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Global solutions and blow-up problems for a nonlinear degenerate parabolic system coupled via nonlocal sources ☆

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Abstract

This paper concerns with a nonlinear degenerate parabolic system coupled via nonlocal sources, subjecting to homogeneous Dirichlet boundary condition. The main aim of this paper is to study conditions on the global existence and/or blow-up in finite time of solutions, and give the estimates of blow-up rates of blow-up solutions.

Keywords: Degenerate parabolic system; Nonlocal sources; Global existence; Blow-up in finite time; Blow-up rates

1. Introduction and main results

The global solutions and blow-up problems for the single parabolic equation with nonlocal nonlinearities had been studied extensively, see [1,4,6,13,23,27] and the references therein. Particularly, in the paper [13], the authors established the critical Fujita exponent for the Cauchy problem

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$$\begin{cases} u_t = \Delta u^m + \left(\int\limits_{R^N} K(y) u^q(y,t)\,\mathrm{d}y \right)^{(p-1)/q} u^{r+1}, & x \in R^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in R^N, \end{cases}$$

where parameters $m, p > 1$, $q > 0$, $r \geqslant 0$, the initial data $u_0(x)$ is a bounded nonnegative function, and the kernel function K is nonnegative and measurable. In the paper [1], the authors considered the following one-dimensional problem:

$$\begin{cases} u_t = \left((u^m)_x + \varepsilon u^n \right)_x + au\|u\|_q^{p-1}, & x \in (0,1),\ t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x \in [0,1]. \end{cases}$$

They proved that the solution blows up in finite time for the large initial data $u_0(x)$ provided that $p > \max\{1, m, n\}$. In the paper [4], the authors studied the following one-dimensional problem:

$$\begin{cases} u_t - (u^m)_{xx} = a \int\limits_{-l}^{l} u^q\,\mathrm{d}x, & x \in (-l,l),\ t > 0, \\ u(-l,t) = u(l,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x \in [-l,l] \end{cases} \tag{1.1}$$

with $l, a > 0$ and $q > m > 1$. They proved that if the solution u of (1.1) blows up in finite time, then the blow-up set is the whole domain $[-l, l]$, and the following estimate holds (here T' denotes the blow-up time):

$$C_1(T'-t)^{-1/(q-1)} \leqslant \max_{x \in [-l,l]} u(x,t) \leqslant C_2(T'-t)^{-1/(q-1)}.$$

In the paper [6], they extended the problem (1.1) to the higher-dimensional case. The author of paper [23] established the uniform blow-up profiles and boundary behavior for a class of diffusion equations with nonlocal nonlinear source.

The global solutions and blow-up problems for the parabolic systems with local nonlinearities, localized nonlinearities and nonlinear boundary conditions had also been studied extensively, see [16–19,22,24–26,28,31] and the references therein.

Motivated by the above works, in this paper, we consider the following nonlinear degenerate parabolic systems coupled via nonlocal sources

$$\begin{cases} u_t = \Delta u^{\alpha_1} + au^p \int\limits_{\Omega} v^q\,\mathrm{d}x, & x \in \Omega,\ t > 0, \\ v_t = \Delta v^{\alpha_2} + bv^m \int\limits_{\Omega} u^n\,\mathrm{d}x, & x \in \Omega,\ t > 0, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where parameters $p, m \geqslant 0$, $a, b, q, n > 0$, $\alpha_1, \alpha_2 > 1$, and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Such initial-boundary value problems model a variety of physical phenomena, which arise, for example, in the study of the flow of a fluid through a homogeneous isotropic rigid porous medium or in studies of population dynamics [2,5,10–12,15,21]. And it has also been suggested that nonlocal growth terms present a more realistic model of a population [2,9].

In this paper, we always assume that the initial data u_0, v_0 satisfy the following (H1)–(H3) or (H4):

(H1) $u_0(x), v_0(x) \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$, $0 < \alpha < 1$;
(H2) $u_0(x), v_0(x) > 0$ in Ω, and $u_0(x), v_0(x) = 0$, $\frac{\partial u_0}{\partial \eta}, \frac{\partial v_0}{\partial \eta} < 0$ on $\partial\Omega$;
(H3) $\Delta u_0^{\alpha_1} + a u_0^p \int_\Omega v_0^q \,\mathrm{d}x$, $\Delta v_0^{\alpha_2} + b v_0^m \int_\Omega u_0^q \,\mathrm{d}x \geqslant 0$, $x \in \Omega$;
(H4) Let δ_0, k_1, k_2 be positive constants (will be given in Section 4), and there exists a constant $\delta > \delta_0$ such that

$$\Delta u_0^{\alpha_1} + a u_0^p \int_\Omega v_0^q \,\mathrm{d}x - \delta u_0^{k_1+1} \geqslant 0, \qquad \Delta v_0^{\alpha_2} + b v_0^m \int_\Omega u_0^n \,\mathrm{d}x - \delta v_0^{k_2+1} \geqslant 0.$$

Our tasks are to investigate the global existence, blow-up in finite time and blow-up rates for the solution of system (1.2). Using the scaling:

$$u^{\alpha_1}(x,t) = u_1(x,\tau), \quad v^{\alpha_2}(x,t) = v_1(x,\tau)(\alpha_1/\alpha_2)^{\alpha_2/(\alpha_2-1)}, \quad t = \tau/\alpha_1,$$

then (1.2) can be written as

$$\begin{cases} u_{1\tau} = u_1^{r_1}\left(\Delta u_1 + a_1 u_1^{p_1} \int_\Omega v_1^{q_1} \,\mathrm{d}x\right), & x \in \Omega,\ \tau > 0, \\ v_{1\tau} = v_1^{r_2}\left(\Delta v_1 + b_1 v_1^{m_1} \int_\Omega u_1^{n_1} \,\mathrm{d}x\right), & x \in \Omega,\ \tau > 0, \\ u_1(x,\tau) = v_1(x,\tau) = 0, & x \in \partial\Omega,\ \tau > 0, \\ u_1(x,0) = u_{01}(x), \quad v_1(x,0) = v_{01}(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $r_1 = (\alpha_1 - 1)/\alpha_1$, $p_1 = p/\alpha_1$, $q_1 = q/\alpha_2$, $a_1 = a(\alpha_1/\alpha_2)^{q/(\alpha_2-1)}$, $u_{01}(x) = u_0^{\alpha_1}$, $r_2 = (\alpha_2 - 1)/\alpha_2$, $m_1 = m/\alpha_2$, $n_1 = n/\alpha_1$, $b_1 = b(\alpha_1/\alpha_2)^{(m-\alpha_2)/(\alpha_2-1)}$, $v_{01}(x) = v_0^{\alpha_2}(\alpha_2/\alpha_1)^{\alpha_2/(\alpha_2-1)}$.

Let $D = (\alpha_1 - p)(\alpha_2 - m) - qn$, $d_1 = qn - (1-p)(1-m)$. Our main results read as follows in detail.

Theorem 1. *The nonnegative solution of* (1.2) *exists locally and is unique.*

Theorem 2. *If* $\alpha_1 > q$, $\alpha_2 > m$ *and* $D > 0$, *then the nonnegative solution of* (1.2) *exists globally.*

Theorem 3. *If* $\alpha_1 < p$, *or* $\alpha_2 < m$, *or* $D < 0$, *then the nonnegative solution of* (1.2) *exists globally for the small initial data* (u_0, v_0), *while blows up in finite time for the large initial data* (u_0, v_0).

Theorem 4. *Assume that* $D = 0$.

(1) *If* $\alpha_1 - p = q$, $\alpha_2 - m = n$, *then the nonnegative solution of* (1.2) *exists globally provided that a and b are small*;
(2) *if* $\alpha_1 - p > q$, $n > \alpha_2 - m$ *or* $q > \alpha_1 - p$, $\alpha_2 - m > n$, *then the nonnegative solution of* (1.2) *exists globally provided that both* $u_0(x)$ *and* $v_0(x)$ *are small.*

Theorem 5. *Assume that $u_0(x), v_0(x)$ satisfy* (H1)–(H4)*, and $n+1-p, q+1-m>0$. The solution (u,v) of* (1.2) *blows up in finite time T^*. Then there exist positive constants C_{iu}, C_{iv} $(i=1,2)$ such that, as $t \to T^*$,*

$$\begin{cases} C_{1u} \leqslant \left(T^*-t\right)^{(q+1-m)/d_1} \max\limits_{\bar{\Omega}} u(x,t) \leqslant C_{2u}, \\ C_{1v} \leqslant \left(T^*-t\right)^{(n+1-p)/d_1} \max\limits_{\bar{\Omega}} v(x,t) \leqslant C_{2v}. \end{cases} \tag{1.4}$$

Theorem 6. *Assume that $u_0(x), v_0(x)$ satisfy* (H1)–(H3)*, $\Delta u_0, \Delta v_0 \leqslant 0$, and $p, m<1$. The solution (u,v) of* (1.2) *blows up in finite time T^*. If* (H4) *holds, or $\frac{n}{1-p} = \frac{q}{1-m}$ and $n > p+\alpha_1$, $q > m+\alpha_2$. Then there exist positive constants C_1, C_2 such that, as $t \to T^*$,*

$$\begin{cases} C_1 \leqslant \left(T^*-t\right)^{(1-p)(q+1-m)/d_1} G_{11}(t) \leqslant C_2, \\ C_1 \leqslant \left(T^*-t\right)^{(1-m)(n+1-p)/d_1} G_{21}(t) \leqslant C_2, \end{cases} \tag{1.5}$$

where $G_{11}(t) = \int_{Q_t} v^q(x,s)\,\mathrm{d}x\,\mathrm{d}s$, $G_{21}(t) = \int_{Q_t} u^n(x,s)\,\mathrm{d}x\,\mathrm{d}s$, $Q_t = \Omega \times (0,t)$.

Theorem 7. *Under the assumptions of Theorem* 6*, if $(\alpha_1 - p)(1-m) < q(n-\alpha_1+1)$ and $(\alpha_2 - m)(1-p) < n(q-\alpha_2+1)$, then*

$$\begin{aligned} &\lim_{t \to T^*} \left(T^*-t\right)^{(q+1-m)/d_1} u(x,t) \\ &\quad = \left(\frac{a|\Omega| d_1 (\alpha_1/\alpha_2)^{q/(\alpha_2-1)}}{\alpha_1\alpha_2}\right)^{-(q+1-m)/d_1} \left(\frac{n+1-p}{\alpha_1}\right)^{q/d_1} \\ &\qquad \times \left(\frac{a(q+1-m)(\alpha_1/\alpha_2)^{(q+\alpha_2-m)/(\alpha_2-1)}}{b\alpha_2}\right)^{(1-m)/d_1}, \\ &\lim_{t \to T^*} \left(T^*-t\right)^{(n+1-p)/d_1} v(x,t) \\ &\quad = (\alpha_1/\alpha_2)^{-1/(\alpha_2-1)} \left(\frac{b|\Omega| d_1 (\alpha_1/\alpha_2)^{(m-\alpha_2)/(\alpha_2-1)}}{\alpha_1\alpha_2}\right)^{-(n+1-p)/d_1} \\ &\qquad \times \left(\frac{q+1-m}{\alpha_2}\right)^{n/d_1} \left(\frac{b(n+1-p)(\alpha_1/\alpha_2)^{(m-\alpha_2-q)/(\alpha_2-1)}}{a\alpha_1}\right)^{(1-p)/d_1} \end{aligned}$$

uniformly on compact subsets of Ω.

