t,x_*(t)+\epsilon^{1/2}x(t)}(du) - \int u^3\mu_{t,x_*(t)}(du)\right|$$
$$=: J_1 + J_2.$$

First, $J_2 \to 0$ ($\epsilon \to 0$) uniformly as $x[0,t]$ over every compact subset of $D_{x_0}[0,t]$, since α^3 is continuous at $x_*(t)$ in its second argument. For J_1, let $C > 0$,

$$J_1 = \int |u|^3\left|f^{(3)}(t,x[0,t]+\epsilon^{1/2}\theta_3^\epsilon u I_{\{t\}})(I_{\{t\}}^{\otimes^3})\right.$$
$$\left. - f^{(3)}(t,x[0,t])(I_{\{t\}}^{\otimes^3})\right|\mu_{t,x_*(t)+\epsilon^{1/2}x(t)}(du)$$
$$= \int_{|u|\le C} |u|^3\left|f^{(3)}(t,x[0,t]+\epsilon^{1/2}\theta_3^\epsilon u I_{\{t\}})(I_{\{t\}}^{\otimes^3})\right.$$
$$\left. - f^{(3)}(t,x[0,t])(I_{\{t\}}^{\otimes^3})\right|\mu_{t,x_*(t)+\epsilon^{1/2}x(t)}(du)$$
$$+ \int_{|u|>C} |u|^3\left|f^{(3)}(t,x[0,t]+\epsilon^{1/2}\theta_3^\epsilon u I_{\{t\}})(I_{\{t\}}^{\otimes^3})\right.$$
$$\left. - f^{(3)}(t,x[0,t])(I_{\{t\}}^{\otimes^3})\right|\mu_{t,x_*(t)+\epsilon^{1/2}x(t)}(du)$$
$$\le \int_{|u|\le C} |u|^3\left|f^{(3)}(t,x[0,t]+\epsilon^{1/2}\theta_3^\epsilon u I_{\{t\}})(I_{\{t\}}^{\otimes^3})\right.$$
$$\left. - f^{(3)}(t,x[0,t])(I_{\{t\}}^{\otimes^3})\right|\mu_{t,x_*(t)+\epsilon^{1/2}x(t)}(du)$$

$$+ \int_{|u|>C} |u|^3 \Big(\text{const}_1 + \text{const}_2 |u|^{j-2} \epsilon^{\frac{j-2}{2}} + \text{const}_3 \cdot |x(t)|^{j-2}\Big) \mu_{t,x_*(t)+\epsilon^{1/2}x(t)}(du)$$
$$=: J_3 + J_4,$$

where J_3 converges to zero (for any C) uniformly in $x[0,T]$ over a compact set according to Step 1. The first part $\int_{|u|>C} |u|^3 \text{const}_1 \mu_{t,x_*(t)+\epsilon^{1/2}x(t)}(du)$ of J_4 goes to zero as $C \to \infty$ uniformly in all $x[0,t]$ (not necessary over compact set) since $|u|^3$ is uniformly integrable with respect to the measures $\mu_{t,x}$ (this can be seen from $\|\beta^4\| < \infty$). The second part $\int_{|u|>C} |u|^3 \text{const}_2 \cdot |u|^{j-2} \cdot \epsilon^{\frac{j-2}{2}} \mu_{t,x_*(t)+\epsilon^{1/2}x(t)}(du)$ also converges to zero uniformly in all $x[0,t]$ because $|u|^{j+1}$ is uniformly integrable with respect to $\mu_{t,x}$. The same uniform integrability of $|u|^{j+1}$ implies that the last part $\int_{|u|>C} |u|^3 \text{const}_3 \cdot |x(t)|^{j-2} \mu_{t,x_*(t)+\epsilon^{1/2}x(t)}(du)$ of J_4 goes to zero uniformly in $x[0,T]$ over any compact set.

At the same time, we note that $\epsilon^{-1/2}\mathfrak{A}^{\eta^\epsilon} f(t, x[0,t])$ is dominated by

$$B'' \left(\text{const}_1 + \text{const}_2 \cdot |x(t)|^j\right),$$

which is uniformly integrable with respect to distributions of η^ϵ because of Lemma 3.1,

$$\sup_{\epsilon \in (0,\epsilon_0)} E_{0,x_0}^{\eta^\epsilon} |\eta^\epsilon(t)|^{j+1} < \infty.$$

Then weak convergence of η^ϵ to η^0 (which was proved by Theorem 1 and Lemma 6 in [7]) completes the proof. □

4. An extension of Theorem 2.1

As mentioned in the remark of Theorem 2.1, the condition (i) of Theorem 2.1 is restrictive to some extent. In this section, we will weaken this condition.

