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Undecidable problems for completely 0-simple semigroups ☆

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ABSTRACT

The decidability of the uniform word problem in a pseudovariety $\mathscr{V}$ of groups is known to be equivalent to the decidability of a number of embedding problems for finite semigroups in classes related to completely 0-simple semigroups with subgroups from $\mathscr{V}$. Unfortunately the existing proof of some of these equivalences turns out to contain an error. We provide an update on these types of properties, correcting the error and amending results accordingly. Several new undecidability properties are exhibited, some superceding previous results. A theme of many of the new contributions is problems that are solvable in polynomial time but become undecidable when a facet of their description is restricted by some finiteness condition.

0. Background and overview

Recall that a semigroup is *0-simple* if it contains only a single nonzero ideal. In a finite 0-simple semigroup, the nonzero idempotent elements satisfy the implication $ef = fe = e \to e = f$, and in general a 0-simple semigroup with this property on idempotents is said to be *completely 0-simple*. As we now recall, the classical Rees-Sushkevich Theorem gives the class of completely 0-simple semigroups a transparent characterisation. (While we recall this construction here, later proofs will assume some familiarity with the isomorphism theory of Rees matrix semigroups, and with Green's relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$, $\mathscr{J}$ and their manifestation on Rees matrix semigroups. For details, refer to the early chapters of any general semigroup theory text; Howie [7] for example.)

Let G be a nonempty set, $\mathbf{G}$ a group (the *structure group*) on G and $0 \notin G$. Let I, J be nonempty sets, and let $P = (P_{j,i})$ be a $J \times I$ matrix (the *sandwich matrix*) over the set $G \cup \{0\}$. If $|J| = \mu$, $|I| = \nu$, we may refer to P as to a matrix of dimension $\mu \times \nu$ or simply as to a $\mu \times \nu$ matrix. The *Rees matrix semigroup with zero* $M^0[\mathbf{G}, P]$ is the semigroup on the set $(I \times G \times J) \cup \{0\}$ with multiplication

$$a \cdot 0 = 0 \cdot a = 0 \quad \text{for all } a \in (I \times G \times J) \cup \{0\}, \text{ and}$$

$$(i_1, g_1, j_1) \cdot (i_2, g_2, j_2) = \begin{cases} 0 & \text{if } P_{j_1,i_2} = 0, \\ (i_1, g_1 P_{j_1,i_2} g_2, j_2) & \text{otherwise.} \end{cases}$$

The Rees-Sushkevich Theorem (see [7, Theorem 3.3.1]) states that, up to isomorphism, the completely 0-simple semigroups are precisely the Rees matrix semigroups with zero and for which each row and each column of the sandwich matrix contains a nonzero element. Note that the maximal subgroups of $M^0[\mathbf{G}, P]$ are all isomorphic to $\mathbf{G}$.

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A *Brandt semigroup* $\mathbf{B}_\kappa(\mathbf{G})$ is a completely 0-simple semigroup isomorphic to $M^0[\mathbf{G}, E_\kappa]$, where E_κ is the $\kappa \times \kappa$ identity matrix (using the group identity for 1). We will say that a completely 0-simple semigroup $\mathbf{S}$ is *pure* if $\mathbf{S}$ is isomorphic to a Rees matrix semigroup $M^0[\mathbf{G}, P]$ (for some group $\mathbf{G}$ with identity 1) such that the sandwich matrix P is over $\{0, 1\}$. An *orthodox* completely 0-simple semigroup is one in which the idempotent elements form a subsemigroup. Such a semigroup is necessarily pure (but the reverse is not true).

