An existence theorem for some semilinear elliptic systems

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Abstract

An existence theorem is obtained for a class of semilinear, second order, uniformly elliptic systems obtained formally from a variational principle and modeled on nonlinear Helmholtz systems. Superlinear growth of the nonlinear term precludes application of standard methods to these systems. Indeed, we permit very rapid growth of the nonlinear term, so the underlying functional is not defined on the Hilbert space within which a solution is naturally sought. Mollification of the nonlinear term nonetheless results in the resulting functional satisfying the Palais–Smale condition; critical points are determined by solution of a dynamical system. The limit of vanishing mollification then produces a weak solution of the original problem.

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1. Main theorem

We consider uniformly elliptic, second order, semilinear systems of the form

\[ \nabla \cdot (A \nabla u) + \psi(u, \cdot) = 0, \quad x \in \Omega \subset \mathbb{R}^n, \]

where \( \Omega \) is open and bounded, \( u(x), \psi(u(x), x) \) are valued in \( \mathbb{R}^m \) and \( A(x) \) is a symmetric \( m \times m \) matrix, piecewise continuous, uniformly bounded and uniformly positive definite with respect to \( x \in \Omega \).

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We assume a piecewise smooth boundary
\[ \partial \Omega = \partial \Omega_D \cup \partial \Omega_B, \] (1.2)
and self-adjoint boundary conditions of the form
\[ \nu \cdot \nabla u + A^{-1} Bu = 0, \quad x \in \partial \Omega_B, \] (1.3)
\[ u = 0, \quad x \in \partial \Omega_D, \] (1.4)
with \( B(x) \) a symmetric, nonnegative \( m \times m \) matrix for \( x \in \partial \Omega_B \). Pure Neumann boundary conditions are specifically excluded, requiring that either \( \partial \Omega_D \) is of nonzero measure or else that \( B \) is positive definite on a set of nonzero measure in \( \partial \Omega_B \). With such boundary conditions, the eigenvalues of the Laplace operator on \( \Omega \) are strictly negative, and
\[ \| v \| = \left[ \int_{\Omega} \nabla v \cdot A \nabla v + \int_{\partial \Omega_B} v \cdot B v \right]^{1/2} \] (1.5)
is an equivalent norm for the space
\[ X \subseteq (H^1(\Omega))^m \]
the completion in \( H^1 \) of smooth, \( m \)-vector valued functions satisfying (1.3), (1.4). Below we use
\[ X^* \subseteq (H^{-1}(\Omega))^m \]
the dual space of \( X \), with the induced norm; dots denote finite-dimensional \( l_2 \) inner products and \( | \cdot | \) finite Euclidean and matrix norms.

Motivated by an application in nonlinear optics, where \( m = 2, n = 2 \) or 3, and \( u \) describes the intensity of radiation in a nonlinear Kerr medium [5,11], we make the following assumptions on the nonlinear term \( \psi \):
\[ \psi(u, \cdot) = \Psi_u(u, \cdot) \] (1.6)
where \( \psi, \Psi \) are continuous in \( u \), piecewise continuous and uniformly bounded with respect to \( x \), and \( \psi_u = \Psi_{uu} \) is of class \( C^1 \) in \( u, x \), and uniformly bounded with respect to \( x \),
\[ \liminf_{|\xi| \to \infty, x \in \Omega} \Psi(\xi, x) = +\infty, \] (1.7)
\[ \liminf_{|\xi| \to \infty, x \in \Omega} \frac{\xi \cdot \psi(\xi, x)}{|\xi||\psi(\xi, x)|} = c_1 > 0, \] (1.8)
\[ \Psi(\xi, x) \leq \alpha \xi \cdot \psi(\xi, x) + c_0, \] (1.9)
for all \( \xi \in \mathbb{R}^m, x \in \Omega \), with
\[ \alpha < \frac{1}{2}. \] (1.10)
Without loss of generality, we assume $\alpha > 0$ and $c_0$ nonnegative in (1.9). Finally, we assume for all $\xi \in \mathbb{R}^m, x \in \Omega$

$$|\xi| \psi(\xi, x) + |\xi|^2 \psi_u(\xi, x) \leq c \left( 1 + g \ell b |\Psi(\xi, y)| + g \ell b \frac{\Psi(\xi, y)^2}{1 + |\xi|^4} \right)$$

(1.11)

with unsubscripted $c$ a generic constant in (1.11) and below.

Our principal result is the following, the proof of which begins in the following section.

**Theorem 1.1.** Under assumptions (1.6)–(1.11) there exists $u \in X$ satisfying

$$\|u\| \leq c + c \|\psi(0, \cdot)\|_{X^*},$$

(1.12)

and

$$|\psi(u, \cdot)| \in L_1(\Omega)$$

(1.13)

and

$$\int_\Omega (\nabla v \cdot A \nabla u - v \cdot \psi(u, \cdot)) + \int_{\partial \Omega_B} v \cdot Bu = 0$$

(1.14)

for all

$$v \in X \cap L_\infty(\Omega).$$

(1.15)

In (1.13), (1.14) and below, by convention $\psi(u(x), x)$ is understood as zero for all $x \in \Omega$ such that $u(x)$ is not defined. From (1.5), for $u \in X$ this happens at most on a set of measure zero in $\Omega$.

Assumptions (1.6)–(1.10) imply that $|\psi(\xi, \cdot)|$ is superlinear in $|\xi|$ for large $\xi \in \mathbb{R}^m$. In particular, from (1.7) there exists $c_2$ such that

$$\Psi(\xi, x) \geq 1 + c_0$$

for all $|\xi| \geq c_2, x \in \Omega$.

(1.16)

From (1.6), (1.9), for $\tau > 0, \xi \in \mathbb{R}^m/0, x \in \Omega$

$$\Psi(\tau \xi, x) \leq c_0 + \alpha \tau \frac{d}{d\tau} \Psi(\tau \xi, x).$$

(1.17)

From (1.16), (1.17), for $|\xi| \geq c_2, x \in \Omega$

$$\Psi(\xi, x) \geq c_0 + \left( \frac{|\xi|}{c_2} \right)^{1/\alpha}$$

(1.18)

whence using (1.9)

$$|\psi(\xi, x)| \geq \frac{|\xi|^{(1-\alpha)/\alpha}}{\alpha c_2^{1/\alpha}}$$

(1.19)

and the superlinear growth of $|\psi|$ follows from (1.10).
In this context, the present result is distinguished from several others [1,2,4,6,7,12,15,16] by the mild restriction (1.11) on the growth of $|\psi(\xi, \cdot)|$ and $|\psi_u(\xi, \cdot)|$ for large $|\xi|$, and on the lack of restrictions on the form of $\Omega$ and of the matrix $A$.