Theorem 4.1. *Under conditions of Theorem 2.1, but with (i) replaced by:*
(i)′ *there is a constant $C > 0$ such that for all $x[0,T], y[0,T] \in D[0,T]$,*

$$|F(x[0,T])| \le C \left(1 + |x(T)|^j + \int_0^T |x(s)|^j ds\right);$$
$$\left|F^{(i)}(x[0,T])(y[0,T]^{\otimes^i})\right| \le (1 + \|y\|^i) C \left(1 + |x(T)|^{j-2} + \int_0^T |x(s)|^{j-2} ds\right),$$
$$i = 1, 2, 3,$$

the following holds: as $\epsilon \downarrow 0$,

$$E_{0,x_0}^{\eta^\epsilon} F(\eta^\epsilon[0,T]) = E_{0,x_0}^{\eta^0} F(\eta^0[0,T]) + \epsilon^{1/2} E_{0,x_0}^{\eta^0} A_1 F(\eta^0[0,T]) + o(\epsilon^{1/2}).$$

Proof. First note that under condition (i)′, by using similar proofs as in Lemmas 3.2 and 3.3, we can prove there are constants C' and C''

$$|f(t, x[0,t])| \le C' \left(1 + |x(t)|^j + \int_0^t |x(s)|^j ds\right);$$
$$\left|f^{(i)}(t, x[0,t])(I_{\{t\}}^{\otimes^i})\right| \le C'' \left(1 + |x(t)|^{j-2} + \int_0^t |x(s)|^{j-2} ds\right).$$

Then we can prove that the compensating operator $\mathfrak{A}^{\eta^\epsilon}$ of $\eta^\epsilon[0,T]$ can be applied to function $f(t, x[0,t])$ given by (14) provided F satisfies condition (i)$'$. To do this, we apply similar arguments as in Proposition 3.4, but this time we define

$$f_n(t, x[0,t]) = f(t, x[0,t]) \cdot h\left(\frac{x(t)}{n}\right) \cdot h\left(\frac{x(0)}{n}\right) \cdot h\left(\frac{\int_0^t (x(s))^{2j}\, ds}{n}\right),$$

for a smooth function h which is equal to 1 in $(-1,1)$ and to 0 outside $(-2,2)$. Similarly we can get (20). The rest of the proof will be almost the same as that of Theorem 2.1. □

Now more classes of examples can be produced based on the condition (i)$'$. The first class of examples takes the form

$$F(x[0,T]) = \phi\left(\int_0^T g(x(s))ds\right)$$

where two smooth functions ϕ and g satisfy suitable upper bound assumptions in order to validate (i)$'$. For instance, we can take $F(x[0,T]) = \left(\int_0^T x^{k_1}(s)ds\right)^{k_2}$ for two positive integers k_1 and k_2. More generally, we can produce functionals combining the end point value $x(T)$ and the integral type functional as follows

$$F(x[0,T]) = \phi\left(x(T), \int_0^T g(x(s))ds\right).$$

It is straightforward to check the differentiability of F given the smoothness of $\phi(x,y)$ and g. Also, the condition (i)$'$ is fulfilled if all the partial derivatives of $\phi(x,y)$ are bounded by $1 + |x|^k + |y|^k$ and the derivatives of $g(x)$ are bounded by $1 + |x|^k$ for some positive integer k. A specific example of this kind is $F(x[0,T]) = \left(\int_0^T x^{k_1}(s)ds\right)^{k_2} \cdot x^{k_3}(T)$ for positive integers k_1, k_2 and k_3.

5. Applications of Theorem 2.1: general expansions

First let us introduce some symbols:

$$\alpha^j_{(i)}(t,x) := \alpha^j_{\underbrace{22\cdots2}_{i}}(t,x).$$

The subscript $_2$ denotes the differentiation in the second spatial argument x. Thus $\alpha^j_{(i)}(t,x)$ means the i times differentiation in x of $\alpha^j(t,x)$.

Theorem 5.1. *Let processes η^ϵ and x_* be in Theorem 2.1. Consider an integer $s \geq 3$. Assume $\alpha^1_2(t,x) \leq C_1$ for some nonnegative constant C_1, and*

$$\|\beta^i\| < \infty \quad \text{for } 3 \leq i \leq 3(s-2)+1; \qquad \|\alpha^2\| < \infty;$$
$$\begin{cases} \|\alpha^1_{22}\|, \|\alpha^1_{222}\|, \ldots, \|\alpha^1_{(s-1)}\| < \infty, \\ \|\alpha^2_2\|, \|\alpha^2_{22}\|, \ldots, \|\alpha^2_{(s-2)}\| < \infty, \\ \|\alpha^3_2\|, \|\alpha^3_{22}\|, \ldots, \|\alpha^3_{(s-3)}\| < \infty, \\ \cdots \\ \|\alpha^{s-1}_2\| < \infty. \end{cases}$$

Assume, in addition, that $\alpha^1_{(s-1)}(t,x), \alpha^2_{(s-2)}(t,x), \ldots, \alpha^{s-1}_2(t,x)$ *and* $\alpha^s(t,x)$ *are continuous in x at the point* $x_*(t)$ *for all t, and* $|u|^{s+2}$ *is uniformly integrable with respect to* $\mu_{t,x}$.