For any non-empty class $\mathscr{K}$ of groups (closed under isomorphism), we let $\mathsf{Br}(\mathscr{K})$ denote the class of all Brandt semigroups with maximal subgroups from $\mathscr{K}$, and we let $\mathsf{CS}^0(\mathscr{K})$ (and $\mathsf{PCS}^0(\mathscr{K})$) denote the class of completely 0-simple semigroups (pure completely 0-simple semigroups, respectively) with maximal subgroups from $\mathscr{K}$. The notation $\mathsf{Br}_m(\mathscr{K})$ denotes the subclass of $\mathsf{Br}(\mathscr{K})$ consisting of Brandt semigroups whose sandwich matrix has dimension $m \times m$. Similarly we define $\mathsf{CS}^0_{m,n}(\mathscr{K})$ to correspond to completely 0-simple semigroups whose sandwich matrices are of dimension $m \times n$. We will refer to semigroups from $\mathsf{Br}_m(\mathscr{K})$, respectively $\mathsf{CS}^0_{m,n}(\mathscr{K})$, as to $m \times m$ Brandt semigroups, respectively $m \times n$ completely 0-simple semigroups, over groups in $\mathscr{K}$. We also let Nil_k denote the class of all k-nilpotent semigroups (that is, semigroups in which every product of length k has the same value 0), and $\mathscr{G}$ denote the class of all groups. A class $\mathscr{V}$ of groups is called a *pseudovariety* if $\mathscr{V}$ is closed under taking homomorphic images of subgroups and forming finitary direct products.[1] We let Triv denote the pseudovariety consisting of the trivial group. In general for a class $\mathscr{K}$, the class $\mathscr{K}_{\text{fin}}$ denotes the finite members of $\mathscr{K}$.

In order to discuss our results it is necessary to recall some basic concepts and notation. We use the notations $\mathbb{S}, \mathbb{P}, \mathbb{P}_u$ to denote the class operators of taking isomorphic copies of subsemigroups, of taking direct products and of taking ultraproducts respectively. The notation $\mathbb{Q}$ abbreviates $\mathbb{SPP}_u$, which is the class operator producing the *quasivariety* generated by a class of semigroups (see Burris and Sankappanavar [2, Section V.2] for further details on this and related concepts; our presentation is tailored to the semigroup theoretic setting). The quasivariety $\mathbb{Q}(\mathscr{K})$ is the class of all semigroups satisfying the *quasiequations* holding in the class $\mathscr{K}$: these are the universally quantified implications of the form $(\&_{1\le i\le n} u_i \approx v_i) \to u \approx v$, where $u, v, u_1, v_1, \ldots$ are semigroup words. The *universal class* generated by $\mathscr{K}$ is the class $\mathbb{SP}_u(\mathscr{K})$, or equivalently, the class of all semigroups satisfying the *universal theory* of $\mathscr{K}$ (the universally quantified prenex-form first order sentences true in $\mathscr{K}$). Note that if the class $\mathscr{K}$ is already closed under taking ultraproducts (as is true for $\mathsf{CS}^0(\mathscr{G})$ and $\mathsf{Br}(\mathscr{G})$ for example), then the quasivariety generated by $\mathscr{K}$ is just $\mathbb{SP}(\mathscr{K})$ while the universal class of $\mathscr{K}$ is just $\mathbb{S}(\mathscr{K})$.

The highly descriptive Rees-Sushkevich Theorem makes the following result of [5] completely unexpected.

Theorem A. *For every pseudovariety of groups $\mathscr{V}$ and for any natural numbers m, n the following are equivalent.*