Of particular interest below and in the application to nonlinear optics is the case where $\psi$ is of polynomial growth with respect to $u$,

$$\psi(u, \cdot) = c|u|^{2N}u + \text{lower order terms in } u$$  \hspace{1cm} (1.20)

with $N > 0$. In this case (1.10) holds with

$$\alpha = \frac{1}{2N + 2}.$$  \hspace{1cm} (1.21)

The regularity of weak solutions of (1.1) then depends on the values of $n, N$. This has been of particular concern in the application to nonlinear optics; see [9] and references therein. This is clarified by the following, which is also used in Section 4 in a slightly more general context.

**Lemma 1.2.** Assume $\psi$ of the form (1.20), and

$$N < \begin{cases} \infty, & n \leq 2, \\ \frac{2}{n-2}, & n \geq 3. \end{cases}$$  \hspace{1cm} (1.22)

Assume $u \in X$ such that

$$\nabla \cdot (A\nabla u) + \psi(u, \cdot) \in L_\infty(\Omega);$$  \hspace{1cm} (1.23)

then $u \in C^1(\Omega)$ and $\psi(u, \cdot) \in L_\infty(\Omega)$.

**Remark.** The case that $u$ satisfies (1.14), (1.15) is included in condition (1.23). The results of [1,6,7] are for $\psi$ of the form (1.20), and with a stronger assumption than (1.22). They nonetheless show that an infinite, unbounded (in $X$) set of solutions of (1.1) is possible. This conclusion is also obtained by considering (1.1) with $n = 1$. When (1.22) fails, the analysis of [8] shows that the solution set may become uncountable.

**Proof of Lemma 1.2.** Under condition (1.22), a function $u \in X$ satisfying (1.23) corresponds to a fixed point of a compact mapping on the space $X$. The familiar “bootstrap” technique, using Sobolev estimates, the standard $L_p$ estimates for second order elliptic systems, and the assumed boundedness of $\psi$ with respect to $x$, shows that any such $u \in W^{2,p}$ for any finite $p$. By application of the Sobolev lemma, $u$ is of class $C^1$; an $L_\infty$ bound on $\psi(u, \cdot)$ then follows from the assumption (1.6). □

2. Mollification

The first step in the proof of Theorem 1.1 is to show that it suffices to consider the case where (1.20) and (1.22) hold, in exchange for proving an additional estimate on the solution obtained. This is achieved by mollification of the nonlinear term in (1.1) and passing to the limit.
Adjusting the constants $c_0$ in (1.9) and $c$ in (1.11) as necessary, it is no loss of generality to assume
\[
\Psi(0, \cdot) = 0. \tag{2.1}
\]

Using (1.10), we fix $N$ so that (1.22) holds and in addition
\[
0 < N < \frac{1}{\alpha} - 2, \quad N \leq 1. \tag{2.2}
\]

For $\varepsilon > 0$, $\xi \in \mathbb{R}^m$, $x \in \Omega$, denote by
\[
D(\xi, x, \varepsilon) = 1 + \varepsilon \Psi(\xi, x)/(1 + |\xi|^2)^N+1. \tag{2.3}
\]

From (2.1) and (2.3),
\[
D(0, \cdot, \cdot) = 1; \tag{2.4}
\]
restricting $\varepsilon$ to sufficiently small positive values, using (2.3) it is no loss of generality to assume
\[
D(\xi, x, \varepsilon) \geq \frac{1}{2}. \tag{2.5}
\]

Denoting by
\[
\Psi_{\varepsilon}(\xi, x) = \frac{\Psi(\xi, x)}{D(\xi, x, \varepsilon)} - \frac{\varepsilon}{2} |\xi|^2 z(x) \tag{2.6}
\]
we compute
\[
\psi_{\varepsilon}(\xi, x) \overset{\text{def}}{=} \frac{\partial}{\partial \xi} \Psi_{\varepsilon}(\xi, x) = \frac{1}{D(\xi, x, \varepsilon)^2} \left[ \psi(\xi, x) + \frac{2\varepsilon(N + 1)\Psi(\xi, x)^2 \xi}{(1 + |\xi|^2)^N+2} \right] - \varepsilon \xi z(x). \tag{2.7}
\]

In (2.6), (2.7), the function $z$ is nonnegative and uniformly bounded, but has jump discontinuities in the interior of $\Omega$, as described in the proof of Lemma 4.2 below. This term is otherwise completely unimportant.

From (2.3) and (2.5)–(2.7), it is clear that $\Psi_{\varepsilon}$, $\psi_{\varepsilon}$ satisfy (1.6)–(1.8) for any positive $\varepsilon$, with the same constant $c_1$ in (1.8). Furthermore, from (2.4) and (2.7)
\[
\psi_{\varepsilon}(0, \cdot) = \psi(0, \cdot). \tag{2.8}
\]

Setting
\[
\tilde{\alpha} = \frac{1}{N + 2} \tag{2.9}
\]
we have from (1.10), (2.2) and (2.9)
\[
\alpha < \tilde{\alpha} < \frac{1}{2}. \tag{2.10}
\]

We next show that $\Psi_{\varepsilon}$, $\psi_{\varepsilon}$ satisfy the equivalent of (1.9).
Lemma 2.1. There exists a constant \( \tilde{c}_0 \) independent of \( \varepsilon \) such that for all \( \xi \in \mathbb{R}^m, x \in \Omega, \varepsilon > 0 \)

\[
\Psi_\varepsilon (\xi, x) \leq \tilde{a} \xi \cdot \psi_\varepsilon (\xi, x) + \tilde{c}_0.
\] (2.11)

Proof. Using (1.6), (2.3) and (2.5)–(2.7), it will suffice to prove (2.11) in the case

\[
|\xi|^2 \geq \frac{N + 2}{N}.
\] (2.12)

From (2.7), using (2.12) and then (1.9), we obtain

\[
\xi \cdot \psi_\varepsilon (\xi, x) = \frac{1}{D(\xi, x, \varepsilon)} \left[ \xi \cdot \psi(\xi, x) + \frac{2\varepsilon(N+1)\Psi(\xi, x)\xi^2|\xi|^2}{(1+|\xi|^2)^{N+2}} \right] - \varepsilon|\xi|^2 z(x)
\]

\[
\geq \frac{1}{D(\xi, x, \varepsilon)^2} \left[ \xi \cdot \psi(\xi, x) + \frac{\varepsilon(N+2)\Psi(\xi, x)^2}{(1+|\xi|^2)^{N+1}} \right] - \varepsilon|\xi|^2 z(x)
\]

(2.13)

In (2.13), we use (2.9), (2.10), then (2.3), (2.5) and (2.6) to obtain

\[
\xi \cdot \psi_\varepsilon (\xi, x) \geq \frac{1}{D(\xi, x, \varepsilon)^2} \left[ \frac{\Psi(\xi, x)}{\tilde{a}} - \frac{c_0}{\alpha} + \frac{\varepsilon \Psi(\xi, x)^2}{\tilde{a}(1+|\xi|^2)^{N+1}} \right] - \varepsilon|\xi|^2 z(x)
\]

\[
= \frac{1}{\tilde{a}} \frac{\Psi(\xi, x)}{D(\xi, x, \varepsilon)} - \frac{c_0}{\alpha} \frac{\Psi(\xi, x)^2}{D(\xi, x, \varepsilon)^2} - \varepsilon|\xi|^2 z(x)
\]

\[
\geq \frac{1}{\tilde{a}} \frac{\Psi_\varepsilon (\xi, x)}{\alpha} - \frac{4c_0}{\alpha} + \varepsilon|\xi|^2 \left( \frac{1}{2\tilde{a}} - 1 \right) - \varepsilon|\xi|^2 z(x).
\] (2.14)

Now as \( z \) is nonnegative, using (2.10) it follows that (2.11) is immediate from (2.14).