Let the functional $F(x[0,T])$ *on* $D_{x_0}[0,T]$ *be* $3(s-2)$ *times differentiable with the following conditions:*

(I) *there is a constant* $B > 0$ *such that for all* $x[0,T], y[0,T] \in D[0,T]$

$$|F(x[0,T])| \le B(1+|x(T)|^s);$$
$$\left|F^{(i)}(x[0,T])(y[0,T]^{\otimes^i})\right| \le (1+\|y\|^i)\cdot B\cdot\left(1+|x(T)|^{s-2}\right),$$
$$i = 1,2,\ldots,3(s-2);$$

(II) $F^{(i)}(x[0,T])(I_{[t,T]}\delta,\ldots,I_{[t,T]}\delta)$, $1 \le i \le 3(s-2)$, *are continuous with respect to* $x[0,T]$ *uniformly as* $x[0,T]$ *changes over an arbitrary compact subset of* $D[0,T]$, t *over* $[0,T]$, *and* $\delta[0,T]$ *over the set of Lipschitz continuous functions with constant* 1, $\|\delta\| \le 1$.

Then as $\epsilon \downarrow 0$,

$$E^{\eta^\epsilon}_{0,x_0}F(\eta^\epsilon[0,T]) = E^{\eta^0}_{0,x_0}F(\eta^0[0,T]) + \sum_{i=1}^{s-2}\epsilon^{\frac{i}{2}}E^{\eta^0}_{0,x_0}A_iF(\eta^0[0,T]) + o(\epsilon^{\frac{s-2}{2}}), \tag{23}$$

where A_1 *is a third-order differential operator defined before,* A_2 *is a sixth-order differential operator given by*

$$\begin{aligned} A_2F(x[0,T]) = {} & \int_0^T A_1\widetilde{F}(x[0,t])dt + \int_0^T\left[\frac{1}{3!}\alpha^1_{222}(t,x_*(t))x(t)^3f^{(1)}\right.\\ & \times(t,x[0,t])(I_{\{t\}}) + \frac{1}{4}\alpha^2_{22}(t,x_*(t))x(t)^2f^{(2)}(t,x[0,t])(I^{\otimes^2}_{\{t\}})\\ & + \frac{1}{3!}\alpha^3_2(t,x_*(t))x(t)f^{(3)}(t,x[0,t])(I^{\otimes^3}_{\{t\}})\\ & \left. + \frac{1}{4!}\alpha^4(t,x_*(t))f^{(4)}(t,x[0,t])(I^{\otimes^4}_{\{t\}})\right]dt \end{aligned}$$

with

$$\begin{aligned} \widetilde{F}(x[0,t]) = {} & \frac{1}{2}\alpha^1_{22}(t,x_*(t))x^2(t)f^{(1)}(t,x[0,t])(I_{\{t\}})\\ & + \frac{1}{2}\alpha^2_2(t,x_*(t))x(t)f^{(2)}(t,x[0,t])(I^{\otimes^2}_{\{t\}})\\ & + \frac{1}{6}\alpha^3(t,x_*(t))f^{(3)}(t,x[0,t])(I^{\otimes^3}_{\{t\}}), \end{aligned} \tag{24}$$

and $A_3,\ldots,A_{s-2}$ *are suitable differential operators defined through derivatives of* f.

Proof. The case when $s = 3$ is actually Theorem 2.1. In what follows, sometimes $f^{(k)}$ will be used to denote $f^{(k)}(t,x[0,t])(I^{\otimes^k}_{\{t\}})$ for short. Now we first prove this for $s = 4$. Applying Taylor's formula we write

$$\begin{aligned} & \mathfrak{A}^{\eta^\epsilon}f(t,x[0,t])\\ & = \epsilon^{1/2}\left\{\left(\frac{1}{2!}\alpha^1_{22}(t,x_*(t))x(t)^2 + \frac{1}{3!}\alpha^1_{222}(t,x_*(t)+\epsilon^{1/2}\theta^\epsilon_1x(t))\epsilon^{1/2}x(t)^3\right)\right.\\ & \quad\times f^{(1)}(t,x[0,t])(I_{\{t\}}) \end{aligned}$$

$$+\frac{1}{2}\left(\alpha_2^2(t,x_*(t))\cdot x(t)+\frac{1}{2!}\alpha_{22}^2(t,x_*+\epsilon^{1/2}\theta_2^\epsilon x(t))\cdot\epsilon^{1/2}x(t)^2\right)$$
$$\cdot f^{(2)}(t,x[0,t])(I_{\{t\}}^{\otimes^2})+\int\left(\frac{1}{3!}u^3 f^{(3)}(t,x[0,t])(I_{\{t\}}^{\otimes^3})\right.$$
$$\left.\left.+\frac{1}{4!}u^4\cdot\epsilon^{1/2}f^{(4)}(t,x[0,t]+\epsilon^{1/2}\theta_3^\epsilon\cdot uI_{\{t\}})(I_{\{t\}}^{\otimes^4})\right)\mu_{t,x_*(t)+\epsilon^{1/2}x(t)}(du)\right\}.$$