(A1) *The uniform word problem in $\mathscr{V}$ is algorithmically solvable.*
(A2) *The set of finite subsemigroups of completely 0-simple semigroups over groups in $\mathscr{V}$ is recursive.*
(A3) *The set of finite subsemigroups of Brandt semigroups over groups in $\mathscr{V}$ is recursive.*
(A4) *The set of finite 4-nilpotent subsemigroups of completely 0-simple semigroups over groups in $\mathscr{V}$ is recursive.*
(A5) *The set of finite 3-nilpotent subsemigroups of Brandt semigroups over groups in $\mathscr{V}$ is recursive.*
(A6) *The set of finite 4-nilpotent subsemigroups of direct products of completely 0-simple semigroups over groups in $\mathscr{V}$ is recursive.*
(A8) *The set of finite subsemigroups of $m \times n$ completely 0-simple semigroups over groups in $\mathscr{V}$ is recursive provided $m, n \ge 3$.*
(A9) *The set of finite subsemigroups of $m \times m$ Brandt semigroups over groups in $\mathscr{V}$ is recursive provided $m \ge 3$.*
(A10) *The set of finite 4-nilpotent subsemigroups of $m \times n$ completely 0-simple semigroups over groups in $\mathscr{V}$ is recursive provided $m, n \ge 4$.*
(A11) *The set of finite 3-nilpotent subsemigroups of $m \times m$ Brandt semigroups over groups in $\mathscr{V}$ is recursive provided $m \ge 3$.*
(A12) *The set of finite 4-nilpotent subsemigroups of direct products of $m \times n$ completely 0-simple semigroups over groups in $\mathscr{V}$ is recursive provided $m, n \ge 4$.*

In [5], Theorem A includes two further conditions.

(A7) *The set of finite 3-nilpotent subsemigroups of direct products of Brandt semigroups over groups in $\mathscr{V}$ is recursive.*
(A13) *The set of finite 3-nilpotent subsemigroups of direct products of $m \times m$ Brandt semigroups over groups in $\mathscr{V}$ is recursive, provided $m \ge 3$.*

Unfortunately, there is an error in the proof of equivalence of conditions (A1)–(A13) given in [5], and we will show that condition (A7) is not equivalent to (A1) after all. The error in [5] takes the form of an unproved observation concerning a certain 3-nilpotent semigroup and leaves the equivalence of (A1) with conditions (A5), (A11) and (A13) also unproved. The error occurs in the second sentence of the proof of case (b) in Lemma 2.3 of [5]; where it is claimed that the idempotents e_1, e_2, e_3 were not used in the proof of case (a). The corresponding steps of proof are also omitted in the proof of case (c) of the same lemma, which is crucial to the proof of the equivalence of condition (A1) with conditions (A4), (A6), (A10), (A12). However in this case the proof steps can indeed be carried out; but while there is no explicit error here, a proof of

[1] While it is now becoming common in the literature to restrict to the notion of pseudovariety to consist of finite algebras only, we do not need this restriction here.

this fact now seems appropriate. In this article we provide the missing details in all of these proofs. With the exception of condition (A13), the proof of the details are all achieved using the original constructions and proof methods presented in [5]. We mention that the equivalence of condition (A1) with conditions (A2), (A3), (A8) and (A9) – arguably the most important parts of Theorem A – are correctly established in [5]. We also stress that the general techniques employed in [5] are correct and have been very influential (see, e.g., [8,9,13,14,17,18] for various applications of these techniques), and we are not aware of any article whose results depend on the false equivalence (A1) $\Leftrightarrow$ (A7).

As well as completing some missing details in the proof of Theorem A, we provide the following new contributions.

Theorem A^+**.** *For every pseudovariety of groups $\mathscr{V}$ and for any natural numbers m, n, the following are equivalent to the conditions of Theorem A.*

(A13) *The set of finite 3-nilpotent subsemigroups of direct products of $m \times m$ Brandt semigroups over groups in $\mathscr{V}$ is recursive, provided $m \geq 3$.*
(A14) *The set of finite 3-nilpotent subsemigroups of direct products of $m \times n$ completely 0-simple semigroups over groups in $\mathscr{V}$ is recursive, provided $m, n \geq 3$.*
(A15) *The set of finite 3-nilpotent subsemigroups of $m \times n$ completely 0-simple semigroups over groups in $\mathscr{V}$ is recursive, provided $m, n \geq 3$.*

Condition (A13) was already mentioned above. A new construction seems to be needed to prove the equivalence of this condition with condition (A1) of Theorem A, and so we have listed it as a new result. Conditions (A14) and (A15) are proper strengthenings of conditions (A12) and (A10), respectively.