From (2.7), using (2.3) and (2.5), we have

\[
\psi_\varepsilon (\xi, x) \to \psi(\xi, x) \quad \text{as} \ \varepsilon \downarrow 0
\] (2.15)

pointwise with respect to \( \xi \in \mathbb{R}^m \), uniformly with respect to \( x \in \Omega \).

Next from (2.6) and (2.7), using (1.11), (2.2), (2.3), and (2.5), elementary computations which we omit in the interest of brevity give

\[
|\Psi_\varepsilon (\xi, \cdot)| \leq c(\varepsilon)(1 + |\xi|^{2N+2}),
\] (2.16)

\[
|\psi_\varepsilon (\xi, \cdot)| \leq c(\varepsilon)(1 + |\xi|^{2N+1}),
\] (2.17)

\[
|\psi_{\varepsilon, u}(\xi, \cdot)| \leq c(\varepsilon)(1 + |\xi|^{2N}).
\] (2.18)

In (2.16)–(2.18), the constants \( c(\varepsilon) \) depend on \( \varepsilon \) but are independent of \( \xi \in \mathbb{R}^m \) and \( x \in \Omega \). From (2.7) and (2.17), for each positive \( \varepsilon \) we have \( \psi_\varepsilon \) satisfying (1.20) and (1.22). Using (1.22)
with (2.16)–(2.18), respectively, familiar Sobolev estimates imply the following:

\begin{align}
\Psi_\varepsilon(u, \cdot) & \quad \text{is a continuous map of } X \to L^1(\Omega), \\
\psi_\varepsilon(u, \cdot) & \quad \text{is a compact map of } X \to X^*, \\
\psi_\varepsilon(u, \cdot) & \quad \text{is a compact map of } X \times X \to X^*.
\end{align}

(2.19) \quad (2.20) \quad (2.21)

Next from (1.18), (1.19), (2.5), (2.6), (2.11), we have lower bounds

\[\Psi_\varepsilon(\xi, x), \xi \cdot \psi_\varepsilon(\xi, x) \geq -c\]

(2.22)

uniformly with respect to \(\varepsilon > 0, \xi \in \mathbb{R}^m, x \in \Omega\).

Subsequent sections are devoted to proving the existence of \(u_\varepsilon \in X\) satisfying

\[\int_\Omega (\nabla v \cdot A\nabla u_\varepsilon - v \cdot \psi_\varepsilon(u_\varepsilon, \cdot)) + \int_{\partial \Omega} v \cdot B u_\varepsilon = 0 \quad \text{for all } v \in X,\]

(2.23)

\[\|u_\varepsilon\| \leq c + c\|\psi(0, \cdot)\|_{X^*} \quad \text{and}\]

(2.24)

\[\int_\Omega u_\varepsilon \cdot \psi_\varepsilon(u_\varepsilon, \cdot) \leq c\]

(2.25)

for each \(\varepsilon > 0\), with constants \(c\) in (2.24), (2.25) independent of \(\varepsilon\). The proof of Theorem 1.1 then follows from the following:

**Lemma 2.2.** Assume the existence of \(u_\varepsilon\) satisfying (2.23)–(2.25) for each \(\varepsilon > 0\). Then there exists \(u \in X\) satisfying (1.12)–(1.15).

**Proof.** From (2.25), using (2.22) and (1.8),

\[\int_\Omega |u_\varepsilon| \psi_\varepsilon(u_\varepsilon, \cdot) | \leq c,\]

(2.26)

so

\[\int_\Omega \psi_\varepsilon(u_\varepsilon, \cdot) | \leq c\]

(2.27)

with constants in (2.26), (2.27) independent of \(\varepsilon\).

Extracting a subsequence of \(\{u_\varepsilon\}\) as necessary, as \(\varepsilon \downarrow 0\) we have \(u \in X\) such that

\[u_\varepsilon \rightharpoonup u \quad \text{weakly in } X,\]

(2.28)

\[u_\varepsilon \to u \quad \text{strongly in } L^2(\Omega), \quad \text{and}\]

(2.29)

\[\psi_\varepsilon(u_\varepsilon, \cdot) \rightharpoonup \zeta\]

(2.30)

weakly in the space of measures on \(\Omega\). From (2.24) and (2.28), the limit \(u\) satisfies (1.12).
We next claim that the weak limit $\zeta$ in (2.30) is actually continuous with respect to Lebesgue measure on $\Omega$. Otherwise, there exists $\kappa > 0$ such that for any $\eta > 0$ there exists $\Omega_\eta \subset \Omega$ such that

$$\text{measure } \Omega_\eta = \eta \quad \text{and}$$

$$\int_{\Omega_\eta} |\zeta| \geq \kappa. \quad (2.31)$$

From (2.30), condition (2.32) would require that

$$\limsup_{\varepsilon \downarrow 0} \int_{\Omega_\eta} \psi_\varepsilon(u_\varepsilon, \cdot) \geq \kappa. \quad (2.33)$$

However, from (2.26) and (2.31), for any large positive $L$,

$$\int_{\Omega_\eta} |\psi_\varepsilon(u_\varepsilon, \cdot)| = \int_{\Omega_\eta \cap \{|u_\varepsilon| \leq L\}} |\psi_\varepsilon(u_\varepsilon, \cdot)| + \int_{\Omega_\eta \cap \{|u_\varepsilon| > L\}} |\psi_\varepsilon(u_\varepsilon, \cdot)|$$

$$\leq c(L) \text{ measure } \Omega_\eta + \frac{1}{L} \int_{\Omega_\eta} \|u_\varepsilon\| \psi_\varepsilon(u_\varepsilon, \cdot) \leq \eta c(L) + c/L. \quad (2.34)$$

Choosing $L$ sufficiently large and then $\eta$ sufficiently small, (2.34) becomes incompatible with (2.33), proving the claim.