Then, if we write $\alpha^3(t,x_*(t)+\epsilon^{1/2}x(t))=\alpha^3(t,x_*(t))+\alpha_2^3(t,x_*(t)+\epsilon^{1/2}\theta x(t))\cdot\epsilon^{1/2}x(t)$, then

$$\epsilon^{-1/2}\left(\epsilon^{-1/2}\mathfrak{A}^{\eta^\epsilon}f(t,x[0,t])-\left[\frac{1}{2!}\alpha_{22}^1(t,x_*(t))\cdot x(t)^2\cdot f^{(1)}\right.\right.$$
$$\left.\left.+1/2\alpha_2^2(t,x_*(t))\cdot x(t)\cdot f^{(2)}+\frac{1}{3!}\alpha^3(t,x_*(t))f^{(3)}(t,x[0,t])(I_{\{t\}}^{\otimes^3})\right]\right)$$
$$\to\left\{\left(\frac{1}{3!}\alpha_{222}^1(t,x_*(t))\cdot x(t)^3\right)\cdot f^{(1)}(t,x[0,t])(I_{\{t\}})\right.$$
$$+\frac{1}{4}\alpha_{22}^2(t,x_*)\cdot x(t)^2\cdot f^{(2)}(t,x[0,t])(I_{\{t\}}^{\otimes^2})$$
$$\left.+\frac{1}{3!}\alpha_2^3(t,x_*(t))x(t)f^{(3)}(t,x[0,t])(I_{\{t\}}^{\otimes^3})+\frac{1}{4!}f^{(4)}(t,x[0,t])(I_{\{t\}}^{\otimes^4})\alpha^4(t,x_*(t))\right\},$$

uniformly as $x[0,t]$ varies over every compact subset of $D_{x_0}[0,t]$ (The proof on uniform convergence can be done similarly as that of Theorem 2.1. The uniform integrability of $|u|^6$ with respect to measures $\mu_{t,x}$ is used and this is a consequence of $\|\beta^7\|<\infty$.) Besides,

$$\epsilon^{-1/2}\left(\epsilon^{-1/2}\mathfrak{A}^{\eta^\epsilon}f(t,x[0,t])-\left[\frac{1}{2!}\alpha_{22}^1(t,x_*(t))\cdot x(t)^2\cdot f^{(1)}\right.\right.$$
$$\left.\left.+1/2\alpha_2^2(t,x_*(t))\cdot x(t)\cdot f^{(2)}+\frac{1}{3!}\alpha^3(t,x_*(t))f^{(3)}(t,x[0,t])(I_{\{t\}}^{\otimes^3})\right]\right)$$

can be dominated by a functional which is uniformly integrable with respect to distributions of η^ϵ. Thus we have

$$\begin{aligned}
&E_{0,x_0}^{\eta^\epsilon}\mathfrak{A}^{\eta^\epsilon}f(t,\eta^\epsilon[0,t])\\
&=\epsilon^{1/2}E_{0,x_0}^{\eta^\epsilon}\left[\frac{1}{2!}\alpha_{22}^1(t,x_*(t))\eta^\epsilon(t)^2f^{(1)}+1/2\alpha_2^2(t,x_*(t))\eta^\epsilon(t)f^{(2)}\right.\\
&\left.+\ \frac{1}{3!}\alpha^3(t,x_*(t))f^{(3)}(t,\eta^\epsilon[0,t])(I_{\{t\}}^{\otimes^3})\right]\\
&+\epsilon\cdot E_{0,x_0}^{\eta^0}\left[\frac{1}{3!}\alpha_{222}^1(t,x_*(t))\cdot\eta^0(t)^3\cdot f^{(1)}+\frac{1}{4}\alpha_{22}^2(t,x_*)\cdot\eta^0(t)^2\cdot f^{(2)}\right.\\
&\left.+\ \frac{1}{3!}\alpha_2^3(t,x_*(t))\eta^0(t)f^{(3)}+\frac{1}{4!}f^{(4)}(t,\eta^0[0,t])(I_{\{t\}}^{\otimes^4})\alpha^4(t,x_*(t))\right]\\
&+o(\epsilon).
\end{aligned}\tag{25}$$

Now we want to apply the case $s = 3$ to write the term $E^{\eta^\epsilon}_{0,x_0}\widetilde{F}(\eta^\epsilon[0,t])$ as $E^{\eta^0}_{0,x_0}\widetilde{F}(\eta^0[0,t]) + \epsilon^{1/2}E^{\eta^0}_{0,x_0}A_1\widetilde{F}(\eta^0[0,t]) + o(\epsilon^{1/2})$, thus we have to apply Theorem 2.1 to functional $\widetilde{F}(x[0,t])$ defined by (24). First we have to check $\widetilde{F}(x[0,t])$ satisfies all conditions of Theorem 2.1: here we just check that the first term of $\widetilde{F}(x[0,t])$ (which is $x^2(t)f^{(1)}(t,x[0,t])(I_{\{t\}})$) satisfies (i) and (ii) of Theorem 2.1. It can be easily seen that $x^2(t)f^{(1)}(t,x[0,t])(I_{\{t\}})$ is differentiable and

$$\begin{aligned}
&\left|x^2(t)f^{(1)}(t,x[0,t])(I_{\{t\}})\right| \le x^2(t)\cdot B''\cdot(1+|x(t)|^{s-2}),\\
&\left[x^2(t)f^{(1)}(t,x[0,t])(I_{\{t\}})\right]^{(1)}(y[0,t])\\
&\quad= \lim_{h\to 0} h^{-1}\left[(x+hy)^2(t)f^{(1)}(t,(x+hy)[0,t])(I_{\{t\}}) - x^2(t)f^{(1)}(t,x[0,t])(I_{\{t\}})\right]\\
&\quad= x^2(t)f^{(2)}(t,x[0,t])(I_{\{t\}},y[0,t]) + 2x(t)y(t)f^{(1)}(t,x[0,t])(I_{\{t\}}).
\end{aligned}$$