This paper is organized as follows. In Section 2, we establish the local existence and uniqueness of the solution of (1.2). Results pertaining to global solution and blow-up in finite time for (1.2) are presented in Section 3. And in Section 4, results regarding to blow-up rates for (1.2) are established.

2. Proof of Theorem 1

We first give a maximum principle, the proof is standard and we omit.

Lemma 1. *Suppose that* $w_1, w_2 \in C^{2+\alpha}(Q_T) \cap C(\bar{Q}_T)$ *and satisfy*

$$\begin{cases} w_{1t} - d_1(x,t)\Delta w_1 \geqslant c_{11}(x,t)w_1 + c_{31}(x,t)\int\limits_{\Omega} c_{21}(x,t)w_2\,\mathrm{d}x, & (x,t)\in Q_T,\\ w_{2t} - d_2(x,t)\Delta w_2 \geqslant c_{12}(x,t)w_2 + c_{32}(x,t)\int\limits_{\Omega} c_{22}(x,t)w_1\,\mathrm{d}x, & (x,t)\in Q_T,\\ w_1(x,t), w_2(x,t) \geqslant 0, & (x,t)\in S_T,\\ w_1(x,0), w_2(x,0)\geqslant 0, & x\in\Omega, \end{cases} \tag{2.1}$$

where c_{ij} $(i=1,2,3;\ j=1,2)$ *are bounded functions and*

$$c_{2j}(x,t)\geqslant 0, \qquad c_{3j}(x,t)\geqslant 0, \qquad d_j(x,t)>0 \quad \text{in } Q_T.$$

Then $w_j(x,t)\geqslant 0$ *on* $\bar{Q}_T$. *Here* $Q_T = \Omega\times(0,T]$, $S_T = \partial\Omega\times(0,T]$.

Since $u = v = 0$ on $\partial\Omega$, (1.2) is not strictly parabolic type. The standard parabolic theory [14,20] cannot be used directly to prove the local existence of solution to (1.2). For any $\varepsilon > 0$, consider the following approximate problem

$$\begin{cases} u_{1\varepsilon t} = \left(u_{1\varepsilon}^{r_1}+\varepsilon\right)\left(\Delta u_{1\varepsilon} + a_1 u_{1\varepsilon}^{p_1}\int\limits_{\Omega} v_{1\varepsilon}^{q_1}\,\mathrm{d}x\right), & x\in\Omega,\ t>0,\\ v_{1\varepsilon t} = \left(v_{1\varepsilon}^{r_2}+\varepsilon\right)\left(\Delta v_{1\varepsilon} + b_1 v_{1\varepsilon}^{m_1}\int\limits_{\Omega} u_{1\varepsilon}^{n_1}\,\mathrm{d}x\right), & x\in\Omega,\ t>0,\\ u_{1\varepsilon}(x,t) = v_{1\varepsilon}(x,t) = 0, & x\in\partial\Omega,\ t>0,\\ u_{1\varepsilon}(x,0) = u_{01}(x), \quad v_{1\varepsilon}(x,0) = v_{01}(x), & x\in\Omega. \end{cases} \tag{2.2}$$

By Schauder's theory and Schauder's fixed-point theorem, we can prove that (2.2) admits a unique positive classical solution $(u_\varepsilon, v_\varepsilon)$, which is defined on $\bar{\Omega}\times[0,T_\varepsilon)$, here $0<T_\varepsilon\leqslant+\infty$. Since T_ε depends on ε, we need to seek a common lower bound for all T_ε. To do this, let $h(x,t) = ke^{-\lambda t}\varphi(x)$, where $\lambda = \max\{\lambda_1(k+1)^{r_1}, \lambda_1(k+1)^{r_2}\}$, and k is small, such that $k\varphi(x)\leqslant u_0(x)$, $k\varphi(x)\leqslant v_0(x)$. Next, let us consider the ordinary differential problem

$$\begin{cases} H'(t) = \hat{a}|\Omega|\left(H(t)+1\right)^{\hat{p}},\\ H(0) = \max\left\{\max\limits_{x\in\bar{\Omega}} u_{01}(x), \max\limits_{x\in\bar{\Omega}} v_{01}(x)\right\}, \end{cases} \tag{2.3}$$

where $\hat{a} = \max\{a_1, b_1\}$, $\hat{p} = \max\{p_1+q_1+r_1, m_1+n_1+r_2\}$. It is obvious that there exists $T_0>0$ such that (2.3) has a bounded solution $H(t)$ defined on $[0,T_0]$. Let $(u_\varepsilon, v_\varepsilon)$ be the solution of (2.2). Using Lemma 1, we obtain the following lemmas (see also [3,7,14]).

Lemma 2. *Under the assumptions* (H1)–(H3), *we have*

$$h(x,t)\leqslant u_\varepsilon(x,t), \qquad v_\varepsilon(x,t)\leqslant H(t) \quad \text{on } \bar{Q}_{T_0}.$$

Lemma 3. *Let* $(u_{\varepsilon_1}, v_{\varepsilon_1})$, $(u_{\varepsilon_2}, v_{\varepsilon_2})$ *be the solutions of* (2.2) *on* $\bar{Q}_{T_0}$, *with* ε_1 *and* ε_2, *respectively. If* $0<\varepsilon_1<\varepsilon_2<1$, *then* $u_{\varepsilon_1}\leqslant u_{\varepsilon_2}$, $v_{\varepsilon_1}\leqslant v_{\varepsilon_2}$ *on* $\bar{Q}_{T_0}$.

Lemma 4. *The solution* $(u_\varepsilon, v_\varepsilon)$ *of* (2.2) *satisfies* $u_{\varepsilon t}, v_{\varepsilon t}\geqslant 0$ *on* $\bar{Q}_{T_0}$.

It follows from Lemmas 2–3 that the point-wise limit

$$u(x,t)=\lim_{\varepsilon\to 0^+}u_\varepsilon(x,t),\qquad v(x,t)=\lim_{\varepsilon\to 0^+}v_\varepsilon(x,t)\tag{2.4}$$

exists for all $(x,t)\in\bar{Q}_{T_0}$. We shall show that the limit function (u,v) is a classical solution of (1.3).

Lemma 5. *Suppose that* u_0, v_0 *satisfy* (H1)–(H3). *Then* (1.3) *has a unique classical solution* (u,v) *defined on* Q_{T_0}.

Proof. It is required to prove that $u,v\in C^{2,1}(Q_{T_0})\cap C(\bar{Q}_{T_0})$. For an arbitrary point $(x_1,t_1)\in\Omega\times(0,T_0)$, choose an open domain $Q=\Omega_0\times(0,t_2)$ such that

$$x_1\in\Omega_0,\qquad \bar{\Omega}_0\subset\Omega,\quad\text{and}\quad 0<t_1<t_2<T_0.$$

Let $k_0=\inf_{(x,t)\in Q}h(x,t)$. By Lemma 2 we have that $u_\varepsilon, v_\varepsilon\geqslant k_0>0$ in Q, and hence $(u_\varepsilon+\varepsilon)^{r_1}\geqslant k_0^{r_1}$, $(v_\varepsilon+\varepsilon)^{r_1}\geqslant k_0^{r_2}$. The standard parabolic theory yields

$$\|u_\varepsilon\|_{C^{2+\alpha,1+(\alpha/2)}(\bar{Q})},\ \|v_\varepsilon\|_{C^{2+\alpha,1+(\alpha/2)}(\bar{Q})}\leqslant K',$$

where K' depends on k_0, not on ε. Therefore, $u,v\in C^{2+\alpha',1+(\alpha'/2)}(Q)$ $(0<\alpha'<\alpha)$ and

$$\|u\|_{C^{2+\alpha',1+(\alpha'/2)}(\bar{Q})},\ \|v\|_{C^{2+\alpha',1+(\alpha'/2)}(\bar{Q})}\leqslant K'.$$

This shows that u,v are in $C^{2,1}(Q_{T_0})$.

Since $0\leqslant \lim_{x\to\partial\Omega}u(x,t)(v(x,t))\leqslant \lim_{x\to\partial\Omega}u_\varepsilon(x,t)(v_\varepsilon(x,t))=0$, we see that u and v are continuous on $\partial\Omega\times(0,T_0)$. This completes the proof of existence.

Next we prove the uniqueness. Suppose that (u_1,v_1) and (u_2,v_2) are two solutions of (1.3). Let

$$A_1(x,t)=\frac{u_1-u_2}{u_1^{1-r_1}-u_2^{1-r_1}},\qquad A_2(x,t)=\frac{v_1-v_2}{v_1^{1-r_2}-v_2^{1-r_2}}.\tag{2.5}$$

It is obvious that $A_1(x,t)$ and $A_2(x,t)$ can be rewritten as

$$A_1(x,t)=\frac{1}{1-r_1}\int_0^1\big[\tau u_1^{1-r_1}(x,t)+(1-\tau)u_2^{1-r_1}(x,t)\big]^{r_1/(1-r_1)}\,\mathrm{d}\tau,$$

$$A_2(x,t)=\frac{1}{1-r_2}\int_0^1\big[\tau v_1^{1-r_2}(x,t)+(1-\tau)v_2^{1-r_2}(x,t)\big]^{r_2/(1-r_2)}\,\mathrm{d}\tau.$$

Since u_i, v_i $(i=1,2)$ are bounded and nonnegative, it follows that $A_i\geqslant 0$ and $A_i\in L^\infty(Q_{T_0})$. Choose two sequences $\{A_{n,i}\}$ of smooth functions such that

$$\frac{1}{n}\leqslant A_{n,i}\leqslant \max_{\bar{\Omega}\times[0,T_0]}A_i+\frac{1}{n},\qquad \frac{A_{n,i}-A_i}{\sqrt{A_{n,i}}}\to 0\quad\text{in } L^2\big(\Omega\times(0,T_0)\big).$$

Let $\chi_i\in C_0^\infty(\Omega)$ $(i=1,2)$ be such that $0\leqslant\chi_i\leqslant 1$. Consider the backward problem

$$\begin{cases}\psi_{i,t}+(1-r_i)A_{n,i}\Delta\psi_i=0, & x\in\Omega,\ t\in(0,T),\\ \psi_i(x,t)=0, & x\in\partial\Omega,\ t\in(0,T),\\ \psi_i(x,T)=\chi_i, & x\in\Omega,\end{cases}\tag{2.6}$$

where $T < T_0$. This is a nondegenerate parabolic equation, and for each n it has a unique solution $\psi_{n,i} \in C^\infty(\bar{Q}_T)$ with the following properties:

(i) $0 \leqslant \psi_{n,i} \leqslant 1$ on $\bar{Q}_T$;
(ii) $\int_{Q_T} A_{n,i}|\Delta\psi_{n,i}|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C_3$;
(iii) $\sup_{0\leqslant t\leqslant T}\int_\Omega (D\psi_{n,i})^2\,\mathrm{d}x \leqslant C_4$;

where the constants C_3 and C_4 depend only on χ_i (see Lemma 4.1 in [10]). Subtracting (1.3) for (u_1, v_1) and (u_2, v_2) yields

$$\left(u_1^{1-r_1} - u_2^{1-r_1}\right)_t = (1-r_1)\left(\Delta(u_1-u_2) + a_1u_1^{p_1}\int_\Omega v_1^{q_1}\,\mathrm{d}x - a_1u_2^{p_1}\int_\Omega v_2^{q_1}\,\mathrm{d}x\right), \tag{2.7}$$

$$\left(v_1^{1-r_2} - v_2^{1-r_2}\right)_t = (1-r_2)\left(\Delta(v_1-v_2) + b_1v_1^{m_1}\int_\Omega u_1^{n_1}\,\mathrm{d}x - b_1v_2^{m_1}\int_\Omega u_2^{n_1}\,\mathrm{d}x\right). \tag{2.8}$$

Multiplying above equations by $\psi_{n,i}$ $(i = 1, 2)$ respectively and integrating the results over $\Omega \times (0, T)$, we have that

$$\begin{aligned}
&\int_{Q_T} \psi_{n,1}\left(u_1^{1-r_1} - u_2^{1-r_1}\right)_t\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= \int_{Q_T}\left(\psi_{n,1}u_1^{1-r_1} - \psi_{n,1}u_2^{1-r_1}\right)_t\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T}\left(u_1^{1-r_1} - u_2^{1-r_1}\right)(\psi_{n,1})_t\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= \int_\Omega \chi_1\left(u_1^{1-r_1}(x,T) - u_2^{1-r_1}(x,T)\right)\mathrm{d}x - \int_{Q_T}\left(u_1^{1-r_1} - u_2^{1-r_1}\right)(\psi_{n,1})_t\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

$$\begin{aligned}
&\int_{Q_T} \psi_{n,1}(1-r_1)\left(\Delta(u_1-u_2) + a_1u_1^{p_1}\int_\Omega v_1^{q_1}\,\mathrm{d}x - a_1u_2^{p_1}\int_\Omega v_2^{q_1}\,\mathrm{d}x\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad= \int_{Q_T} \psi_{n,1}(1-r_1)\left(\Delta(u_1-u_2) + a_1u_1^{p_1}\int_\Omega \left(v_1^{q_1} - v_2^{q_1}\right)\mathrm{d}x\right.\\
&\qquad\left. + a_1\left(u_1^{p_1} - u_2^{p_1}\right)\int_\Omega v_2^{q_1}\,\mathrm{d}x\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad= \int_{Q_T}(1-r_1)\Delta(u_1-u_2)\psi_{n,1}\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+ (1-r_1)\int_{Q_T}\left(a_1u_1^{p_1}\int_\Omega\left(v_1^{q_1} - v_2^{q_1}\right)\mathrm{d}y + a_1\left(u_1^{p_1} - u_2^{p_1}\right)\int_\Omega v_2^{q_1}\,\mathrm{d}y\right)\psi_{n,1}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Noting the definitions of A_i, we can get

$$\int_\Omega\left(u_1^{1-r_1}(x,T) - u_2^{1-r_1}(x,T)\right)\chi_1\,\mathrm{d}x$$

$$
\begin{aligned}
&- \int_{Q_T} \big(u_1^{1-r_1} - u_2^{1-r_1}\big)\big((\psi_{n,1})_t + (1-r_1)A_1 \Delta \psi_{n,1}\big)\,\mathrm{d}x\,\mathrm{d}t \\
&= (1-r_1) \int_{Q_T} \left(a_1 u_1^{p_1} \int_\Omega \big(v_1^{q_1} - v_2^{q_1}\big)\,\mathrm{d}y + a_1 \big(u_1^{p_1} - u_2^{p_1}\big) \int_\Omega v_2^{q_1}\,\mathrm{d}y \right) \psi_{n,1}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\int_\Omega \big(v_1^{1-r_2}(x,T) - v_2^{1-r_2}(x,T)\big)\chi_2\,\mathrm{d}x \\
&\quad - \int_{Q_T} \big(v_1^{1-r_2} - v_2^{1-r_2}\big)\big((\psi_{n,2})_t + (1-r_2)A_2 \Delta \psi_{n,2}\big)\,\mathrm{d}x\,\mathrm{d}t \\
&= (1-r_2) \int_{Q_T} \left(b_1 v_1^{m_1} \int_\Omega \big(u_1^{n_1} - u_2^{n_1}\big)\,\mathrm{d}y + b_1 \big(v_1^{m_1} - v_2^{m_1}\big) \int_\Omega u_2^{n_1}\,\mathrm{d}y \right) \psi_{n,2}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

By the property (i) of the solution of (2.6), and noticing that

$$
\begin{aligned}
M_1(x,t) &= \frac{u_1^{p_1} - u_2^{p_1}}{u_1^{1-r_1} - u_2^{1-r_1}} \geqslant 0, \qquad M_2(x,t) = \frac{v_1^{q_1} - v_2^{q_1}}{v_1^{1-r_2} - v_2^{1-r_2}} \geqslant 0, \\
M_3(x,t) &= \frac{u_1^{n_1} - u_2^{n_1}}{u_1^{1-r_1} - u_2^{1-r_1}} \geqslant 0, \qquad M_4(x,t) = \frac{v_1^{m_1} - v_2^{m_1}}{v_1^{1-r_2} - v_2^{1-r_2}} \geqslant 0
\end{aligned}
$$

are all bounded functions (see the definitions of A_i), we have

$$
\begin{aligned}
&\int_\Omega \big(u_1^{1-r_1}(x,T) - u_2^{1-r_1}(x,T)\big)\chi_1\,\mathrm{d}x + \int_\Omega \big(v_1^{1-r_2}(x,T) - v_2^{1-r_2}(x,T)\big)\chi_2\,\mathrm{d}x \\
&\quad + (1-r_1) \int_{Q_T} \big(u_1^{1-r_1} - u_2^{1-r_1}\big)(A_{n,1} - A_1)\Delta \psi_{n,1}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + (1-r_2) \int_{Q_T} \big(v_1^{1-r_2} - v_2^{1-r_2}\big)(A_{n,2} - A_2)\Delta \psi_{n,2}\,\mathrm{d}x\,\mathrm{d}t \\
&= (1-r_1) \int_{Q_T} \left(a_1 u_1^{p_1} \int_\Omega \big(v_1^{q_1} - v_2^{q_1}\big)\,\mathrm{d}y + a_1 \big(u_1^{p_1} - u_2^{p_1}\big) \int_\Omega v_2^{q_1}\,\mathrm{d}y \right) \psi_{n,1}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + (1-r_2) \int_{Q_T} \left(b_1 v_1^{m_1} \int_\Omega \big(u_1^{n_1} - u_2^{n_1}\big)\,\mathrm{d}y + b_1 \big(v_1^{m_1} - v_2^{m_1}\big) \int_\Omega u_2^{n_1}\,\mathrm{d}y \right) \psi_{n,2}\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant C_5 \int_{Q_T} \left(u_1^{p_1} \int_\Omega \big(v_1^{1-r_2} - v_2^{1-r_2}\big)_+\,\mathrm{d}y + \big(u_1^{1-r_1} - u_2^{1-r_1}\big)_+ \int_\Omega v_2^{q_1}\,\mathrm{d}y \right) \psi_{n,1}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C_6 \int_{Q_T} \left(v_1^{m_1} \int_\Omega \big(u_1^{1-r_1} - u_2^{1-r_1}\big)_+\,\mathrm{d}y + \big(v_1^{1-r_2} - v_2^{1-r_2}\big)_+ \int_\Omega u_2^{n_1}\,\mathrm{d}y \right) \psi_{n,2}\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

$$\leqslant C_7 \int_{Q_T} \big(\big(v_1^{1-r_2} - v_2^{1-r_2}\big)_+ + \big(u_1^{1-r_1} - u_2^{1-r_1}\big)_+ \big) \psi_{n,1} \, \mathrm{d}x \, \mathrm{d}t$$
$$+ C_8 \int_{Q_T} \big(\big(u_1^{1-r_1} - u_2^{1-r_1}\big)_+ + \big(v_1^{1-r_2} - v_2^{1-r_2}\big)_+ \big) \psi_{n,2} \, \mathrm{d}x \, \mathrm{d}t,$$

where $w_+ = \max\{w, 0\}$, C_5, C_6, C_7 and C_8 are positive constants. By the definitions of $A_{n,i}$ and the properties of the solution of (2.6), we have

$$\int_{Q_T} |A_i - A_{n,i}| \big|\Delta \psi_n^i\big| \, \mathrm{d}x \, \mathrm{d}t \leqslant \left\| \frac{A_i - A_{n,i}}{\sqrt{A_{n,i}}} \right\|_{L^2(Q_T)} \big\| \sqrt{A_{n,i}} \Delta \psi_{n,i} \big\|_{L^2(Q_T)}$$
$$\leqslant \sqrt{C_3} \left\| \frac{A_i - A_{n,i}}{\sqrt{A_{n,i}}} \right\|_{L^2(Q_T)} \to 0 \quad \text{as } n \to \infty.$$

Thus

$$\int_{\Omega} \big(u_1^{1-r_1}(x,T) - u_2^{1-r_1}(x,T)\big) \chi_1 \, \mathrm{d}x + \int_{\Omega} \big(v_1^{1-r_2}(x,T) - v_2^{1-r_2}(x,T)\big) \chi_2 \, \mathrm{d}x$$
$$\leqslant (C_7 + C_8) \int_{Q_T} \big(\big(v_1^{1-r_2} - v_2^{1-r_2}\big)_+ + \big(u_1^{1-r_1} - u_2^{1-r_1}\big)_+ \big) \, \mathrm{d}x \, \mathrm{d}t$$

for any functions χ_1 and χ_2. It follows that

$$\int_{\Omega} \big(u_1^{1-r_1}(x,T) - u_2^{1-r_1}(x,T)\big) \, \mathrm{d}x + \int_{\Omega} \big(v_1^{1-r_2}(x,T) - v_2^{1-r_2}(x,T)\big) \, \mathrm{d}x$$
$$\leqslant (C_7 + C_8) \int_{Q_T} \big(\big(v_1^{1-r_2} - v_2^{1-r_2}\big)_+ + \big(u_1^{1-r_1} - u_2^{1-r_1}\big)_+ \big) \, \mathrm{d}x \, \mathrm{d}t.$$

Clearly, T can be replaced by any $t < T_0$. By Gronwall's inequality, we see that $u_1 \leqslant u_2$, $v_1 \leqslant v_2$ on Q_{T_0}. Similarly, we can prove $u_1 \geqslant u_2$, $v_1 \geqslant v_2$ on Q_{T_0}. The proof is completed. □

By the continuation method we have that (1.3) has a unique maximal defined solution (u, v) and we denote T^* the maximal existence time. If $T^* < \infty$, then (u, v) blows up in T^*.

In view of Lemmas 3–5 we have the following

Lemma 6. *Suppose that* (u_0, v_0) *satisfies* (H1)–(H3)*, then the solution* (u, v) *of* (1.3) *satisfies* $u_t, v_t \geqslant 0$ *in any compact subsets of* $\Omega \times (0, T^*)$.