Thus from (2.30) and (2.27),

$$\zeta \in L^1(\Omega). \quad (2.35)$$

From (2.29) and (2.15), as $\varepsilon \downarrow 0$,

$$\psi_\varepsilon(u_\varepsilon, \cdot) \to \psi(u, \cdot) \quad (2.36)$$

almost everywhere in $\Omega$. In particular, from (2.30) and (2.36),

$$\zeta = \psi(u, \cdot) \quad (2.37)$$

except possibly on a set of measure zero in $\Omega$.

By convention, $\psi(u, \cdot)$ vanishes where $u$ is not defined; thus from (2.35), the identification (2.37) holds in $L^1(\Omega)$. This proves (1.13).

Next we claim that as $\varepsilon \downarrow 0$,

$$\psi_\varepsilon(u_\varepsilon, \cdot) \to \psi(u, \cdot) \quad \text{strongly in } L^1(\Omega). \quad (2.38)$$

For any positive $L$ such that

$$\text{measure}\{x \in \Omega: |u(x)| = L\} = 0 \quad (2.39)$$
we have
\[
\int_\Omega \left| \psi_\varepsilon(u_\varepsilon, \cdot) - \psi(u, \cdot) \right| \\
\leq \int_{|u_\varepsilon| \leq L, |u| \leq L} \left| \psi_\varepsilon(u_\varepsilon, \cdot) - \psi(u, \cdot) \right| + \int_{|u_\varepsilon| \leq M, |u| > L} \left| \psi_\varepsilon(u_\varepsilon, \cdot) \right| + \int_{M < |u_\varepsilon| \leq L, |u| > L} \left| \psi_\varepsilon(u_\varepsilon, \cdot) \right| \\
+ \int_{|u_\varepsilon| > L} \left| \psi_\varepsilon(u_\varepsilon, \cdot) \right| + \int_{|u_\varepsilon| > L, |u| \leq L} \left| \psi(u, \cdot) \right| + \int_{|u| > L} \left| \psi(u, \cdot) \right|. \\
(2.40)
\]

As \( L \to \infty \), the final right-hand term in (2.40) is \( o(1) \) from (1.13), and the fourth term is \( O(1/L) \) from (2.26). Using (1.13), the next to last term is majorized by
\[
c(L) \text{ measure} \{ x \in \Omega: |u(x)| \leq L < |u_\varepsilon(x)| \},
\]
which has zero limit as \( \varepsilon \downarrow 0 \) for any fixed \( L \) satisfying (2.39), using (2.29).

From (2.36) it follows that for any fixed \( L \), the first term in (2.40) vanishes in the limit \( \varepsilon \downarrow 0 \).

The third right-hand term of (2.40) is \( O(1/M) \) from (2.26). We shall choose \( M \) depending on \( L \) such that \( M \to \infty \) as \( L \to \infty \), and such that the second right-hand term of (2.40), which is majorized by
\[
c(M) \text{ measure} \{ x \in \Omega: |u(x)| > L \}
\]
has zero limit as \( L \to \infty \).

Thus choosing \( L \) sufficiently large, \( M \) depending on \( L \) as described, and then \( \varepsilon \) sufficiently small, the right side of (2.40) is made arbitrarily small. This proves the claim (2.38).

Now (1.14), (1.15) follow easily from (2.23), using (2.28) and (2.38). \( \square \)

3. Preliminary estimates

Henceforth we simplify the notation by dropping the \( \varepsilon \)-subscripts and the explicit \( x \)-dependence from \( u, \psi, \Psi \), wherever no ambiguity arises. In particular, all generic constants \( c \) are independent of \( \varepsilon \). Comparing (1.9) with (2.11) and (1.10) with (2.10), no ambiguity arises from dropping the tilde on \( \alpha \) and \( c_0 \). Finally, only transparent changes in notation result from replacing \( A \) throughout by the identity matrix.

Given \( \psi, \Psi \) satisfying (1.6)–(1.10) and (2.16)–(2.21), we seek \( u \in X \) satisfying (2.23)–(2.25) as a critical point of a functional of class \( C^2 \) on \( X \) \([3,4,10,13–17]\)
\[
I(u) = \int_\Omega \left( \frac{1}{2} |\nabla u|^2 - \Psi(u) \right) + \frac{1}{2} \int_{\partial\Omega_B} u \cdot Bu. \\
(3.1)
\]

It follows from (2.19) that \( I \) is defined on all of \( X \). Using (2.20), it can be shown that \( I \) satisfies the Palais–Smale condition \([16]\). While the superlinear growth of \( \psi \) precludes application of methods such as the saddle point lemma to this problem, the proof that \( I \) satisfies the Palais–Smale condition provides important estimates on solutions.
Differentiating (3.1), using (1.6) and a partial integration using (1.2)–(1.4) we obtain for all \(u, v \in X\)

\[
I'(u)v = \int_{\Omega} (\nabla u \cdot \nabla v - v \psi(u)) + \int_{\partial \Omega_B} v \cdot B u = -\int_{\Omega} v \cdot \rho(u) \tag{3.2}
\]

with the left side of (1.1) appearing as the “residual” \(\rho(u) \in X^*\) given by

\[
\rho(u) = \Delta u + \psi(u). \tag{3.3}
\]

Combining (3.1) and (3.2) with \(v = u\), we find

\[
I(u) - \left(\frac{\alpha}{2} + \frac{1}{4}\right) I'(u) u = \left(\frac{1 - 2\alpha}{4}\right) \left(\int_{\Omega} |\nabla u|^2 + \int_{\partial \Omega_B} u \cdot B u\right) + \left(\frac{\alpha}{2} + \frac{1}{4}\right) \int_{\Omega} u \cdot \psi(u) - \int_{\Omega} \Psi(u) \geq \left(\frac{1 - 2\alpha}{4}\right) \left(\int_{\Omega} |\nabla u|^2 + \int_{\partial \Omega_B} u \cdot B u + \int_{\Omega} u \cdot \psi(u)\right) - c_0 \tag{3.4}
\]

using (1.10) in the last step.

From (3.4), using (1.5) and (2.22) we obtain

\[
\|u\|^2 + \int_{\Omega} u \cdot \psi(u) \leq c + c \left( I(u) - \left(\frac{\alpha}{2} + \frac{1}{4}\right) I'(u) u \right). \tag{3.5}
\]

Next we set

\[
\Gamma = [1, 2] \tag{3.6}
\]

and for \(\gamma \in \Gamma\), denote a continuous, symmetric map \(P_\gamma : X^* \to X\) by

\[
P_\gamma = \left(1 - \frac{\Delta}{\gamma}\right)^{-1} \tag{3.7}
\]

the inverse in (3.7) determined with the boundary conditions (1.2)–(1.4).