Proceeding in this way, we can find $\left[x^2(t)f^{(1)}\right]^{(2)}$ and $\left[x^2(t)f^{(1)}\right]^{(3)}$. For the derivative $[x^2(t)f^{(1)}]^{(1)}$, there is a term involving $x^2(t)f^{(2)}$, and condition (I) of Theorem 5.1 implies $|x^2(t)f^{(2)}| \le \text{const}_1 + \text{const}_2|x^{2+s-2}(t)|$. Thus in order to make the first derivative satisfy condition (i) of Theorem 2.1, we should require, in case $s = 4$, $\|\beta^i\| < \infty, 3 \le i \le 7$. For condition (ii) of Theorem 2.1, we need uniform continuity of $\widetilde{F}^{(3)}$ in some sense: note that $\widetilde{F}^{(3)}$ consists of $f^{(i)}, 2 \le i \le 3(s-2) = 6$, and this is the reason why we assume condition (II) of Theorem 5.1, i.e. uniform continuity of $F^{(i)}, 2 \le i \le 3(s-2) = 6$ (here we actually do not need uniform continuity of $F^{(1)}$, but we will see that uniform continuity of $F^{(1)}$ is needed when we prove the case when $s = 5$).

Applying Theorem 2.1 with t in lieu of T to $\widetilde{F}$ and then putting the result back to (25), we get a new version of (25) and put it in the following equality

$$E^{\eta^\epsilon}_{0,x_0}F(\eta^\epsilon[0,T]) = E^{\eta^0}_{0,x_0}F(\eta^0[0,T]) + \int_0^T E^{\eta^\epsilon}_{0,x_0}\mathfrak{A}^{\eta^\epsilon}f(t,\eta^\epsilon[0,t])dt.$$

This proves (23) when $s = 4$, and gives A_2, which is a sixth-order differential operator. In conclusion, for $s = 4$, we require F to be six times differentiable and $\|\beta^i\| < \infty$ for $3 \le i \le 7$.

Now we prove the case $s = 5$. We use Taylor's formula to write one term further,

$$\begin{aligned}
\mathfrak{A}^{\eta^\epsilon}f(t,x[0,t]) = \epsilon^{1/2}\Bigg\{&\Bigg(\frac{1}{2!}\alpha^1_{22}x(t)^2 + \frac{1}{3!}\alpha^1_{222}\epsilon^{1/2}x(t)^3\\
&+ \frac{1}{4!}\alpha^1_{(4)}(t,x_*(t)+\epsilon^{1/2}\theta^\epsilon_1 x(t))\epsilon x(t)^4\Bigg)f^{(1)}(t,x[0,t])(I_{\{t\}})\\
&+1/2\Bigg(\alpha^2_2 x(t) + \frac{1}{2!}\alpha^2_{22}\epsilon^{1/2}x(t)^2\\
&+ \frac{1}{3!}\alpha^2_{222}(t,x_*+\epsilon^{1/2}\theta^\epsilon_2 x(t))\epsilon x(t)^3\Bigg)f^{(2)}(t,x[0,t])(I^{\otimes 2}_{\{t\}})\\
&+\int\Bigg(\frac{1}{3!}u^3 f^{(3)} + \frac{1}{4!}u^4\epsilon^{1/2}f^{(4)}\\
&+ \frac{1}{5!}u^5\epsilon f^{(5)}(t,x[0,t]+\epsilon^{1/2}\theta^\epsilon_3 u I_{\{t\}})(I^{\otimes 5}_{\{t\}})\Bigg)\\
&\times\mu_{t,x_*(t)+\epsilon^{1/2}x(t)}(du)\Bigg\}.
\end{aligned}$$

Then, by expanding α^3 and α^4, we get

$$\epsilon^{-1}\Bigg(\epsilon^{-1/2}\mathfrak{A}^{\eta^\epsilon} f(t, x[0,t]) - \Bigg[\frac{1}{2!}\alpha^1_{22}(t, x_*(t)) \cdot x(t)^2 \cdot f^{(1)}$$

$$+ 1/2\alpha^2_2(t, x_*(t)) \cdot x(t) \cdot f^{(2)} + \frac{1}{3!}\alpha^3(t, x_*(t)) f^{(3)}(t, x[0,t])(I^{\otimes 3}_{\{t\}})\Bigg]$$

$$- \epsilon^{1/2}\left[\frac{1}{3!}\alpha^1_{222} x(t)^3 f^{(1)} + \frac{1}{2!}\alpha^2_{22} x(t)^2 f^{(2)} + \frac{1}{3!}\alpha^3_2 x(t) f^{(3)} + \frac{1}{4!}\alpha^4 f^{(4)}\right]\Bigg)$$

$$\to \Bigg\{\left(\frac{1}{4!}\alpha^1_{(4)}(t, x_*(t)) \cdot x(t)^4\right) \cdot f^{(1)}(t, x[0,t])(I_{\{t\}})$$