3. Proof of Theorems 2–4

In this section, we use the upper and lower solutions method to deduce conditions on the global existence or blow-up in finite time for solutions. Denote

$$A = \begin{pmatrix} \alpha_1 - p & -q \\ -n & \alpha_2 - m \end{pmatrix}, \qquad D = (\alpha_1 - p)(\alpha_2 - m) - qn.$$

Lemma 7. [3] *If $\alpha_1 > p$, $\alpha_2 > m$ and $D > 0$, then the equation $A(l_1, l_2)^T = (1, 1)^T$ has a unique solution $(l_1, l_2)^T$ and satisfies $l_1, l_2 > 0$. It is obvious that $A(cl_1, cl_2)^T > (0, 0)^T$ for any $c > 0$.*

Lemma 8. [3] *If $\alpha_1 < p$, or $\alpha_2 < m$, or $D < 0$, then there exist $l_1, l_2 > 0$ such that $A(l_1, l_2)^T < (0, 0)^T$. It is obvious that $A(cl_1, cl_2)^T < (0, 0)^T$ for any $c > 0$.*

For convenience, we will denote

$$R_1(w, s) = w_t - \Delta w^{\alpha_1} - aw^p \int_\Omega s^q \,\mathrm{d}x, \qquad R_2(w, s) = s_t - \Delta s^{\alpha_2} - bs^m \int_\Omega w^n \,\mathrm{d}x.$$

Proof of Theorem 2. Let $\varphi(x)$ be the unique positive solution of the linear elliptic problem

$$-\Delta\varphi(x) = 1, \quad x \in \Omega; \qquad \varphi(x) = 0, \quad x \in \partial\Omega. \tag{3.1}$$

Denote $C = \max_{\bar\Omega} \varphi(x)$ and

$$\bar u = \big(K\big(\varphi(x) + 1\big)\big)^{l_1}, \qquad \bar v = \big(K\big(\varphi(x) + 1\big)\big)^{l_2}, \tag{3.2}$$

where $l_1, l_2 < 1$ satisfy $\alpha_1 l_1, \alpha_2 l_2 < 1$ and K to be determined. Clearly, $(\bar u, \bar v)$ is bounded for any $t > 0$, and $\bar u \geqslant K^{l_1}$, $\bar v \geqslant K^{l_2}$. The direct computation gives

$$\begin{aligned} R_1(\bar u, \bar v) &= -\Delta\big(K\big(\varphi(x) + 1\big)\big)^{\alpha_1 l_1} - a\big(K\big(\varphi(x) + 1\big)\big)^{pl_1} \int_\Omega K\big(\varphi(x) + 1\big)^{ql_2} \,\mathrm{d}x \\ &= -\alpha_1 l_1(\alpha_1 l_1 - 1)\big(\varphi(x) + 1\big)^{\alpha_1 l_1 - 2}|\nabla\varphi|^2 K^{\alpha_1 l_1} + \alpha_1 l_1\big(\varphi(x) + 1\big)^{\alpha_1 l_1 - 1} K^{\alpha_1 l_1} \\ &\quad - aK^{pl_1 + ql_2}\big(\varphi(x) + 1\big)^{pl_1} \int_\Omega \big(\varphi(x) + 1\big)^{ql_2} \,\mathrm{d}x \\ &\geqslant \alpha_1 l_1 K^{\alpha_1 l_1}(C + 1)^{\alpha_1 l_1 - 1} - \big(K(C + 1)\big)^{pl_1 + ql_2} a|\Omega|, \end{aligned} \tag{3.3}$$

$$R_2(\bar u, \bar v) \geqslant \alpha_2 l_2 K^{\alpha_2 l_2}(C + 1)^{\alpha_2 l_2 - 1} - \big(K(C + 1)\big)^{ml_1 + nl_2} b|\Omega|. \tag{3.4}$$

Denote

$$K_1 = \big((\alpha_1 l_1)^{-1}(C + 1)^{pl_1 + ql_2 - \alpha_1 l_1 + 1} a|\Omega|\big)^{1/(\alpha_1 l_1 - pl_1 - ql_2)},$$
$$K_2 = \big((\alpha_2 l_2)^{-1}(C + 1)^{ml_1 + nl_2 - \alpha_2 l_2 + 1} b|\Omega|\big)^{1/(\alpha_2 l_2 - ml_1 - nl_2)}.$$

If $\alpha_1 > p$, $\alpha_2 > m$ and $D > 0$, by Lemma 7, there exist positive constants l_1, l_2 such that

$$\alpha_1 l_1 > pl_1 + ql_2, \quad \alpha_2 l_2 > ml_1 + nl_2, \quad \text{and} \quad \alpha_1 l_1, \alpha_2 l_2 < 1. \tag{3.5}$$

One can choose K so large that $K \geqslant \max\{K_1, K_2\}$ and

$$K\big(\varphi(x) + 1\big)^{l_1} \geqslant u_0(x), \qquad K\big(\varphi(x) + 1\big)^{l_2} \geqslant v_0(x). \tag{3.6}$$

From (3.3)–(3.6) we see that $(\bar u, \bar v)$ is a positive upper solution of (1.2). Hence $u \leqslant \bar u$, $v \leqslant \bar v$, which implies that (u, v) exists globally. This completes the proof. □

Proof of Theorem 3. By Lemma 8, there exist positive constants l_1, l_2 such that

$$\alpha_1 l_1 < pl_1 + ql_2, \quad \alpha_2 l_2 < ml_1 + nl_2, \quad \text{and} \quad \alpha_1 l_1, \alpha_2 l_2 < 1. \tag{3.7}$$

Therefore, we can choose K so small that $K \leqslant \min\{K_1, K_2\}$. Furthermore, if (u_0, v_0) is small enough to satisfy (3.6), then $(\bar{u}, \bar{v})$ defined by (3.2) is a positive upper solution of (1.2). Hence (u, v) exists globally.

Now we prove that the solution will blow up in finite time if the initial data is large. Without loss of generality, we may assume that $0 \in \Omega$. Then there exists a ball $B_R(0) \Subset \Omega$. Let

$$w(r) = \frac{R^3}{12} - \frac{R}{4}r^2 + \frac{1}{6}r^3, \quad 0 \leqslant r \leqslant R,$$

where $r = |x|$. Clearly, $0 \leqslant w(r) \leqslant R^3/12$ and $w(r)$ is nonincreasing. Moreover,

$$-\Delta w = \frac{N}{2}R - \frac{N+1}{2}r.$$

Hence

$$\begin{cases} -\Delta w \leqslant 0 & \text{for } \dfrac{N}{N+1}R \leqslant r \leqslant R, \\ 0 \leqslant -\Delta w \leqslant \dfrac{N}{2}R, \quad w \geqslant \dfrac{(3N+1)R^3}{12(N+1)^3} & \text{for } 0 \leqslant r < \dfrac{NR}{N+1}. \end{cases} \tag{3.8}$$

Set

$$V(r,t) = T - \left(\frac{12w(r)}{R^3}\right)^{\sigma} t, \quad 0 < t < T,$$

where $0 < T < 1$ is small and $\sigma > 1$ will be determined later. And denote

$$s(r,t) = w(r)/V(r,t).$$

Then $s(r,t)$ becomes unbounded when $t \to T^-$ at the point $r = 0$. By Lemma 8, there exist positive constants l_1, l_2 such that

$$pl_1 + ql_2 > \alpha_1 l_1 + 1, \quad ml_1 + nl_2 > \alpha_2 l_2 + 1, \quad \text{and} \quad \alpha_1 l_1, \alpha_2 l_2 < 1. \tag{3.9}$$

Let

$$\underline{u} = s^{l_1}, \qquad \underline{v} = s^{l_2}, \quad \text{in } B_R(0) \times (0,T).$$

A routine calculation yields

$$\underline{u}_t = \left(s^{l_1}\right)_t = \left(\frac{w}{V}\right)^{l_1}_t = \frac{l_1 (12/R^3)^{\sigma} w^{l_1+\sigma}}{V^{1+l_1}},$$

$$-\Delta \underline{u}^{\alpha_1} \leqslant \frac{\alpha_1 l_1 w^{\alpha_1 l_1 - 1}(V + \sigma(\frac{12w(r)}{R^3})^{\sigma} t)(-\Delta w)_+}{V^{\alpha_1 l_1 + 1}}$$

$$\leqslant \frac{\alpha_1 l_1 \sigma T w^{\alpha_1 l_1 - 1}(-\Delta w)_+}{V^{\alpha_1 l_1 + 1}}.$$

Fix $\sigma = \max\{1, pl_1 - l_1, ml_2 - l_2\}$. From (3.8) and (3.9), we have that, for $\frac{N}{N+1}R \leqslant r \leqslant R$,

$$V \geqslant T - \left[\frac{12w(\frac{NR}{N+1})}{R^3}\right]^{\sigma} t = T - \left[\frac{3N+1}{(N+1)^3}\right]^{\sigma} t \geqslant T - \frac{t}{2^{\sigma}} \geqslant \frac{T}{2}.$$

It follows that, for $0 < T \ll 1$,

$$R_1(\underline{u},\underline{v}) \leqslant \frac{l_1(12/R^3)^\sigma w^{l_1+\sigma}}{V^{1+l_1}} - a\frac{w^{l_1 p}}{V^{l_1 p}} \int_{NR/(N+1)}^{R} \frac{w^{l_2 q}}{V^{l_2 q}}\,\mathrm{d}r$$
$$\leqslant \frac{l_1(12/R^3)^\sigma w^{l_1+\sigma}}{V^{1+l_1}} - \frac{a w^{l_1 p}}{T^{l_2 q} V^{l_1 p}} \int_{NR/(N+1)}^{R} w^{l_2 q}\,\mathrm{d}r$$
$$\leqslant w^{l_1 p}\left(\frac{(12 l_1/R^3)^\sigma w^{l_1+\sigma-l_1 p}}{(T/2)^{1+l_1}} - \frac{a\int_{NR/(N+1)}^{R} w^{l_2 q}\,\mathrm{d}r}{T^{p l_1+q l_2}}\right) \leqslant 0.$$

Next, we consider the case that $0 \leqslant r < \frac{NR}{N+1}$, $0 < T \ll 1$. Note that $s(0,t) = R^3/(12(T-t))$ and

$$R_1(\underline{u},\underline{v})|_{r=0} \leqslant \frac{l_1(R^3/12)^{l_1}}{(T-t)^{l_1+1}} + \frac{\alpha_1 l_1 \sigma T(R^3/12)^{\alpha_1 l_1-1} NR}{2(T-t)^{\alpha_1 l_1+1}} - \frac{a(w(\frac{NR}{N+1}))^{p l_1+q l_2}}{(N+1)(T-t)^{l_1 p+l_2 q}} < 0,$$

if $0 < T \ll 1$. By the continuity of $R_1(\underline{u},\underline{v})$, there exists $r_0 > 0$ such that $R_1(\underline{u},\underline{v}) \leqslant 0$ in $\overline{B_{r_0}(0)} \times (0,T)$ if $0 < T \ll 1$. For $r \in (r_0, \frac{NR}{N+1})$, we have

$$R_1(\underline{u},\underline{v}) \leqslant \frac{l_1(R^3/12)^{l_1}}{V^{l_1+1}} + \frac{\alpha_1 l_1 \sigma T(R^3/12)^{\alpha_1 l_1-1} NR}{2V^{\alpha_1 l_1+1}} - \frac{aNR[w(\frac{NR}{N+1})]^{p l_1+q l_2}}{(N+1)V^{l_1 p}(T-\xi t)^{l_2 q}}$$
$$\leqslant \frac{l_1(R^3/12)^{l_1}}{(T-t)^{l_1+1}} + \frac{\alpha_1 l_1 \sigma T(R^3/12)^{\alpha_1 l_1-1} NR}{2(T-t)^{\alpha_1 l_1+1}}$$
$$- \frac{aNR[w(\frac{NR}{N+1})]^{p l_1+q l_2}}{(N+1)(T-t)^{l_1 p}(T-(\frac{3N+1}{(N+1)^3})^\sigma t)^{l_2 q}} \leqslant 0,$$

where $0 < T \ll 1$, $\xi \in ((\frac{3N+1}{(N+1)^3})^\sigma, (\frac{12 w(r_0)}{R^3})^\sigma)$.