Recall that a class $\mathscr{K}$ of semigroups has *polynomial time finite membership problem*, if there is a polynomial p and an algorithm that when presented with a finite semigroup **S**, decides membership of **S** in $\mathscr{K}$ in at most $p(|S|)$ steps. The next result demonstrates why the conditions of Theorem A^+ are particularly surprising: each of the cases (B1)–(B3) shows that with the absence of the bounds on the "dimension" of the completely 0-simple semigroup or Brandt semigroup, not only do the analogous classes have polynomial time finite membership problems, they are essentially independent of the choice of the pseudovariety $\mathscr{V}$ (compare conditions (A13), (A14), (A15) with (B1), (B2) and (B3) respectively).

Theorem B. *Let $\mathscr{V}$ be a pseudovariety of groups.*

(B1) *The quasivariety $\mathsf{Nil}_3 \cap \mathbb{Q}(\mathsf{Br}(\mathscr{V}))$ of all 3-nilpotent semigroups in the quasivariety generated by the Brandt semigroups over $\mathscr{V}$ coincides with the quasivariety $\mathsf{Nil}_3 \cap \mathbb{Q}(\mathsf{Br}(\mathsf{Triv}))$, has no finite axiomatisation but has polynomial time finite membership problem.*
(B2) *The quasivariety $\mathsf{Nil}_3 \cap \mathbb{Q}(\mathsf{CS}^0(\mathscr{V}))$ of all 3-nilpotent semigroups in the quasivariety generated by the completely 0-simple semigroups over $\mathscr{V}$ coincides with the quasivariety $\mathsf{Nil}_3 \cap \mathbb{Q}(\mathsf{CS}^0(\mathsf{Triv}))$, can be axiomatised within Nil_3 by*

$$xy \approx yz \rightarrow xy \approx 0,$$

and hence has polynomial time finite membership problem.
(B3) *(Hall et al. [5]) If $\mathscr{V}$ is non-trivial, then the universal class $\mathsf{Nil}_3 \cap \mathbb{SP}_{\mathrm{u}}(\mathsf{CS}^0(\mathscr{V}))$ of all 3-nilpotent semigroups in the universal class generated by completely 0-simple semigroups over $\mathscr{V}$ does not depend on $\mathscr{V}$, can be axiomatised within Nil_3 by*

$$xy \approx 0 \vee yz \approx 0,$$

and thus has polynomial time finite membership problem.
(B3′) *The universal class $\mathsf{Nil}_3 \cap \mathbb{SP}_{\mathrm{u}}(\mathsf{CS}^0(\mathsf{Triv}))$ of all 3-nilpotent semigroups in the universal class generated by the completely 0-simple semigroups with trivial subgroups has no finite axiomatisation but has polynomial time finite membership problem.*

Condition (B1) here is why condition (A7) is not equivalent to the other conditions in Theorems A and A^+. Observe that aside from the one exception (B3′), conditions (B1), (B2) and (B3) give classes that are independent of the choice of the pseudovariety $\mathscr{V}$. Condition (B3′) is also something of a surprise since not only is the corresponding class distinct from that obtained for all other pseudovarieties, it is not finitely axiomatisable! As a further contrast with condition (B3), we will show that the class of finite 3-nilpotent subsemigroups of pure completely 0-simple semigroups over a pseudovariety $\mathscr{V}$ of groups is not recursive provided that $\mathscr{V}$ has undecidable uniform word problem. So, for example, the class of finite 3-nilpotent subsemigroups of orthodox completely 0-simple semigroups is not recursive. This shows that the polynomial time decidability described in Theorem B part (B3) strongly depends on the ability to place distinct group elements in the sandwich matrix of the embedding completely 0-simple semigroups.

Completely 0-simple semigroup with trivial subgroups are commonly referred to as *combinatorial completely 0-simple semigroups*. Because the uniform word problem in Triv is trivially decidable, the forward implications of Theorem A show that membership in either the quasivariety or universal class generated by $\mathsf{CS}^0(\mathsf{Triv})$ or by $\mathsf{Br}(\mathsf{Triv})$ is decidable. The following theorem gives a more precise description of these classes and the complexity of their finite membership problem.