$$+ \frac{1}{12}\alpha^2_{222}(t, x_*) \cdot x(t)^3 \cdot f^{(2)}(t, x[0,t])(I^{\otimes 2}_{\{t\}})$$

$$+ \frac{1}{3!}\alpha^3_{22}(t, x_*) \cdot x(t)^2 \cdot f^{(3)}(t, x[0,t])(I^{\otimes 3}_{\{t\}})$$

$$+ \frac{1}{4!}\alpha^4_2(t, x_*) \cdot x(t) \cdot f^{(4)}(t, x[0,t])(I^{\otimes 4}_{\{t\}})$$

$$+ \frac{1}{5!} f^{(5)}(t, x[0,t])(I^{\otimes 5}_{\{t\}})\alpha^5(t, x_*(t))\Bigg\},$$

uniformly as $x[0,t]$ varies over every compact subset of $D_{x_0}[0,t]$. Just like before, it follows

$$E^{\eta^\epsilon}_{0,x_0}\mathfrak{A}^{\eta^\epsilon} f(t, \eta^\epsilon[0,t])$$

$$= \epsilon^{1/2} \cdot E^{\eta^\epsilon}_{0,x_0}\Bigg[\frac{1}{2!}\alpha^1_{22}(t, x_*(t)) \cdot \eta^\epsilon(t)^2 \cdot f^{(1)} + 1/2\alpha^2_2(t, x_*(t)) \cdot \eta^\epsilon(t) \cdot f^{(2)}$$

$$+ \frac{1}{3!}\alpha^3(t, x_*(t)) f^{(3)}(t, \eta^\epsilon[0,t])(I^{\otimes 3}_{\{t\}})\Bigg]$$

$$+ \epsilon \cdot E^{\eta^\epsilon}_{0,x_0}\left[\frac{1}{3!}\alpha^1_{222}\eta^\epsilon(t)^3 f^{(1)} + \frac{1}{2!}\alpha^2_{22}\eta^\epsilon(t)^2 f^{(2)} + \frac{1}{3!}\alpha^3_2\eta^\epsilon(t) f^{(3)} + \frac{1}{4!}\alpha^4 f^{(4)}\right]$$

$$+ \epsilon^{3/2} \cdot E^{\eta^0}_{0,x_0}\Bigg[\frac{1}{4!}\alpha^1_{(4)}(t, x_*(t)) \cdot \eta^0(t)^4 \cdot f^{(1)} + \frac{1}{12}\alpha^2_{222}(t, x_*) \cdot \eta^0(t)^3 \cdot f^{(2)}$$

$$+ \frac{1}{3!}\alpha^3_{22}(t, x_*) \cdot \eta^0(t)^2 \cdot f^{(3)} + \frac{1}{4!}\alpha^4_2(t, x_*) \cdot \eta^0(t) \cdot f^{(4)}$$

$$+ \frac{1}{5!} f^{(5)}(t, \eta^0[0,t])(I^{\otimes 5}_{\{t\}})\alpha^5(t, x_*(t))\Bigg] + o(\epsilon^{3/2}).$$

For $\epsilon^{1/2} \cdot E^{\eta^\epsilon}_{0,x_0}$, we want to apply the result of case $s = 4$ to write

$$\epsilon^{1/2} \cdot E^{\eta^\epsilon}_{0,x_0}\widetilde{F}\eta^\epsilon[0,t] = \epsilon^{1/2}\Big(E^{\eta^0}_{0,x_0}\widetilde{F}\eta^0[0,t] + \epsilon^{1/2} E^{\eta^0}_{0,x_0} A_1\widetilde{F}\eta^0[0,t]$$

$$+ \ \epsilon E^{\eta^0}_{0,x_0} A_2\widetilde{F}\eta^0[0,t] + o(\epsilon)\Big),$$

so we need $\widetilde{F}$ to be six times differentiable. Thus F should be nine times differentiable. Furthermore, in order to use the case $s = 4$, we have to assume the uniform continuity of $\widetilde{F}^{(i)}, 2 \le i \le 6$, which consist of $f^{(i)}, 1 \le i \le 3(s-2) = 9$ (this is why we also include uniform continuity of $F^{(1)}$ in assumption (II) of Theorem 5.1). Noticing that the derivatives of $\widetilde{F}$ contain a term $\eta^\epsilon(t)^2 \cdot f^{(i)}$, we get from condition (I) of Theorem 5.1 that

$$|\eta^\epsilon(t)^2 f^{(i)}| \le \text{const}_1 + \text{const}_2|\eta^\epsilon(t)|^{2+s-2} = \text{const}_1 + \text{const}_2|\eta^\epsilon(t)|^5.$$

Thus in order to use the result of case $s = 4$, we need $\|\beta^i\| < \infty$ for $3 \le i \le 10$ (we notice that when $s = 4$, if condition (I) is modified to be $\|y\|^i \cdot B \cdot \left(1 + |x(T)|^5\right)$, then we just need $\|\beta^i\| < \infty$ for $3 \le i \le 10$).