Similarly, we can prove $R_2(\underline{u},\underline{v}) \leqslant 0$ in $B_R(0) \times (0,T)$ if $T \ll 1$.

Choose $u_0(x), v_0(x)$ so large that $u_0(x) \geqslant \underline{u}(x,0)$, $v_0(x) \geqslant \underline{v}(x,0)$ on $\overline{B_R(0)}$. Noticing that $\underline{u}(x,t) = \underline{v}(x,t) = 0$ on $\partial\Omega$. Hence, $(u,v) \geqslant (\underline{u},\underline{v})$ by Lemma 1. Therefore, (u,v) blows up in finite time. The proof is completed. □

Proof of Theorem 4. Let $\psi(x)$ be the unique solution of the following elliptic problem:

$$\Delta\psi(x) = -1, \quad x \in \Omega; \qquad \psi(x) = K_0 > 1, \quad x \in \partial\Omega, \tag{3.10}$$

where K_0 is a positive number. Obviously, we have $\max_{x\in\bar{\Omega}} \psi(x) = K \geqslant K_0 > 1$.

Case 1 *(when $\alpha_1 - p = q$, $\alpha_2 - m = n$).* Set

$$w_1(x,t) = M\psi^{1/\alpha_1}(x), \qquad w_2(x,t) = M\psi^{1/\alpha_2}(x),$$

where M is to be determined later. A series of computations yields

$$w_{1t} - \Delta w_1^{\alpha_1} - a w_1^p \int_\Omega w_2^q\,\mathrm{d}x = M^{\alpha_1} - aM^{p+q}(\psi)^{p/\alpha_1} \int_\Omega \psi^{q/\alpha_2}\,\mathrm{d}x$$
$$\geqslant M^{\alpha_1} - aM^{p+q} K^{p/\alpha_1+q/\alpha_2}|\Omega|,$$

$$
\begin{aligned}
w_{2t} - \Delta w_2^{\alpha_2} - b w_2^m \int_\Omega w_1^n \, \mathrm{d}x &= M^{\alpha_2} - a M^{m+n} (\psi)^{m/\alpha_2} \int_\Omega \psi^{n/\alpha_1} \, \mathrm{d}x \\
&\geqslant M^{\alpha_2} - b M^{m+n} K^{m/\alpha_2 + n/\alpha_1} |\Omega|.
\end{aligned}
$$

We choose $a < (K^{p/\alpha_1+q/\alpha_2}|\Omega|)^{-1}$, $b < (K^{m/\alpha_2+n/\alpha_1}|\Omega|)^{-1}$. Then

$$
w_{1t} - \Delta w_1^{\alpha_1} - a w_1^p \int_\Omega w_2^q \, \mathrm{d}x \geqslant 0,
$$

$$
w_{2t} - \Delta w_2^{\alpha_2} - b w_2^m \int_\Omega w_1^n \, \mathrm{d}x \geqslant 0.
$$

Therefore, (w_1, w_2) is an upper solution of (1.2) provided that

$$
M \geqslant \max\left\{K_0^{-1/\alpha_1} u_0(x), K_0^{-1/\alpha_2} v_0(x)\right\}.
$$

By Lemma 1, $w_1(x,t) \geqslant u(x,t)$, $w_2(x,t) \geqslant v(x,t)$. Therefore, (u, v) exists globally.

Case 2 *(when $\alpha_1 - p > q$, $n > \alpha_2 - m$).* Set $\bar{w}(x,t) = L\psi^{1/\alpha_2}(x)$ with $\psi(x)$ given by (3.10) and $L = (bK^{(m+n)/\alpha_2}|\Omega|)^{-1/(m+n-\alpha_2)}$. A series of computations yields

$$
\begin{aligned}
w_{1t} - \Delta w_1^{\alpha_1} - a w_1^p \int_\Omega \bar{w}^q \, \mathrm{d}x &= M^{\alpha_1} - a M^p \psi^{p/\alpha_1} \int_\Omega L^q \psi^{q/\alpha_2} \, \mathrm{d}x \\
&\geqslant M^{\alpha_1} - a M^p L^q K^{p/\alpha_1 + q/\alpha_2} |\Omega|, \\
\bar{w}_t - \Delta \bar{w}^{\alpha_2} - b \bar{w}^m \int_\Omega w_1^n \, \mathrm{d}x &= L^{\alpha_2} - b L^m \psi^{m/\alpha_2} \int_\Omega M^n \psi^{n/\alpha_1} \, \mathrm{d}x \\
&\geqslant L^{\alpha_2} - b L^m M^n K^{m/\alpha_2 + n/\alpha_1} |\Omega|.
\end{aligned}
$$

Choose M, L such that $M^{\alpha_1} - aM^pL^qK^{p/\alpha_1+q/\alpha_2}|\Omega| = 0$, $L^{\alpha_2} - bL^mM^nK^{m/\alpha_2+n/\alpha_1}|\Omega| = 0$. If $u_0(x), v_0(x)$ are small enough and satisfy $MK_0^{1/\alpha_1} \geqslant u_0(x)$, $LK_0^{1/\alpha_2} \geqslant v_0(x)$, then by Lemma 1, $(w_1, \bar{w})$ is the supper solution of (1.2). Therefore, (u, v) exists globally.

When $q > \alpha_1 - p$ and $\alpha_2 - m > n$, similar as above, we also can prove that the solution of (1.2) exists globally provided that both $u_0(x)$ and $u_0(x)$ are small. □

4. Proof of Theorems 5–7

In this section we will rely on (1.3) to prove Theorems 5–7. And we write the solution of (1.3) as (u, v), instead of (u_1, v_1). We also assume that the solution (u, v) blows up in finite time T^*. Denote $U(t) = \max_{x\in\bar{\Omega}} u(x,t)$, $V(t) = \max_{x\in\bar{\Omega}} v(x,t)$, which are Lipschitz continuous [8]. By (1.3), we see that $U(t), V(t)$ satisfy

$$
U_t \leqslant a_1|\Omega|U^{r_1+p_1}V^{q_1}, \quad V_t \leqslant b_1|\Omega|V^{r_2+m_1}U^{n_1} \quad \text{a.e. } t \in \big(0, T^*\big). \tag{4.1}
$$

Let $\beta_1 = n_1 + 1 - r_1 - p_1$, $\beta_2 = q_1 + 1 - r_2 - m_1$. Then

$$
\begin{aligned}
\big(U^{\beta_1} + V^{\beta_2}\big)_t &\leqslant (\beta_1 a_1 + \beta_2 b_1)|\Omega|U^{n_1}V^{q_1} \\
&\leqslant C_{10}\big(U^{\beta_1}\big)^{n_1/\beta_1}\big(V^{\beta_2}\big)^{q_1/\beta_2} \\
&\leqslant C_{10}\big(U^{\beta_1} + V^{\beta_2}\big)^{(n_1\beta_2+q_1\beta_1)/(\beta_1\beta_2)}.
\end{aligned}
$$

Integrating above inequality over (t, T^*), we obtain

$$U^{\beta_1} + V^{\beta_2} \geqslant C_{11}\left(T^* - t\right)^{-(\beta_1\beta_2)/d}, \tag{4.2}$$

where $d = q_1 n_1 - (1 - r_1 - p_1)(1 - r_2 - m_1)$. Let $k_1 = d/\beta_2$, $k_2 = d/\beta_1$.

Lemma 9. *Suppose that* $u_0(x), v_0(x)$ *satisfy* (H1)–(H4) *and* $\beta_1, \beta_2 > 0$, *then*

$$u_t - \delta u^{k_1+1} \geqslant 0, \qquad v_t - \delta v^{k_2+1} \geqslant 0 \quad \text{in } Q_{T^*}. \tag{4.3}$$

Proof. Set $J_1(x,t) = u_t - \delta u^{k_1+1}$, $J_2(x,t) = v_t - \delta v^{k_2+1}$. Then

$$\lim_{x\to\partial\Omega} J_1(x,t) = \lim_{x\to\partial\Omega} J_2(x,t) \geqslant 0, \quad J_1(x,0), J_2(x,0) \geqslant 0,\ x \in \Omega.$$

A series of computations yields $J_{1t} = u_{tt} - \delta(k_1+1)u^{k_1}u_t$ and

$$\begin{aligned} u_{tt} = {} & r_1 u^{-1}\left(J_1^2 + 2\delta u^{k_1+1}J_1 + \delta^2 u^{2k_1+2}\right) + u^{r_1}\Delta J_1 + \delta(k_1+1)k_1 u^{k_1-1+r_1}|\nabla u|^2 \\ & + \delta(k_1+1)u^{k_1+r_1}\Delta u + a_1 p_1 u^{p_1-1+r_1} J_1 \int_\Omega v^{q_1}\,\mathrm{d}x + a_1 p_1 \delta u^{p_1+k_1+r_1} \int_\Omega v^{q_1}\,\mathrm{d}x \\ & + a_1 q_1 u^{p_1+r_1} \int_\Omega v^{q_1-1} J_2\,\mathrm{d}x + a_1 q_1 \delta u^{p_1+r_1} \int_\Omega v^{q_1+k_2}\,\mathrm{d}x. \end{aligned}$$

Therefore,

$$\begin{aligned} J_{1t} = {} & u^{r_1}\Delta J_1 + \left(2r_1\delta u^{k_1} + a_1 p_1 u^{p_1-1-r_1} \int_\Omega v^{q_1}\,\mathrm{d}x\right) J_1 + a_1 q_1 u^{p_1+r_1} \int_\Omega v^{q_1-1} J_2\,\mathrm{d}x \\ & + r_1 u^{-1} J_1^2 r_1 \delta^2 u^{2k_1+1} + \delta k_1(k_1+1)u^{k_1-1+r_1}|\nabla u|^2 + a_1 p_1 \delta u^{p_1+k_1+r_1} \int_\Omega v^{q_1}\,\mathrm{d}x \\ & + a_1 \delta q_1 u^{p_1+r_1} \int_\Omega v^{q_1+k_2}\,\mathrm{d}x - a_1\delta(k_1+1)u^{k_1+p_1+r_1} \int_\Omega v^{q_1}\,\mathrm{d}x. \end{aligned}$$