From the assumptions on the boundary conditions, as described in Section 1, and the form of \(P_\gamma\), it follows that for \(g \in X^* \setminus 0\),

\[
\int_{\Omega} g \cdot P_\gamma g
\]

is a strictly increasing function of \(\gamma\), and for \(g, h \in X^*\),
\[ \int_{\Omega} g \cdot P_{\gamma} h \]

is real analytic in \( \gamma \) for \( \gamma > 0 \). For \( u \in X, \gamma \in \Gamma \), denote by

\[ w = w(u, \gamma) = \frac{1}{\gamma} P_{\gamma} \rho(u); \tag{3.8} \]

from (3.7) and (3.8), it follows that \( w \in X \) and satisfies

\[ -\Delta w + \gamma w = \rho(u). \tag{3.9} \]

From (3.8), (3.3) and (2.20), the map \( u, \gamma \mapsto w(u, \gamma) \) is a continuous map of \( X \times \Gamma \rightarrow X \). From (3.2), (3.3), (3.8) and (3.9), for any \( u \in X \) a partial integration gives

\[
-I'(u)u = \int_{\Omega} u \cdot (-\Delta w + \gamma w) = \int_{\Omega} \nabla u \cdot \nabla w + \int_{\partial \Omega_B} u \cdot Bw + \gamma \int_{\Omega} u \cdot w \leq \beta \|u\|^2 + \frac{c}{\beta} \|w\|^2 + c \frac{\gamma^2}{\beta} \int_{\Omega} |w|^2 \tag{3.10}
\]

for any \( \beta > 0 \). Choosing \( \beta \) sufficiently small in (3.10) and using (3.6), from (3.5) we have for any \( u \in X, \gamma \in \Gamma \),

\[
\|u\|^2 + \int_{\Omega} u \cdot \psi(u) \leq c + c J_\gamma(u) \quad \text{with} \quad J_\gamma(u) = I(u) + \frac{\gamma}{2} \|w(u, \gamma)\|^2 + \frac{\gamma^2}{2} \int_{\Omega} |w(u, \gamma)|^2. \tag{3.11}
\]

Using (1.5), (3.8), (3.9), we rewrite (3.12), using a partial integration, in the form

\[
J_\gamma(u) = I(u) + \frac{\gamma}{2} \int_{\Omega} (|\nabla w|^2 + \gamma |w|^2) + \frac{\gamma}{2} \int_{\partial \Omega_B} w \cdot Bw
\]

\[
= I(u) + \frac{\gamma}{2} \int_{\Omega} w \cdot (-\Delta w + \gamma w) = I(u) + \frac{\gamma}{2} \int_{\Omega} w \cdot \rho(u)
\]

\[
= I(u) + \frac{1}{2} \int_{\Omega} \rho(u) \cdot P_{\gamma} \rho(u). \tag{3.13}
\]

From (3.12) and (2.22) we have a lower bound

\[
\inf_{u \in X, \gamma \in \Gamma} J_\gamma(u) \geq c > -\infty. \tag{3.14}
\]
From (3.7) and (3.13), it follows that for fixed $u$, $J_\gamma(u)$ is nondecreasing with respect to $\gamma$. For fixed $\gamma \in \Gamma$, $J_\gamma$ is a functional of class $C^1$ on $X$; differentiating (3.13), for $u, v \in X$,

$$J'_\gamma(u)v = -\int_\Omega v \cdot T_\gamma(u),$$

obtained after partial integrations, with $T_\gamma(u) \in X^*$ given by

$$T_\gamma(u) = \rho(u) - \left(\Delta + \psi_\gamma(u)\right)P_\gamma \rho(u).$$

Using (3.7) and (3.16), an alternative expression for $T_\gamma$ is

$$T_\gamma(u) = (1 + \gamma)\rho(u) - \left(\gamma + \psi_\gamma(u)\right)P_\gamma \rho(u).$$

From (3.16) or (3.17), it is evident that a critical point $u$ of $J_\gamma$ for some fixed $\gamma$ need not correspond to a solution, i.e. $u$ such that $\rho(u)$ vanishes. However in Lemma 4.2 below, we shall show that if $T_\gamma(u) = 0$ for almost all $\gamma \in \Gamma$, then $u$ is a solution. To find such, we introduce moments of the $J_\gamma$ with respect to $\gamma$ and use a descent algorithm.

Denote by $Q$ the set of all nonnegative measures on $\Gamma$ of unit mass, and for $\phi \in Q$, denote by

$$K_\phi(u) = \int_\Gamma \phi J_\gamma(u)$$

$$= I(u) + \frac{1}{2} \int_\Omega \rho(u) \cdot \int_\Gamma \phi P_\gamma \rho(u).$$

Taking the moment of (3.11) with $\phi$, using (3.18) we have

$$\|u\|^2 + \int_\Omega u \cdot \psi(u) \leq c + cK_\phi(u)$$

implying a lower bound of the form

$$\inf_{u \in X, \phi \in Q} K_\phi(u) \geq c > -\infty.$$  

For fixed $\phi \in Q$, $K_\phi$ is also a functional of class $C^1$ on $X$, with derivative $K'_\phi(u) \in X^*$ obtained from (3.18) and (3.15), then (3.16)

$$K'_\phi(u)v = -\int_\Omega \int_\Gamma \phi T_\gamma(u)v$$

$$= -\int_\Omega v \cdot \left[\rho(u) - \left(\Delta + \psi_\gamma(u)\right)\int_\Gamma \phi P_\gamma \rho(u)\right].$$
Alternatively, using (3.17) in the final step we obtain

\[ K'_\phi(u)v = -\int_\Omega v \cdot \left[ \left(1 + \int_\Gamma \phi_\gamma \right) \rho(u) - \int_\Gamma \phi(\gamma + \psi_\mu(u)) P_\gamma \rho(u) \right]. \] (3.22)

4. Two lemmas

Here we present two lemmas central to the subsequent argument, but such that the proofs may be skipped by uninterested readers without liability thereafter. The first result is an essential use of the compactness conditions (2.20), (2.21).

**Lemma 4.1.** Assume a sequence \( u_j \in X, \phi_j \in Q, j = 1, \ldots, \) such that as \( j \to \infty \)

\[ K_{\phi_j}(u_j) \to K \quad \text{and} \quad K'_{\phi_j}(u_j) \to 0 \quad \text{strongly in } X^*. \] (4.1)

Then there exists a subsequence (also denoted by \( u_j, \phi_j \)) such that as \( j \to \infty \)

\[ u_j \to u \quad \text{strongly in } X \quad \text{and} \]

\[ \phi_j \to \phi \quad \text{weakly in } Q. \] (4.2)

**Proof.** From (3.19) and (2.22)

\[ \|u_j\| \leq c \] (4.7)

with \( c \) independent of \( j \). Extracting a subsequence as necessary, we have (4.4) holding simultaneously with

\[ u_j \to u \quad \text{weakly in } X. \] (4.8)

From (4.8) and (2.20), again extracting a subsequence as necessary,

\[ \psi(u_j) \to \psi(u) \quad \text{strongly in } X^*, \] (4.9)

whence from (3.3) and (4.8),

\[ \rho(u_j) \to \rho(u) \quad \text{weakly in } X^*. \] (4.10)

Thus from (3.7),

\[ P_\gamma \rho(u_j) \to P_\gamma \rho(u) \quad \text{weakly in } X, \] (4.11)

uniformly with respect to \( \gamma \in \Gamma \).
Again extracting a subsequence as necessary,

\[ P_{\gamma} \rho(u_j) \to P_{\gamma} \rho(u) \quad \text{strongly in } X^*, \quad (4.12) \]

uniformly with respect to \( \gamma \in \Gamma \).