For term $\epsilon \cdot E^{\eta^\epsilon}_{0,x_0}$, we will apply the case $s = 3$ so that we can get $o(\epsilon^{3/2})$. When we check condition (i) of Theorem 2.1 for the third derivative, we will meet a term $x^3(t) f^{(4)}$, and condition (I) of Theorem 5.1 implies $|x^3(t) f^{(4)}| \le \text{const}_1 + \text{const}_2|x(t)|^6$. In order to make the derivative satisfy condition (i) of Theorem 2.1, $\|\beta^i\| < \infty$ for $3 \le i \le 10$ is enough.

In a word, for $s = 5$, we need F to be nine times differentiable and $\|\beta^i\| < \infty$ for $3 \le i \le 10$.

For general $s \ge 6$, we notice that the order of differentiability of F will increase by 3 each time, and finiteness of $\|\beta^i\|, 3 \le i \le 3(s-2)+1$ will be required. Thus the proof can be done by writing more and more terms in Taylor's formulas. □

For future use in the theory of large deviations, we state an extension of Theorem 5.1 as follows. The proof copies that of Theorem 5.1 by using Theorem 4.1 instead of Theorem 2.1 everywhere.

Theorem 5.2. *Under conditions of Theorem 5.1, but with (I) replaced by:*
(I)′ *there is a constant $C > 0$ such that for all $x[0,T], y[0,T] \in D[0,T]$*

$$|F(x[0,T])| \le C\left(1 + |x(T)|^s + \int_0^T |x(t)|^s dt\right);$$

$$\left|F^{(i)}(x[0,T])(y[0,T]^{\otimes^i})\right| \le (1 + \|y\|^i)C\left(1 + |x(T)|^{s-2} + \int_0^T |x(t)|^{s-2} dt\right)$$

for $i = 1, 2, \ldots, 3(s-2)$, the following holds: as $\epsilon \downarrow 0$.

$$E^{\eta^\epsilon}_{0,x_0} F(\eta^\epsilon[0,T]) = E^{\eta^0}_{0,x_0} F(\eta^0[0,T]) + \sum_{i=1}^{s-2} \epsilon^{\frac{i}{2}} E^{\eta^0}_{0,x_0} A_i F(\eta^0[0,T]) + o(\epsilon^{\frac{s-2}{2}}),$$

where $A_1, A_2, \ldots, A_{s-2}$ are same as in Theorem 5.1.

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Appendix. Compensating operators after transformations

In this section, we give the details on deriving compensating operators for stochastic processes defined through linear transformations of some processes. Recall that the family of locally infinitely divisible processes $(\xi^\epsilon(t), P^{\xi^\epsilon}_{s,x})$ in Section 1.1 has generating operators $A^{\xi^\epsilon}_t$ given by

$$A^{\xi^\epsilon}_t f(x) = \alpha^1(t,x)f'(x) + \frac{\epsilon}{2}a(t,x)f''(x) + \frac{1}{\epsilon}\int \left[f(x+\epsilon u) - f(x) - \epsilon u f'(x)\right]\mu_{t,x}(du)$$

for twice continuously differentiable functions f that are bounded together with their first and second derivatives. It was mentioned that $\eta^\epsilon(t) = \epsilon^{-1/2}(\xi^\epsilon(t) - x_*(t))$ converges as $\epsilon \to 0$ in distribution to a Gaussian diffusion process η^0 with generating operator

$$A^{\eta^0}_t f(x) = \alpha^1_2(t, x_*(t)) \cdot x \cdot f'(x) + \frac{1}{2}\alpha^2(t, x_*(t)) \cdot f''(x)$$

where the subscript $_2$ means differentiation in second spatial argument. Here x_* is the unique solution of ordinary differential equation $x'_*(t) = \alpha^1(t, x_*(t))$. We will show here how to get the generating operators $A^{\eta^\epsilon}_t$ of processes η^ϵ and that $\lim_{\epsilon\downarrow 0} A^{\eta^\epsilon}_t f(x) = A^{\eta^0}_t f(x)$ for some class of functions f under suitable conditions on η^ϵ.

The compensating operators $\mathfrak{A}^{\xi^\epsilon}$ of processes ξ^ϵ are defined as $\mathfrak{A}^{\xi^\epsilon} f(t,x) = \partial f(t,x)/\partial t + A^{\xi^\epsilon}_t f(t,\cdot)(x)$. To find the generating operators $A^{\eta^\epsilon}_t$, we assume the initial position of process ξ^ϵ is $\xi^\epsilon_s = x$. From definitions of compensating operator and generating operator, it follows:

$$P^{s,t}_{\xi^\epsilon} f(x) - f(x) = \int_s^t P^{s,v}_{\xi^\epsilon} A^{\xi^\epsilon}_v f(x)dv,$$

$$P^{s,t}_{\xi^\epsilon} f(t,\cdot)(x) - f(s,x) = \int_s^t P^{s,v}_{\xi^\epsilon}\mathfrak{A}^{\xi^\epsilon} f(v,\cdot)(x)dv,$$