Theorem C. *The universal classes $\mathbb{S}(\mathsf{CS}^0(\mathsf{Triv}))$ and $\mathbb{S}(\mathsf{Br}(\mathsf{Triv}))$ generated by combinatorial completely 0-simple semigroups and combinatorial Brandt semigroups (respectively) have polynomial time finite membership problems. However no class containing the universal class $\mathsf{Nil}_3 \cap \mathbb{S}(\mathsf{Br}(\mathsf{Triv}))$ of 3-nilpotent subsemigroups of combinatorial Brandt semigroups and contained within $\mathbb{S}(\mathsf{CS}^0(\mathsf{Triv}))$ has a finite axiomatisation for its universal theory.*

In Section 6 we describe axiomatisations for both $\mathsf{CS}^0(\mathsf{Triv})$ and $\mathsf{Br}(\mathsf{Triv})$ in the universal first order logic of semigroups with 0.

We also give present a number of other undecidability results concerning basic embeddability problems for finite semigroups. These results are not observed in [5], however researchers familiar with the methods of that article will immediately recognise the results as corollaries of the techniques presented in [5]. Since we are here revisiting the article [5], it seems appropriate to state these corollaries now. In the following, the *height* of a finite regular semigroup $\mathbf{S}$ is the height of the ordered set of $\mathscr{J}$-classes of $\mathbf{S}$; in a finite inverse semigroup this is equal to the height of the semilattice of idempotents. A subsemigroup $\mathbf{S}$ of a semigroup $\mathbf{T}$ is said to be *full* if $\mathbf{S}$ contains all idempotent elements of $\mathbf{T}$. Let Inv denote the class of inverse semigroups (in the type $\langle 2\rangle$) and Reg denote the class of regular semigroups.

Theorem D. *Let $\mathscr{U}$ be any class of regular semigroups containing $\mathsf{Br}_3(\mathscr{G}_{\mathrm{fin}})$, and $\mathscr{V}$ be any class of finite regular semigroups containing $\mathsf{Inv}_{\mathrm{fin}}$. For any $n \geq 1$, there is no algorithm to decide membership of finite semigroups in any of the following classes:*

(D1) *the class of full subsemigroups of semigroups in $\mathscr{U}$;*
(D2) *the class of subsemigroups of members of $\mathscr{V}$ of height at most n;*
(D3) *the class of subsemigroups of members of $\mathscr{V}$ with at most $n + 1$ distinct $\mathscr{J}$-classes.*

Theorem D is in stark contrast to basic semigroup theoretic facts, in a way analogous to the contrast between Theorems A$^+$ and B: without the bounds (fullness, height, number of $\mathscr{J}$-classes), the result fails to be true for many of the obvious choices of $\mathscr{U}$ and $\mathscr{V}$. For example, $\mathscr{U}$ can be any of $\mathsf{Inv}_{\mathrm{fin}}$, Inv, $\mathsf{Reg}_{\mathrm{fin}}$, Reg, while $\mathscr{V}$ can be either of $\mathsf{Inv}_{\mathrm{fin}}$ or $\mathsf{Reg}_{\mathrm{fin}}$. Indeed, it is one of the most well known facts in semigroup theory that every finite semigroup embeds into a finite regular semigroup (the semigroup of all transformations on a finite set), whence the finite subsemigroups of members of $\mathsf{Reg}_{\mathrm{fin}}$ and Reg are nothing other than the class of finite semigroups. For the inverse case, results of Schein show that the classes of the finite subsemigroups of members of $\mathsf{Inv}_{\mathrm{fin}}$ and Inv coincide and have polynomial time membership problems (cf. [19,20]; see also [21] for details, proofs and further references on this topic, and see [23] for an alternative algorithm). We mention that when $\mathscr{U} = \mathsf{Inv}$, the result (D1) is established by Gould and Kambites [4].