Noting that $\frac{k_1}{2k_1+1-r_1-p_1} + \frac{q_1}{q_1+k_2} = 1$, by virtue of Young's inequality

$$u^{k_1}\left(\int_\Omega v^{q_1+k_2}\mathrm{d}x\right)^{q_1/(q_1+k_2)} \leqslant \frac{k_1(\varepsilon u^{k_1})^{(2k_1+1-r_1-p_1)/k_1}}{2k_1+1-r_1-p_1} + \frac{q_1\int_\Omega v^{q_1+k_2}\,\mathrm{d}x}{(q_1+k_2)\varepsilon^{(q_1+k_2)/q_1}}, \tag{4.4}$$

where $\varepsilon = (\frac{p_1(k_1-p_1+1)}{(p_1+k_2)q_1}|\Omega|^{k_2/(q_1+k_2)})^{q_1/(q_1+k_2)}$. Thus, using the Hölder inequality and (4.4) we have

$$\begin{aligned} & J_{1t} - u^{r_1}\Delta J_1 - \left(2r_1\delta u^{k_1} + a_1 p_1 u^{p_1-1-r_1} \int_\Omega v^{q_1}\,\mathrm{d}x\right) J_1 - a_1 q_1 u^{p_1+r_1} \int_\Omega v^{q_1-1} J_2\,\mathrm{d}x \\ & \quad \geqslant r_1\delta^2 u^{2k_1+1} + a_1\delta q_1 u^{p_1+r_1} \int_\Omega v^{q_1+k_2}\,\mathrm{d}x - a_1\delta(k_1-p_1+1)u^{k_1+p_1+r_1} \int_\Omega v^{q_1}\,\mathrm{d}x \\ & \quad \geqslant r_1\delta^2 u^{2k_1+1} + a_1\delta q_1 u^{p_1+r_1} \int_\Omega v^{q_1+k_2}\,\mathrm{d}x - a_1\delta(k_1-p_1+1)u^{k_1+p_1+r_1} \end{aligned}$$

$$\times\left(\int_\Omega v^{q_1+k_2}\,\mathrm{d}x\right)^{\frac{q_1}{q_1+k_2}}|\Omega|^{\frac{k_2}{q_1+k_2}}$$

$$\geqslant r_1\delta^2u^{2k_1+1}+a_1\delta q_1u^{p_1+r_1}\int_\Omega v^{q_1+k_2}\,\mathrm{d}x$$

$$-\frac{q_1a_1\delta(k_1-p_1+1)u^{p_1+r_1}}{(p_1+k_2)\varepsilon^{(q_1+k_2)/q_1}}\int_\Omega v^{q_1+k_2}\,\mathrm{d}x|\Omega|^{\frac{k_2}{q_1+k_2}}$$

$$-\frac{k_1a_1\delta(k_1-p_1+1)u^{p_1+r_1}}{2k_1+1-r_1-p_1}\varepsilon^{(2k_1+1-r_1-p_1)/k_1}u^{2k_1+1-r_1-p_1}|\Omega|^{\frac{k_2}{q_1+k_2}}$$

$$\geqslant r_1\delta^2u^{2k_1+1}+a_1\delta q_1u^{p_1+r_1}\int_\Omega v^{q_1+k_2}\,\mathrm{d}x$$

$$-\frac{q_1a_1\delta(k_1-p_1+1)u^{p_1+r_1}}{(p_1+k_2)\varepsilon^{(q_1+k_2)/q_1}}\int_\Omega v^{q_1+k_2}\,\mathrm{d}x|\Omega|^{\frac{k_2}{q_1+k_2}}$$

$$-\frac{k_1a_1\delta(k_1-p_1+1)}{2k_1+1-r_1-p_1}\varepsilon^{(2k_1+1-r_1-p_1)/k_1}u^{2k_1+1}|\Omega|^{\frac{k_2}{q_1+k_2}}$$

$$\geqslant r_1\delta(\delta-\delta_{0u})u^{2k_1+1}\geqslant 0,$$

where

$$\delta_{0u}=\frac{k_1a_1(k_1-p_1+1)}{r_1(2k_1+1-r_1-p_1)}|\Omega|^{\frac{k_2}{q_1+k_2}}\varepsilon^{(2k_1+1-r_1-p_1)/k_1}.$$

We can determine a number δ_{0v} by the similar way. Let $\delta_0=\max\{\delta_{0u},\delta_{0v}\}$, similar to the above one has

$$J_{2t}-v^{r_2}\Delta J_2-\left(2r_2\delta v^{k_2}+b_1m_1v^{m_1-1+r_2}\int_\Omega u^{n_1}\,\mathrm{d}x\right)J_2-b_1n_1v^{m_1+r_2}\int_\Omega u^{n_1-1}J_1\,\mathrm{d}x\geqslant 0.$$

By Lemma 1, we have $J_1, J_2\geqslant 0$. This completes the proof. □

It follows (4.3) that

$$U_t\geqslant\delta U^{k_1+1},\qquad V_t\geqslant\delta V^{k_2+1}\quad\text{in }Q_{T^*}.\tag{4.5}$$

Combining with (4.1), we can obtain

$$\left(\frac{\delta}{a_1|\Omega|}\right)^{\beta_2/q_1}U^{\beta_1}\leqslant V^{\beta_2},\qquad\left(\frac{\delta}{b_1|\Omega|}\right)^{\beta_1/n_1}V^{\beta_2}\leqslant U^{\beta_1}.\tag{4.6}$$

And integrating (4.3) over $(0,T^*)$, we conclude that

$$\begin{cases}U(t)\leqslant C_{12}\big(T^*-t\big)^{-1/k_1}=C_{12}\big(T^*-t\big)^{-\beta_2/d},\\ V(t)\leqslant C_{13}\big(T^*-t\big)^{-1/k_2}=C_{13}\big(T^*-t\big)^{-\beta_1/d}.\end{cases}\tag{4.7}$$

From (4.2), (4.6) and (4.7), we can obtain the following lemma.

Lemma 10. *Suppose that* u_0, v_0 *satisfy* (H1)–(H4) *and* $\beta_1, \beta_2 > 0$*, then*

$$\begin{cases} C_{14} \leqslant U(t)\big(T^*-t\big)^{1/k_1} = U(t)\big(T^*-t\big)^{\beta_2/d} \leqslant C_{12}, \\ C_{15} \leqslant V(t)\big(T^*-t\big)^{1/k_2} = V(t)\big(T^*-t\big)^{\beta_1/d} \leqslant C_{13}, \end{cases} \tag{4.8}$$

where C_{12}, C_{13} *are given by* (4.7)*,* C_{14} *and* C_{15} *are positive constants.*

By Lemma 10, we can obtain Theorem 5 immediately (note: we write (u_1, v_1) as (u, v)).
Next, we will prove Theorem 6. First of all, we give some lemmas. Denote

$$g_1(t) = a_1 \int_\Omega v_1^{q_1}\,\mathrm{d}x, \qquad G_1(t) = \int_0^t g_1(s)\,\mathrm{d}s,$$

$$g_2(t) = a_2 \int_\Omega u_1^{n_1}\,\mathrm{d}x, \qquad G_2(t) = \int_0^t g_2(s)\,\mathrm{d}s.$$

Lemma 11. *Suppose that* u_0, v_0 *satisfy* (H1)–(H3)*. If* $p_1 + r_1 < 1$*,* $m_1 + r_2 < 1$ *hold, then*

$$\lim_{t\to T^*} g_i(t) = \lim_{t\to T^*} G_i(t) = \infty \quad (i = 1, 2) \text{ as } t \to T^*. \tag{4.9}$$

Proof. Assume that $\lim_{t\to T^*} g_i(t) < \infty$, then $u(x,t)$ and $v(x,t)$ exist globally in time for any $u_0(x), v_0(x)$ since $p_1, m_1 < 1$ (see [29,30]). This leads to a contradiction. Therefore $\lim_{t\to T^*} g_i(t) = \infty$.

Next, we will infer $\lim_{t\to T^*} G_i(t) = \infty$ $(i = 1, 2)$. Let $U(t), V(t)$ be as above, then

$$U'(t) \leqslant U^{p_1+r_1}(t) g_1(t), \qquad V'(t) \leqslant V^{m_1+r_2}(t) g_2(t), \quad \text{a.e. } t \in \big[0, T^*\big). \tag{4.10}$$

Integrating (4.10) over $(0, t)$, we obtain

$$\frac{U^{1-r_1-p_1}(t)}{1-r_1-p_1} \leqslant G_1(t) + \frac{U^{1-r_1-p_1}(0)}{1-r_1-p_1} \quad \text{if } p_1 + r_1 < 1. \tag{4.11}$$

Noticing that $\lim_{t\to T^*} U(t) = \infty$, it follows that $\lim_{t\to T^*} G_1(t) = \infty$. Similarly, $\lim_{t\to T^*} G_2(t) = \infty$. This completes the proof. □

Lemma 12. *Assume that* $\frac{n_1}{1-r_1-p_1} = \frac{q_1}{1-r_2-m_1}$*. If* $n_1 - p_1 - 1, q_1 - m_1 - 1 > 0$ *hold, then*

$$\int_\Omega u^{\beta_1}\,\mathrm{d}x, \int_\Omega v^{\beta_2}\,\mathrm{d}x \leqslant C_{16}\big(T^*-t\big)^{-(\beta_1\beta_2)/d}. \tag{4.12}$$

Proof. Set

$$F_1(t) = \frac{1}{\beta_1}\int_\Omega u^{\beta_1}\,\mathrm{d}x, \qquad F_2(t) = \frac{1}{\beta_2}\int_\Omega u^{\beta_2}\,\mathrm{d}x. \tag{4.13}$$

A series of calculations yields

$$F_1'(t)=\int_\Omega u^{n_1-r_1-p_1}u_t\,\mathrm{d}x=\int_\Omega u^{n_1-p_1}\left(\Delta u+a_1u^{p_1}\int_\Omega v^{q_1}\,\mathrm{d}x\right)\mathrm{d}x,$$

$$\begin{aligned}F_1''(t)&=\int_\Omega (n_1-p_1)u^{n_1-p_1-1}u_t\left(\Delta u+a_1u^{p_1}\int_\Omega v^{q_1}\,\mathrm{d}x\right)\mathrm{d}x\\&\quad+\int_\Omega u^{n_1-p_1}\left(\Delta u_t+a_1p_1u^{p_1-1}u_t\int_\Omega v^{q_1}\,\mathrm{d}x+a_1u^{p_1}\int_\Omega q_1v^{q_1-1}v_t\,\mathrm{d}x\right)\mathrm{d}x\\&=(n_1-p_1)\int_\Omega u^{n_1-p_1-1-r_1}u_t^2\,\mathrm{d}x+\int_\Omega u^{n_1-p_1}\Delta u_t\,\mathrm{d}x\\&\quad+a_1p_1\int_\Omega v^{q_1}\,\mathrm{d}x\int_\Omega u^{n_1-1}u_t\,\mathrm{d}x+a_1q_1\int_\Omega u^{n_1}\,\mathrm{d}x\int_\Omega v^{q_1-1}v_t\,\mathrm{d}x.\end{aligned}$$