From (4.8), (2.21) and (4.11), again extracting a subsequence as necessary and uniformly with respect to \( \gamma \),

\[ \psi_u(u_j) P_{\gamma} \rho(u_j) \to \psi_u(u) P_{\gamma} \rho(u) \quad \text{strongly in } X^*. \quad (4.13) \]

From (3.22)

\[
-K_{\phi_j}'(u_j) = \int_{\Gamma} \phi_j T_{\gamma}(u_j) \\
= \left(1 + \int_{\Gamma} \phi_j \gamma\right) \rho(u_j) - \int_{\Gamma} \phi_j \gamma P_{\gamma} \rho(u_j) - \int_{\Gamma} \phi_j \psi_u(u_j) P_{\gamma} \rho(u_j). \quad (4.14)
\]

We apply (4.2) to (4.14), using (4.10), (4.12) and (4.13), to infer

\[ \rho(u_j) \to \rho(u) \quad \text{strongly in } X^*. \quad (4.15) \]

Now (4.3) follows from (4.15), using (3.3) and (4.9).

Given (4.3), from (3.13) we have

\[ J_{\gamma}(u_j) \to J_{\gamma}(u) \quad (4.16) \]

and from (3.17)

\[ T_{\gamma}(u_j) \to T_{\gamma}(u) \quad \text{strongly in } X^*, \quad (4.17) \]

both uniformly with respect to \( \gamma \). Using (4.3) and (4.4), (4.5) follows from (4.16) and (4.6) follows from (4.17), recalling (3.21).

The following lemma is the mechanism by which we ultimately find a solution.

**Lemma 4.2.** Assume \( u \in X, \omega \in Q \), such that

\[ \omega \neq \delta(\gamma - 1) \quad \text{or} \quad \delta(\gamma - 2) \quad \text{and} \quad (4.18) \]

\[ K_{\phi}'(u) = 0 \quad (4.19) \]

for every \( \phi \in Q \) such that

\[ K_{\phi}(u) = K_{\omega}(u); \quad (4.20) \]

then

\[ \rho(u) = 0. \quad (4.21) \]
Proof. From (3.13), condition (4.20) is equivalent to

$$\int_I \phi \int_\Omega \rho(u) \cdot P_\gamma \rho(u) = \int_I \omega \int_\Omega \rho(u) \cdot P_\gamma \rho(u).$$  \hspace{1cm} (4.22)

Assume \( u \in X \) fixed such that \( \rho(u) \) does not vanish in \( X^* \); then it follows from (3.7) that

$$\int_\Omega \rho(u) P_\gamma \rho(u)$$

is a smooth, strictly increasing function of \( \gamma \), and that \( T_\gamma(u) \) depends smoothly on \( \gamma \). Condition (4.18) assures that the set of \( \phi \) satisfying (4.20) is not trivial, whence (4.19) can only be satisfied with \( T_\gamma(u) \) of the form

$$T_\gamma(u) = f \left( 1 - \frac{1}{b} \int_\Omega \rho(u) \cdot P_\gamma \rho(u) \right) \quad \text{with}$$  \hspace{1cm} (4.23)

$$b = \int_I \omega \int_\Omega \rho(u) P_\gamma \rho(u) > 0,$$  \hspace{1cm} (4.24)

and some \( f \in X^* \) independent of \( \gamma \). In the interest of continuity, we defer the justification of (4.23) to the end of the proof.

Comparing (3.16) with (4.24) we have

$$f \left( 1 - \frac{1}{b} \int_\Omega \rho(u) \cdot P_\gamma \rho(u) \right) = \rho(u) - \left( \Delta + \psi(u) \right) P_\gamma \rho(u).$$  \hspace{1cm} (4.25)

for each \( \gamma \in \Gamma \).

However, each term in (4.25) is real analytic in \( \gamma \) for \( \gamma > 0 \), so (4.25) must remain valid in the limit \( \gamma \downarrow 0 \). Since the eigenvalues of the Laplace operator are negative definite, given the form of the boundary conditions (1.2)–(1.4), from (3.7) we readily obtain

$$\lim_{\gamma \downarrow 0} P_\gamma \rho(u) = 0.$$  \hspace{1cm} (4.26)

Using (4.26), it follows that (4.25) can only hold with \( f = \rho(u) \), and (4.25) simplifies to

$$\left( \Delta + \psi(u) \right) P_\gamma \rho(u) = \frac{\rho(u)}{b} \int_\Omega \rho(u) \cdot P_\gamma \rho(u).$$  \hspace{1cm} (4.27)

From (4.27), we have the existence of a decomposition of \( P_\gamma \rho(u) \) of the form

$$P_\gamma \rho(u) = \mu \int_\Omega \rho(u) \cdot P_\gamma \rho(u) + \sigma(\gamma) \quad \text{with}$$  \hspace{1cm} (4.28)

$$\mu \in X, \quad \sigma \in C(\Gamma \to X)$$  \hspace{1cm} (4.29)
satisfying
\[
(\Delta + \psi_u(u))\mu = \frac{\rho(u)}{b} \quad \text{and} \quad (\Delta + \psi_u(u))\sigma(\gamma) = 0. \tag{4.30}
\]

With boundary conditions of the form (1.2)–(1.4) the operator \(\Delta + \psi_u(u)\) is symmetric on \(X\), so from (4.30), using (4.31), for any \(\gamma \in \Gamma\)
\[
\int_{\Omega} \sigma(\gamma) \cdot \rho(u) = b \int_{\Omega} \sigma(\gamma) \cdot (\Delta + \psi_u(u))\mu = b \int_{\Omega} (\Delta + \psi_u(u))\sigma(\gamma) \cdot \mu = 0. \tag{4.32}
\]

Using (4.32), the inner product of (4.28) with \(\rho(u)\) gives
\[
\int_{\Omega} \rho(u) \cdot \mu = 1. \tag{4.33}
\]

We denote by
\[
Y = \text{span}\{\sigma(\gamma), \gamma \in \Gamma\} \tag{4.34}
\]
from (4.31)
\[
Y \subseteq \ker(\Delta + \psi_u(u)). \tag{4.35}
\]

By appeal to (2.21), \(Y\) is a finite-dimensional subspace of \(X\).