for suitable $f(x)$ and $f(t,x)$, where the family $P^{s,t}_{\xi^\epsilon}$ is the multiplicative family of linear operators of Markov processes ξ^ϵ defined by $P^{s,t}_{\xi^\epsilon} f(x) = E^{\xi^\epsilon}_{s,x} f(\xi^\epsilon(t))$. Term $P^{s,t}_{\xi^\epsilon} f(t,\cdot)(x)$ means that $P^{s,t}_{\xi^\epsilon}$ is applied to function $f(t,x)$ in its section argument x, and $P^{s,v}_{\xi^\epsilon}\mathfrak{A}^{\xi^\epsilon} f(v,\cdot)(x)$ means that $P^{s,v}_{\xi^\epsilon}$ is applied to function $h(v,x) := \mathfrak{A}^{\xi^\epsilon} f(v,x)$ in its second argument x. We derive two connections between multiplicative families of ξ^ϵ and η^ϵ as follows:

$$\begin{aligned} P^{s,t}_{\xi^\epsilon} f(x) &= E^{\xi^\epsilon}_{s,x} f(\xi^\epsilon(t)) = E^{\eta^\epsilon}_{s,\epsilon^{-1/2}(x-x_*(s))} f\left(\eta^\epsilon(t)\epsilon^{1/2} + x_*(t)\right) \\ &= P^{s,t}_{\eta^\epsilon} F(t,\cdot)\left(\epsilon^{-1/2}(x - x_*(s))\right), \qquad F(t,x) = f\left(x\epsilon^{1/2} + x_*(t)\right), \end{aligned}$$

$$\begin{aligned} P^{s,t}_{\eta^\epsilon} f(x) &= E^{\eta^\epsilon}_{s,x} f(\eta^\epsilon(t)) = E^{\xi^\epsilon}_{s,x\epsilon^{1/2}+x_*(s)} f\left(\epsilon^{-1/2}(\xi^\epsilon(t) - x_*(t))\right) \\ &= P^{s,t}_{\xi^\epsilon} G(t,\cdot)\left(x\epsilon^{1/2} + x_*(s)\right), \qquad G(t,x) = f\left(\epsilon^{-1/2}(x - x_*(t))\right). \end{aligned}$$

The generating operator $A^{\eta^\epsilon}_t$ can be found in the following way.

$$\begin{aligned} P^{s,t}_{\eta^\epsilon} f(x) - f(x) &= P^{s,t}_{\xi^\epsilon} G(t,\cdot)\left(x\epsilon^{1/2} + x_*(s)\right) - G\left(s, x\epsilon^{1/2} + x_*(s)\right) \\ &= \int_s^t P^{s,v}_{\xi^\epsilon}\mathfrak{A}^{\xi^\epsilon} G(v,\cdot)\left(x\epsilon^{1/2} + x_*(s)\right)dv \\ &= \int_s^t P^{s,v}_{\xi^\epsilon} g(v,\cdot)\left(x\epsilon^{1/2} + x_*(s)\right)dv, \end{aligned}$$

$$\text{set } g(v,x) = \mathfrak{A}^{\xi^\epsilon} G(v,\cdot)(x) = \int_s^t P_{\eta^\epsilon}^{s,v} \widetilde{g}(v,\cdot)(x)dv, \quad \text{where } \widetilde{g}(v,x) = g\left(v, x\epsilon^{1/2} + x_*(v)\right).$$

It can be easily found that

$$\begin{aligned} g(v,x) &= \epsilon^{-1/2} f'\left(\epsilon^{-1/2}(x - x_*(v))\right)\left(\alpha^1(v,x) - x_*'(v)\right) + \frac{1}{2}a(v,x)f'' \\ &\times \left(\epsilon^{-1/2}(x - x_*(v))\right) \\ &+ \frac{1}{\epsilon}\int\left[f\left(\epsilon^{-1/2}(x + \epsilon u - x_*(v))\right) - f\left(\epsilon^{-1/2}(x - x_*(v))\right)\right. \\ &- \left.\epsilon^{1/2} u f'\left(\epsilon^{-1/2}(x - x_*(v))\right)\right]\mu_{v,x}(du) \end{aligned}$$

from which we get

$$\begin{aligned} A_t^{\eta^\epsilon} f(x) &= \widetilde{g}(t,x) = \epsilon^{-1/2} f'(x)\left(\alpha^1(t, x\epsilon^{1/2} + x_*(t)) - \alpha^1(t, x_*(t))\right) \\ &+ \frac{1}{2}a(t, x\epsilon^{1/2} + x_*(t))f''(x) \\ &+ \frac{1}{\epsilon}\int\left[f\left(x + \epsilon^{1/2}u\right) - f(x) - \epsilon^{1/2}uf'(x)\right]\mu_{t,x\epsilon^{1/2}+x_*(t)}(du). \end{aligned}$$

Now we take the limit $\epsilon \to 0$ to have

$$\lim_{\epsilon\to 0} A_t^{\eta^\epsilon} f(x) = \alpha_2^1(t, x_*(t)) \cdot x \cdot f'(x) + \frac{1}{2}\alpha^2(t, x_*(t)) \cdot f''(x)$$

for all $f(x)$ that are bounded and continuous together with their first and second derivatives (suitable conditions on η^ϵ should be imposed). And this limit is the generating operator for some Gaussian process η^0.

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