Since

$$\begin{aligned}&\int_\Omega u^{n_1-p_1}\Delta u_t\,\mathrm{d}x\\&\quad=\int_\Omega \Delta\big(u^{n_1-p_1}\big)u_t\,\mathrm{d}x\\&\quad=\int_\Omega (n_1-p_1)(n_1-p_1-1)u^{n_1-p_1-2}|\nabla u|^2u_t\,\mathrm{d}x+\int_\Omega (n_1-p_1)u^{n_1-p_1-1}\Delta uu_t\,\mathrm{d}x\\&\quad=\int_\Omega (n_1-p_1)(n_1-p_1-1)u^{n_1-p_1-2}|\nabla u|^2u_t\,\mathrm{d}x+(n_1-p_1)\int_\Omega u^{n_1-p_1-1-r_1}u_t^2\,\mathrm{d}x\\&\qquad-a_1(n_1-p_1)\int_\Omega v^{q_1}\,\mathrm{d}x\int_\Omega u^{n_1-1}u_t\,\mathrm{d}x,\end{aligned}$$

we have

$$\begin{aligned}F_1''(t)&\geqslant 2(n_1-p_1)\int_\Omega u^{\beta_1}(u_t)^2\,\mathrm{d}x+(2a_1p_1-a_1n_1)\int_\Omega u^{n_1-1}u_t\,\mathrm{d}x\int_\Omega v^{q_1}\,\mathrm{d}x\\&\quad+a_1q_1\int_\Omega u^{n_1}\,\mathrm{d}x\int_\Omega v^{q_1-1}v_t\,\mathrm{d}x.\end{aligned}$$

Similarly,

$$F_2'(t)=\int_\Omega v^{q_1-m_1}\left(\Delta v+b_1v^{m_1}\int_\Omega u^{n_1}\,\mathrm{d}x\right)\mathrm{d}x,$$

$$\begin{aligned}F_2''(t)&\geqslant 2(q_1-m_1)\int_\Omega v^{q_1-m_1-1-r_2}v_t^2\,\mathrm{d}x+(2b_1m_1-b_1q_1)\int_\Omega u^{n_1}\,\mathrm{d}x\int_\Omega v^{q_1-1}v_t\,\mathrm{d}x\\&\quad+b_1n_1\int_\Omega u^{n_1-1}u_t\,\mathrm{d}x\int_\Omega v^{q_1}\,\mathrm{d}x.\end{aligned}$$

Therefore

$$\begin{aligned}
&(b_1F_1+a_1F_2)'' \\
&\quad\geqslant 2b_1(n_1-p_1)\int_\Omega u^{n_1-p_1-1-r_1}u_t^2\,\mathrm{d}x+2a_1(q_1-m_1)\int_\Omega v^{q_1-m_1-1-r_2}v_t^2\,\mathrm{d}x \\
&\qquad+2a_1b_1p_1\int_\Omega u^{n_1-1}u_t\,\mathrm{d}x\int_\Omega v^{q_1}\,\mathrm{d}x+2a_1b_1m_1\int_\Omega u^{n_1}\,\mathrm{d}x\int_\Omega v^{q_1-1}v_t\,\mathrm{d}x \\
&\quad\geqslant 2b_1(n_1-p_1)\int_\Omega u^{n_1-p_1-1-r_1}u_t^2\,\mathrm{d}x+2a_1(q_1-m_1)\int_\Omega v^{q_1-m_1-1-r_2}v_t^2\,\mathrm{d}x.
\end{aligned}$$

Using the Hölder inequality we have

$$\begin{aligned}
F_1'(t)&\leqslant\left(\int_\Omega u^{n_1-r_1-p_1+1}\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega u^{n_1-p_1-1-r_1}u_t^2\,\mathrm{d}x\right)^{1/2}, \\
F_2'(t)&\leqslant\left(\int_\Omega v^{q_1-r_2-m_1+1}\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega v^{q_1-m_1-1-r_2}v_t^2\,\mathrm{d}x\right)^{1/2}.
\end{aligned}$$

Denote $\mu=\frac{d}{\beta_1\beta_2}$. Noticing $\frac{n_1}{1-r_1-p_1}=\frac{q_1}{1-r_2-m_1}$, it follows that

$$\begin{aligned}
&\frac{1}{\mu+1}(b_1F_1+a_1F_2)(b_1F_1+a_1F_2)'' \\
&\quad\geqslant\frac{1}{\mu+1}\left(\frac{b_1}{\beta_1}\int_\Omega u^{\beta_1}\,\mathrm{d}x+\frac{a_1}{\beta_2}\int_\Omega v^{\beta_2}\,\mathrm{d}x\right) \\
&\qquad\times\left(2b_1(n_1-p_1)\int_\Omega u^{n_1-p_1-1-r_1}u_t^2\,\mathrm{d}x+2a_1(q_1-m_1)\int_\Omega v^{q_1-m_1-1-r_2}v_t^2\,\mathrm{d}x\right) \\
&\quad\geqslant\frac{1}{\mu+1}\left(\frac{b_1}{\beta_1}\int_\Omega u^{\beta_1}\,\mathrm{d}x+\frac{a_1}{\beta_2}\int_\Omega v^{\beta_2}\,\mathrm{d}x\right) \\
&\qquad\times\left(2b_1(n_1-p_1)\frac{F_1'^2(t)}{\int_\Omega u^{\beta_1}\,\mathrm{d}x}+2a_1(q_1-m_1)\frac{F_2'^2(t)}{\int_\Omega v^{\beta_2}\,\mathrm{d}x}\right) \\
&\quad\geqslant\big(b_1F_1'+a_1F_2'\big)^2,
\end{aligned}$$

which yields

$$\begin{aligned}
\big((b_1F_1+a_1F_2)^{-\mu}\big)'&=-\mu(b_1F_1+a_1F_2)^{-\mu-1}(b_1F_1+a_1F_2)', \\
\big((b_1F_1+a_1F_2)^{-\mu}\big)''&=(-\mu)(-\mu-1)(b_1F_1+a_1F_2)^{-\mu-2}\big((b_1F_1+a_1F_2)'\big)^2 \\
&\quad-\mu(b_1F_1+a_1F_2)^{-\mu-1}(b_1F_1+a_1F_2)'' \\
&\leqslant\mu(b_1F_1+a_1F_2)^{-\mu-2}(b_1F_1+a_1F_2)(b_1F_1+a_1F_2)'' \\
&\quad-\mu(b_1F_1+a_1F_2)^{-\mu-1}(b_1F_1+a_1F_2)''=0.
\end{aligned}$$

Therefore

$$\big((b_1F_1 + a_1F_2)^{-\mu}\big)' \leqslant \big((b_1F_1 + a_1F_2)^{-\mu}\big)'\big|_{t=0} = -C_{17}, \tag{4.14}$$

where C_{17} is a positive constant. Integrating (4.14) over (t, T^*), we obtain

$$(b_1F_1 + a_1F_2)^{-\mu} \geqslant C_{17}\big(T^* - t\big).$$

Equation (4.12) thus follows. This completes the proof. □

Lemma 13. *Suppose that u_0, v_0 satisfy* (H1)–(H4) *and $\Delta u_0, \Delta v_0 \leqslant 0$, (u, v) is the solution of* (1.3) *defined on $\Omega \times (0, T^*)$, then $\Delta u, \Delta v \leqslant 0$ in any compact subset of $\Omega \times (0, T^*)$.*

Proof. The proof is similar to [6]. We omit the details. □

Lemma 14. *Assume that $u_0(x), v_0(x)$ satisfy* (H1)–(H3)*, $\Delta u_0, \Delta v_0 \leqslant 0$ and $p_1+r_1, m_1+r_2 < 1$. If either* (H4) *holds, or $\frac{n_1}{1-r_1-p_1} = \frac{q_1}{1-r_2-m_1}$ and $n_1 - p_1 - 1, q_1 - m_1 - 1 > 0$, then there exist positive constants c_1 and c_2, such that*

$$\begin{cases} c_1 \leqslant \big(T^* - t\big)^{(1-r_1-p_1)(q_1+1-r_2-m_1)/d} G_1(t) \leqslant c_2, \\ c_1 \leqslant \big(T^* - t\big)^{(1-r_2-m_1)(n_1+1-r_1-p_1)/d} G_2(t) \leqslant c_2, \end{cases} \tag{4.15}$$

where $d = q_1n_1 - (1 - r_1 - p_1)(1 - r_2 - m_1)$.

Proof. If u_0, v_0 satisfy (H1)–(H4) and $\Delta u_0, \Delta v_0 \leqslant 0$, then (4.15) follows from (4.8) and (4.11).

If $\frac{n_1}{1-r_1-p_1} = \frac{q_1}{1-r_2-m_1}$, using the inequality

$$(b_1 + b_2)^k \leqslant C(k)\big(b_1^k + b_2^k\big), \quad b_1, b_2 \geqslant 0,\ m > 1,\ C(k) = 2^{k-1}, \tag{4.16}$$

it follows from (4.11) that

$$u^{n_1} \leqslant \big[(1 - r_1 - p_1)G_1(t) + U^{1-r_1-p_1}(0)\big]^{\frac{n_1}{1-r_1-p_1}} \leqslant C_{18} + C_{19}G_1^{\frac{n_1}{1-r_1-p_1}}(t). \tag{4.17}$$

Integrating (4.17) over Ω, we get

$$G_2'(t) = g_2(t) = b_1 \int_\Omega u^{n_1}\,\mathrm{d}x \leqslant b_1|\Omega|\big[C_{18} + C_{19}G_1^{n_1/(1-r_1-p_1)}(t)\big].$$

By the proof of Lemma 11, there exist $t_0 \in (0, T^*)$ and positive constants C_{20} and C_{21} such that

$$G_2'(t) \leqslant C_{20}G_1^{\frac{n_1}{1-r_1-p_1}}(t), \quad G_1'(t) \leqslant C_{21}G_2^{\frac{q_1}{1-r_2-m_1}}(t), \quad \forall t \in \big(t_0, T^*\big).$$

Noting that $\frac{n_1}{1-r_1-p_1} = \frac{q_1}{1-r_2-m_1}$, we have

$$\big((G_1 + G_2)^{1-\frac{q_1}{1-r_2-m_1}}\big)' \geqslant -C_{22}\frac{q_1 + r_2 + m_1 - 1}{1 - r_2 - m_1}, \tag{4.18}$$

where $C_{22} = \max\{C_{20}, C_{21}\}$. Integrating (4.18) over (t, T^*), it yields

$$G_1(t) + G_2(t) \geqslant C_{23}\big(T^* - t\big)^{-(1-r_2-m_1)/(q_1+r_2+m_1-1)}, \tag{4.19}$$

where $C_{23} = (C_{22}\frac{q_1+r_2+m_1-1}{1-r_2-m_1})^{-\frac{1-r_2-m_1}{q_1+r_2+m_1-1}}$.