Without loss of generality, we remove the remaining ambiguity in \(\mu\) by requiring
\[
\int_{\Omega} \mu \cdot v = 0 \quad \text{for all} \ v \in Y. \tag{4.36}
\]

The space
\[
Z = Y \oplus \text{span}\{\mu\} \tag{4.37}
\]
is also finite-dimensional, and from (4.28) we have
\[
P_\gamma \rho(u) \in Z \quad \text{for all} \ \gamma \in \Gamma. \tag{4.38}
\]

We employ an eigenfunction expansion
\[
\rho(u) = \sum_i a_i r_i, \quad a_i \neq 0, \ r_i \in X/0, \quad \text{with} \ 
\Delta r_i = -\lambda_i r_i, \quad \int_{\Omega} r_i \cdot r_j = \delta_{ij} \quad \text{and} \ 
\lambda_i > 0. \tag{4.39}
\]
By choosing the $a_i$ appropriately, it is no loss of generality to assume
\[ \lambda_i \neq \lambda_j \quad \text{for all } i \neq j. \]  
(4.42)

From (3.7) and (4.40)
\[ P_\gamma r_i = \frac{\gamma}{\gamma + \lambda_i} r_i, \]  
(4.43)
so from (4.39)
\[ P_\gamma \rho(u) = \sum_i a_i \frac{\gamma}{\gamma + \lambda_i} r_i. \]  
(4.44)

Now (4.38) and (4.44) are compatible only if the sums in (4.39) and (4.44) are finite; then comparing (4.39) and (4.44), it follows that
\[ \rho(u) \in Z. \]  
(4.45)

Now from (4.45), using (4.32), (4.33) and (4.36), we obtain
\[ \mu = \rho(u) \left/ \int_\Omega |\rho(u)|^2 \right. \]  
(4.46)
We use (4.46) to rewrite (4.30) as
\[ \psi_u(u) \rho(u) = \left( \int_\Omega |\rho(u)|^2 \right) b \rho(u) - \Delta \rho(u). \]  
(4.47)
Knowing that the sum in (4.39) is finite, it follows that the function $\rho(u)$ is in $C^\infty(\Omega)$ and vanishes at most on a set of measure zero in the interior of $\Omega$. The same is true for the right side of (4.47).

By appeal to Lemma 1.2, $u \in C^1(\Omega)$, so from the continuity assumptions (1.6) on $\psi_u$, it follows from (2.6) that
\[ \psi_u(u) \rho(u) = W - \varepsilon z \rho(u) \]  
(4.48)
with $W$ of class $C^1$ as a function of $x$ in $\Omega$. Since $z$ has jump discontinuities in the interior of $\Omega$, however, for $\varepsilon > 0$ the right side of (4.48) cannot be of class $C^1(\Omega)$.

This incompatibility between (4.47) and (4.48) proves the lemma.

It remains to justify (4.23). Using the first line of (3.21) and (3.18), conditions (4.19), (4.20) implies
\[ \int_T \int_\Omega T_\gamma v = 0 \]  
(4.49)
for all \( v \in X \) and all \( \phi \in Q \) such that
\[
\int_{\Gamma} \phi \int_{\Omega} \rho(u) \cdot P_{\gamma} \rho(u) = \int_{\Gamma} \omega \int_{\Omega} \rho(u) \cdot P_{\gamma} \rho(u).
\] (4.50)

We expand
\[
T_{\gamma}(u) = f + f_1 \int_{\Omega} \rho(u) \cdot P_{\gamma} \rho(u) + f_2(\gamma)
\] (4.51)

with \( f, f_1, f_2(\gamma) \in X^* \), \( f_2 \) depending smoothly on \( \gamma \) and satisfying
\[
\int_{\Gamma} f_2(\gamma) = \int_{\Gamma} f_2(\gamma) \int_{\Omega} \rho(u) \cdot P_{\gamma} \rho(u) = 0.
\] (4.52)

We claim that
\[
f_2(\gamma) = 0.
\] (4.53)

From (4.18), the set \( \{ \gamma \mid \omega(\gamma) \geq \epsilon > 0 \} \) for some sufficiently small \( \epsilon \) contains either both endpoints \( \gamma = 1 \) and \( \gamma = 2 \) or else contains an interior point of \( \Gamma \). In either case, given that \( \int_{\Omega} \rho(u) \cdot P_{\gamma} \rho(u) \) is positive and increasing with respect to \( \gamma \), there exists \( \tilde{\phi} \in C[1, 2] \) satisfying
\[
\int_{\Gamma} \tilde{\phi} = \int_{\Gamma} \tilde{\phi} \int_{\Omega} \rho(u) \cdot P_{\gamma} \rho(u) = 0,
\] (4.54)
\[
\int_{\Gamma} \tilde{\phi} f_2(\gamma) \neq 0 \quad \text{(in } X^*) \quad \text{and}
\] (4.55)
\[
\{ \gamma \mid \tilde{\phi}(\gamma) < 0 \} \subseteq \{ \gamma \mid \omega(\gamma) \geq \epsilon > 0 \}
\] (4.56)

for some sufficiently small \( \epsilon \). From (4.54) and (4.56)
\[
\phi = \omega + \epsilon \tilde{\phi} \in Q
\] (4.57)
for \( |\epsilon'| > 0 \) sufficiently small. Using this \( \phi \) and \( T_{\gamma} \) obtained from (4.51) in (4.49), we obtain a contradiction from (4.50), (4.54) and (4.55).

Given (4.53), setting \( \phi = \omega \) and using (4.51) in (4.49) shows that
\[
f_1 = -f/b,
\] (4.58)

\( b \) given in (4.24). Now (4.23) follows from (4.51), using (4.53) and (4.58). \( \square \)
5. Constructive algorithm

We determine \( u(t) \in X, \phi(t) \in Q \) continuous with respect to \( t > 0 \) such that

\[
K_{\phi(t)}(u(t))
\]

is decreasing on balance, if not monotonically, as \( t \) increases. From (3.19) and (2.22), such a process must necessarily terminate. The algorithm is chosen such that the conditions of Lemma 4.2 are satisfied when this occurs.

The initial values \( u(0), \phi(0) \) are arbitrary, subject to the condition

\[
K'_{\phi(0)}(u(0)) = -\int_{\Gamma} \phi(0)T_\gamma(u(0)) \neq 0. \tag{5.1}
\]

If \( u \in X \) is such that

\[
T_\gamma(u) = 0 \quad \text{for all} \quad \gamma \in \Gamma, \tag{5.2}
\]

then the assumptions of Lemma 4.2 hold for any \( \omega \in Q \) satisfying (4.18), so \( u \) is a solution. For any \( u \in X \) not satisfying (5.2), we have \( \theta(u) \in Q \) determined from

\[
\theta(u) = \int_\Omega T_\gamma(u) \cdot P_1 T_\gamma(u). \tag{5.3}
\]

Then given \( u(0), \phi(0) \), we determine \( u(t), \phi(t) \) for \( t > 0 \) from a dynamical system

\[
\frac{du(t)}{dt} = -P_1 K'_{\phi(t)}(u(t)), \tag{5.4}
\]

\[
\frac{d\phi(t)}{dt} = \delta e^{-t} \int_0^t e^s H(s) ds \left( \theta(u(t)) - \phi(t) \right) \tag{5.5}
\]

where \( \delta > 0 \) is chosen sufficiently small below, and

\[
H(s) = \int_\Omega K'_{\phi(s)}(u(s)) \cdot P_1 K'_{\phi(s)}(u(s)). \tag{5.6}
\]

As long as (5.2) is not satisfied, existence and uniqueness of the solution of (5.3)–(5.6) follows from the Picard theorem.