The paper is structured as follows. We begin in Section 1 by detailing background information on the uniform word problem for groups, and how it is to be interpreted in finite semigroups via the notion of a partial group. Section 2 explains the forward implications (A1) $\Rightarrow$ (Ai) for $i = 1, \ldots, 15$. Section 3 uses the constructions and methods of [5] to establish the remaining unproved implications of Theorem A (as explained above), with the exception of (A7) and (A13). In Section 4 we prove Theorem A$^+$ (that is, conditions (A13)–(A15)). In Section 5 we skip to the proof of Theorem D, whose proof is along similar lines to those of the previous sections. Theorem C is proved in Section 6; part (B3′) of Theorem B is a special case of this. The remaining three parts of Theorem B are proved in the final Section 7.

1. Preliminaries: Partial algebras and the uniform word problem

We formulate our notion of a group presentation in the language of monoids. Let A be a set of symbols and A' be a disjoint copy of A (with bijective correspondence $a \mapsto a'$, say). A group presentation Π is a pair $\langle A \cup A' \mid R\rangle$, where R is a set of equalities of the form $u = 1$, for some semigroup word u over the alphabet $A \cup A'$, and where R includes the equalities $aa' = 1$ and $a'a = 1$ for each $a \in A$. The presentation is finite if the sets A and R are finite. It is clear that every group presentation Π (in the language of groups in the type $\langle 2, 1, 0\rangle$, say) is equivalent to a monoid presentation of this form (in the type $\langle 2, 0\rangle$), where each a and a' are interpreted as inverse of each other and $(a_1 \ldots a_n)^{-1}$ is interpreted as $a'_n a'_{n-1} \ldots a'_1$. An *interpretation* of the presentation $\langle A \cup A' \mid R\rangle$ in a group $\mathbf{G}$ is a homomorphism ϕ from the free monoid on the generators $A \cup A'$ into the group $\mathbf{G}$ for which the formal equalities in R become true in $\mathbf{G}$ (it is obvious that each a' must be mapped to the inverse of a, since $\phi(a)\phi(a') = \phi(a')\phi(a) = 1$ holds).

We also use the following definition of the *uniform word problem* for a class $\mathscr{K}$ of groups.

INSTANCE: a pair Π, w, where $\Pi = \langle A \cup A' \mid R\rangle$ is a finite group presentation and w is a semigroup word in the alphabet $A \cup A'$.

QUESTION: Is w equal to 1 in every interpretation of Π in the groups from $\mathscr{K}$?

Following [5], a *partial group* $\mathbf{A}$ is a set A together with a partially defined binary operation $\cdot$ of multiplication (often written as concatenation) and with an element 1 acting as a multiplicative identity element. In this article, we make the additional assumption that a partial group contains no violations of associativity: in other words, if the products $(ab)c$ and $a(bc)$ are both defined in $\mathbf{A}$, then they are equal. A *homomorphism* of a partial group $\mathbf{A}$ into a group $\mathbf{G}$ is a map from A into G satisfying $\phi(a \cdot b) = \phi(a)\phi(b)$ whenever $a \cdot b$ is defined in $\mathbf{A}$. Since the element 1 is idempotent in $\mathbf{A}$ this element will necessarily be mapped to the identity element of $\mathbf{G}$ under such a homomorphism. A homomorphism will be called an *embedding* if it is injective. Note that not every partial group is embeddable in a group however every partially defined groupoid with identity element that is embeddable in a group is necessarily a partial group.

The following important fact is due to Evans [3] (here particularised to the present setting).

Evans' Connection. *Let $\mathscr{V}$ be a pseudovariety of groups. The uniform word problem for $\mathscr{V}$ is decidable if and only if the set of finite partial groups embeddable in groups from $\mathscr{G}$ is recursive.*

Our restriction to partial groups having no violations of associativity is no hinderance to this fact, since the property is necessary and algorithmically verifiable. We briefly recall part of the idea of the proof of Evans' Connection: with each instance Π, w of the uniform word problem, one can effectively associate a finite set of partial groups $\mathscr{P}$ such that there is an interpretation of Π in a group $\mathbf{G}$ in which $w \neq 1$ if and only if at least one member of $\mathscr{P}$ embeds into $\mathbf{G}$. (The association is based on forming partial algebras on various quotients of the finite set of subwords of all words appearing in the presentation Π or in w.)