By the Hölder inequality and $\frac{n_1}{1-r_1-p_1}=\frac{q_1}{1-r_2-m_1}$, we obtain

$$g_1(t)=a_1\int_\Omega v^{q_1}\,\mathrm{d}x\leqslant a_1\Big(\int_\Omega v^{q_1+1-r_2-m_1}\Big)^{q_1/(q_1+1-r_2-m_1)}|\Omega|^{1-\frac{q_1+1-r_2-m_1}{q_1}}$$
$$\leqslant C_{24}\big(T^*-t\big)^{-\frac{(n_1+1-r_1-p_1)q_1}{d}}=C_{24}\big(T^*-t\big)^{-\frac{n_1(q_1+1-r_2-m_1)}{d}},$$

where C_{24} is a positive constant. Integrating above inequality over $(0,t)$ yields

$$G_1(t)\leqslant C_{24}\int_0^t\big(T^*-s\big)^{-\frac{n_1(q_1+1-r_2-m_1)}{d}}\,\mathrm{d}s\leqslant C_{25}\big(T^*-t\big)^{-\frac{1-r_2-m_1}{q_1-1+r_2+m_1}}. \tag{4.20}$$

Similarly,

$$G_2(t)\leqslant C_{25}\big(T^*-t\big)^{-\frac{1-r_2-m_1}{q_1-1+r_2+m_1}}. \tag{4.21}$$

From (1.3), (4.3) and Lemma 13, we have

$$a_1\int_\Omega v^{q_1}\,\mathrm{d}x\geqslant\delta u^{k_1+1-r_1-p_1},\qquad b_1\int_\Omega u^{n_1}\,\mathrm{d}x\geqslant\delta v^{k_2+1-r_2-m_1}. \tag{4.22}$$

Noting $\frac{n_1}{1-r_1-p_1}=\frac{q_1}{1-r_2-m_1}$, we obtain $k_1+1-r_1-p_1=n_1$ and $k_2+1-r_2-m_1=q_1$. Integrating (4.22) over $\Omega\times(0,t)$, we obtain

$$|\Omega|G_1(t)\geqslant\delta G_2(t),\qquad |\Omega|G_2(t)\geqslant\delta G_1(t). \tag{4.23}$$

From (4.19), (4.20), (4.21) and (4.23), the assertion of Lemma 14 follows. The proof is completed. □

By Lemma 14, we can obtain Theorem 6 immediately. The proof of Theorem 7 will be given by the following Lemma 16.

Lemma 15. *Under the assumptions of Lemma* 14*, if* $(1-p_1)(1-r_2-m_1)<q_1(n_1-r_1)$ *and* $(1-m_1)(1-r_1-p_1)<n_1(q_1-r_2)$*, then*

$$\lim_{t\to T^*}\frac{u^{1-p_1-r_1}(x,t)}{(1-r_1-p_1)G_1(t)}=\lim_{t\to T^*}\frac{\|u(\cdot,t)\|_\infty^{1-p_1-r_1}}{(1-r_1-p_1)G_1(t)}=1, \tag{4.24}$$
$$\lim_{t\to T^*}\frac{v^{1-r_2-m_1}(x,t)}{(1-r_2-m_1)G_2(t)}=\lim_{t\to T^*}\frac{\|v(\cdot,t)\|_\infty^{1-r_2-m_1}}{(1-r_2-m_1)G_2(t)}=1 \tag{4.25}$$

uniformly on compact subsets of Ω.

Proof. Here we consider the first eigenvalue problem

$$-\Delta\varphi=\lambda_1\varphi,\quad x\in\Omega;\qquad \varphi(x)=0,\quad x\in\partial\Omega.$$

Normalize $\varphi(x)$ as $\varphi(x)>0$ in Ω and $\int_\Omega\varphi(x)\,\mathrm{d}x=1$. Define

$$z(x,t)=G_1(t)-\frac{1}{1-r_1-p_1}u^{1-r_1-p_1}(x,t)$$

and

$$\gamma(t) = \int_\Omega z(y,t)\varphi(y)\,\mathrm{d}y.$$

A series of computations yields

$$\begin{aligned}
\gamma'(t) &= \int_\Omega \big(g_1(t) - u^{-p_1-r_1}u_t\big)\varphi(y)\,\mathrm{d}y = -\int_\Omega u^{-p_1}(y,t)\Delta u(y,t)\varphi(y)\,\mathrm{d}y \\
&= \lambda_1 \int_\Omega u^{1-p_1}(y,t)\varphi(y)\,\mathrm{d}y \\
&= \lambda_1(1-r_1-p_1)^{(1-p_1)/(1-r_1-p_1)} \int_\Omega \big(G_1(t) - z(y,t)\big)^{(1-p_1)/(1-r_1-p_1)}\varphi(y)\,\mathrm{d}y \\
&\leqslant \lambda_1(1-r_1-p_1)^{(1-p_1)/(1-r_1-p_1)} \int_\Omega \big(G_1(t) + z^-(y,t)\big)^{(1-p_1)/(1-r_1-p_1)}\varphi(y)\,\mathrm{d}y \\
&\leqslant C_{26}\left(G_1^{(1-p_1)/(1-r_1-p_1)}(t) + \int_\Omega \big(z^-(y,t)\big)^{(1-p_1)/(1-r_1-p_1)}\varphi(y)\,\mathrm{d}y\right),
\end{aligned}$$

where $z^- = \max\{-z, 0\}$. By (4.11), we know that

$$\inf_\Omega z(x,t) \geqslant -C, \tag{4.26}$$

which means $z^-(x,t) \leqslant C$. Then

$$\gamma'(t) \leqslant C_{27}G_1^{(1-p_1)/(1-r_1-p_1)}(t) + C_{28}. \tag{4.27}$$

Integrating (4.27) between 0 to t yields

$$\gamma(t) \leqslant C_{29}\left(1 + \int_0^t G_1^{(1-p_1)/(1-r_1-p_1)}(s)\,\mathrm{d}s\right). \tag{4.28}$$

Thus (4.26) and (4.28) implies

$$\int_\Omega \big|z(y,t)\big|\varphi(y)\,\mathrm{d}y \leqslant C_{30}\left(1 + \int_0^t G_1^{(1-p_1)/(1-r_1-p_1)}(s)\,\mathrm{d}s\right). \tag{4.29}$$

Define $K_\rho = \{y \in \Omega\colon \operatorname{dist}(y, \partial\Omega) \geqslant \rho\}$. Since $-\Delta z \leqslant 0$ in $\Omega \times (0, T^*)$. Using Lemma 4.5 in [23], we obtain

$$\sup_{K_\rho} z(x,t) \leqslant \frac{C_{30}}{\rho^{N+1}}C_{30}\left(1 + \int_0^t G_1^{(1-p_1)/(1-r_1-p_1)}(s)\,\mathrm{d}s\right). \tag{4.30}$$

It follows from (4.11) and (4.30) that

$$-\frac{c}{G_1(t)} \leqslant 1 - \frac{u^{1-r_1-p_1}(x,t)}{(1-r_1-p_1)G_1(t)} \leqslant \frac{C(1 + \int_0^t G_1^{(1-p_1)/(1-r_1-p_1)}(s)\,\mathrm{d}s)}{G_1(t)}, \tag{4.31}$$

for any $x \in K_\rho$, where c and C are positive constants. We know from (4.15) that

$$C_{31} \leqslant G_1^{(1-p_1)/(1-r_1-p_1)}(t)\big(T^* - t\big)^{(1-p_1)(q_1+1-r_2-m_1)/d} \leqslant C_{32},$$
$$C_{33} \leqslant G_1'(t)\big(T^* - t\big)^{(1-r_1-p_1)(q_1+1-r_2-m_1)/d+1} \leqslant C_{34},$$

where C_{31}, C_{32}, C_{33} and C_{34} are positive constants. Noting that

$$(1-p_1)(q_1+1-r_2-m_1)/d < (1-r_1-p_1)(q_1+1-r_2-m_1)/d+1$$
$$\iff \quad (1-p_1)(1-r_2-m_1) < q_1(n_1-r_1),$$

then

$$\lim_{t\to T^*} \frac{\int_0^t G_1(s)^{(1-p_1)/(1-r_1-p_1)}\,\mathrm{d}s}{G_1(t)} = \lim_{t\to T^*} \frac{G_1(t)^{(1-p_1)/(1-r_1-p_1)}}{G_1'(t)} = 0.$$

Thus

$$\lim_{t\to T^*} \frac{u^{1-r_1-p_1}(x,t)}{(1-r_1-p_1)G_1(t)} = \lim_{t\to T^*} \frac{\|u(\cdot,t)\|_\infty^{1-r_1-p_1}}{(1-r_1-p_1)G_1(t)} = 1.$$

Similarly,

$$\lim_{t\to T^*} \frac{v^{1-r_2-m_1}(x,t)}{(1-r_2-m_1)G_2(t)} = \lim_{t\to T^*} \frac{\|v(\cdot,t)\|_\infty^{1-r_2-m_1}}{(1-r_2-m_1)G_2(t)} = 1. \qquad \square$$

Lemma 16. *Under the assumptions of Lemma* 15*, we have that*

$$\lim_{t\to T^*}\big(T^*-t\big)^{(q_1+1-r_2-m_1)/d}u(x,t) = (a_1|\Omega|d)^{-(q_1+1-r_2-m_1)/d}(n_1+1-r_1-p_1)^{q_1/d} \times \left(\frac{a_1(q_1+1-r_2-m_1)}{b_1}\right)^{(1-r_2-m_1)/d},$$
$$\lim_{t\to T^*}\big(T^*-t\big)^{(n_1+1-r_1-p_1)/d}v(x,t) = (b_1|\Omega|d)^{-(n_1+1-r_1-p_1)/d}(q_1+1-r_2-m_1)^{n_1/d} \times \left(\frac{b_1(n_1+1-r_1-p_1)}{a_1}\right)^{(1-r_1-p_1)/d}$$

uniformly on compact subsets of Ω, where $d = q_1n_1 - (1-r_1-p_1)(1-r_2-m_1)$.

Proof. By Lemma 15, it follows that for any $x \in \Omega$,

$$\lim_{t\to T^*} \frac{u^{n_1}}{((1-r_1-p_1)G_1(t))^{n_1/(1-r_1-p_1)}} = 1,$$
$$\lim_{t\to T^*} \frac{v^{q_1}}{((1-r_2-m_1)G_2(t))^{q_1/(1-r_2-m_1)}} = 1,$$

which implies that as $t \to T^*$,

$$G_1'(t) = a_1\int_\Omega v^{q_1}\,\mathrm{d}x \sim a_1|\Omega|\big((1-r_2-m_1)G_2(t)\big)^{q_1/(1-r_2-m_1)},$$
$$G_2'(t) = b_1\int_\Omega u^{n_1}\,\mathrm{d}x \sim b_1|\Omega|\big((1-r_1-p_1)G_1(t)\big)^{n_1/(1-r_1-p_1)},$$

where the notation $u \sim v$ means $\lim_{t\to T^*} u(t)/v(t) = 1$. Hence, we obtain

$$\begin{aligned} G_1(t) \sim{}& (1-r_1-p_1)^{-1}(a_1|\Omega|d)^{-(1-r_1-p_1)(q_1+1-r_2-m_1)/d} \\ &\times (1+n_1-r_1-p_1)^{q_1(1-r_1-p_1)/d} \\ &\times \left(\frac{a_1(q_1+1-r_2-m_1)}{b_1}\right)^{(1-r_1-p_1)(1-r_2-m_1)/d} \\ &\times (T^*-t)^{-(1-r_1-p_1)(q_1+1-r_2-m_1)/d}, \\ G_2(t) \sim{}& (1-r_2-m_1)^{-1}(b_1|\Omega|d)^{-(1-r_2-m_1)(n_1+1-r_1-p_1)/d} \\ &\times (1+q_1-r_2-m_1)^{n_1(1-r_2-m_1)/d} \\ &\times \left(\frac{b_1(n_1+1-r_1-p_1)}{a_1}\right)^{(1-r_2-m_1)(1-r_1-p_1)/d} \\ &\times \left(T^*-t\right)^{-(1-r_2-m_1)(n_1+1-r_1-p_1)/d}. \end{aligned}$$

Combining with Lemma 15, we obtain the result of Lemma 16 immediately. □

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