From (3.18), using (5.4)–(5.6), for any \( T > 0 \),

\[
K_{\phi(T)}(u(T)) - K_{\phi(0)}(u(0)) = \int_0^T \frac{d}{dt} K_{\phi(t)}(u(t)) dt = \int_0^T \left[ \int_{\Gamma} \left( \frac{d\phi(t)}{dt} J_\gamma(u(t)) \right) + K'_{\phi(t)}(u(t)) \frac{du(t)}{dt} \right] dt.
\]
\[
\begin{align*}
&= \int_0^T H(s) \left[ \delta \int_s^\Gamma e^{s-t} \int_\Gamma \left( \theta(u(t)) - \phi(t) \right) J_y(u(t)) \, dt \right. - 1 \bigg] \, ds \\
&\leq \int_0^T H(s) \left[ 2\delta \int_s^T e^{s-t} J_y(u(t)) \, dt \right. - 1 \bigg] \, ds \leq \int_0^T H(s) \left[ 2\delta \sup_{s \leq r \leq T} J_y(u(t)) \right. - 1 \bigg] \, ds. \quad (5.7)
\end{align*}
\]

From (3.13) and (3.7),

\[J_y(u) \leq 2J_1(u) - 2I(u). \quad (5.8)\]

In (5.8), we use (3.1) and (1.9) to get an upper bound for \(-I(u)\)

\[-I(u) = -\frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{2} \int_\partial \Omega B(u) + \int_\Omega \Psi(u) \leq \alpha \int_\Omega u \cdot \psi(u) + c_0. \quad (5.9)\]

Using (3.19) and (2.22), we obtain from (5.9)

\[-I(u) \leq c + cK_\phi(u) \quad (5.10)\]

with constants \(c\) independent of \(\phi \in Q\).

Now using (5.8) and (5.10) in (5.7), we obtain

\[
K_{\phi(T)}(u(T)) - K_{\phi(0)}(u(0)) \leq \left[ c_3 \delta \left( 1 + \sup_{\tau \in [0,T]} K_{\phi(\tau)}(u(\tau)) \right) \right] \int_0^T H(s) \, ds \\
\leq -\frac{1}{2} \int_0^T H(s) \, ds \quad (5.11)
\]

for any \(\delta > 0\) satisfying

\[\delta < 1/\left[ 2c_3 \left( 1 + K_{\phi(0)}(u(0)) \right) \right]. \quad (5.12)\]

As \(T\) may be chosen arbitrarily large, assuming that (5.2) is not satisfied at some value of \(t\), it follows from (3.20) and (5.11) that there exists a sequence \(\{t_j\}\) such that as \(t_j \to \infty\),

\[H(t_j) \to 0, \quad K_{\phi(t_j)}(u(t_j)) \to K. \quad (5.13)\]

Then from (5.13) and (5.6), as \(t_j \to \infty\),

\[K'_{\phi(t_j)}(u(t_j)) \to 0 \quad \text{strongly in } X^*. \quad (5.14)\]
By application of Lemma 4.1, we have

\[ u(t_j) \to u \text{ strongly in } X, \quad (5.15) \]

\[ \phi(t_j) \rightharpoonup \omega \text{ weakly in } Q, \quad (5.16) \]

\[ K'_\omega(u) = 0 \quad \text{and} \quad (5.17) \]

\[ K_\omega(u) = K. \quad (5.18) \]

We claim that \( \omega \) necessarily satisfies (4.18). Indeed, from (5.5), at each \( t \), \( \phi \) “moves towards \( \theta \),” and thus the only way (4.18) could fail is that

\[ \theta(u(t_j)) \rightharpoonup \delta(\gamma - 1) \quad \text{or} \quad \delta(\gamma - 2) \quad (5.19) \]

as \( t_j \to \infty \). However from (5.3), for this to occur \( T_\gamma(u(t_j)) \) would have to be sharply peaked, as a function of \( \gamma \), near \( \gamma = 1 \) or \( \gamma = 2 \); from (3.17), with \( \|u(t)\| \) uniformly bounded with respect to \( t \) as implied by (5.11), (3.19), this is impossible.

Now given \( u, K, \omega \) as obtained from Lemma 4.1, we restart the algorithm with

\[ u(0) = u \quad (5.20) \]

and any \( \phi(0) \in Q \), \( \phi(0) \neq \delta(\gamma - 1) \) or \( \delta(\gamma - 2) \) such that

\[ K'_{\phi(0)}(u) \neq 0 \quad \text{and} \quad (5.21) \]

\[ K_{\phi(0)}(u) = K. \quad (5.22) \]

As \( J_\gamma(u) \) is nondecreasing with respect to \( \gamma \) and \( \omega \) satisfies (4.18), we can always choose \( \phi(0) \neq \delta(\gamma - 1) \) or \( \delta(\gamma - 2) \) satisfying (5.22). In this manner we continue to reduce

\[ K_{\phi(t)}(u(t)) \]

so it must become impossible to continue. Either (5.2) suddenly becomes satisfied, and we have a solution, or else it becomes impossible to choose \( \phi(0) \) satisfying (5.21) and (5.22). In the latter case, \( u \) is a solution by appeal to Lemma 4.2.

Thus we find a solution \( u \) with

\[ K_{\phi}(u) \leq K_{\phi(0)}(u(0)) \quad (5.23) \]

for some \( \phi \in Q \).

To obtain the estimates (2.24), (2.25), we use (5.23), (3.19), (2.22), obtaining

\[ \|u\|^2 + \left| \int_\Omega u \cdot \psi(u) \right| \leq c + cK_{\phi}(u) \leq c + cK_{\phi(0)}(u(0)). \quad (5.24) \]

Now from (3.18),

\[ K_{\phi(0)}(u(0)) \leq c + c|I(u(0))| + c\|\rho(u(0))\|_{X^*}^2 \leq c + c\|\psi(0)\|_{X^*}^2 \quad (5.25) \]
making the specific choice $u(0) = 0$, and using (2.1), (2.6), (2.8), and (3.3). The estimates (2.24), (2.25) follow from (5.24), (5.25). This completes the proof of Theorem 1.1.

References