A partial group $\mathbf{A}$ is said to be *symmetric* if for each a there is a unique a' such that $aa' = 1 = a'a$. A *symmetric extension* $\mathbf{B}$ of a partial group $\mathbf{A}$ is a symmetric partial group $\mathbf{B}$ on a set $B \supseteq A$ (and whose multiplication extends that of $\mathbf{A}$) in which for each $b \in B$ either b or b' is contained in A. The following facts are obvious: there are at most $2|A|$ elements in a symmetric extension of $\mathbf{A}$, and hence only finitely many symmetric extensions of $\mathbf{A}$ (and they can be effectively constructed); if $\mathbf{A}$ embeds into a group $\mathbf{G}$ then there is a symmetric extension of $\mathbf{A}$ embedding into $\mathbf{G}$; if there is a symmetric extension of $\mathbf{A}$ embedding into a group $\mathbf{G}$, then $\mathbf{A}$ embeds into $\mathbf{G}$. Hence the following fact is also true.

Evans' Connection for symmetric partial groups. *Let $\mathscr{V}$ be a pseudovariety of groups. The uniform word problem for $\mathscr{V}$ is decidable if and only if the set of finite symmetric partial groups embeddable in groups from $\mathscr{G}$ is recursive.*

This is the basic fact we will use in most of the undecidability results in this article. In one case however we use the following embellishment of Evans' Connection for symmetric partial groups.

Lemma 1.1. *There are recursively inseparable subsets $I, J \subseteq \mathbb{N}$ and a computable function associating with each natural number $n \in \mathbb{N}$ a finite set $\mathscr{P}_n$ of finite symmetric partial groups such that if $i \in I$ there is a member of $\mathscr{P}_i$ embeddable in a finite group and if $j \in J$ then no member of $\mathscr{P}_j$ embeds into any group.*

Proof. We use the proof of the undecidability of the uniform word problem for groups (by Slobodskoĭ [22]; see also [11]). In that proof, there is a fixed finite presentation Π, a word w_n (in the alphabet of Π) effectively constructed for each $n \in \mathbb{N}$, and recursively inseparable subsets $I, J \subseteq \mathbb{N}$ with the properties:

- if $i \in I$ then there is a finite group interpreting Π in which w_i is not equal to 1;
- if $j \in J$ then every group satisfying Π has $w_j = 1$.

Now recall that, following the proof of Evans' Connection, we can effectively associate with each of these uniform word problem instances Π, w_n, a finite family of partial groups $\mathscr{P}_n$ (which we can assume to consist of symmetric partial groups) with the property that if $\mathbf{G}$ is a group interpreting Π, but with $w_n \neq 1$, then some member of $\mathscr{P}_n$ embeds into $\mathbf{G}$, while if every group satisfying Π has $w_n = 1$, then no member of $\mathscr{P}_n$ embeds into a group. The function $n \mapsto \mathscr{P}_n$ now has the desired properties. □

In order to encode partial groups into semigroups, we need a further refinement of the concepts (also introduced in [5]).

Let $\mathbf{B}$ be a partial group with identity element 1 and A be a subset of B containing 1. For each $i = 1, 2, \ldots$ define the set A^i by $A^1 := A$ and

$$A^{i+1} := \{xy \mid x \in A, y \in A^i \text{ and } xy \text{ is defined in } \mathbf{B}\}.$$

For $k > 1$, the partial group $\mathbf{B}$ is said to be an *extension of rank k* of the partial group $\mathbf{A}$ if the universe A of $\mathbf{A}$ is a subset of the universe of $\mathbf{B}$, the partial multiplication on $\mathbf{A}$ is a restriction of the partial multiplication of $\mathbf{B}$ (that is, $a \cdot b = c$ in $\mathbf{A}$ implies $a \cdot b = c$ in $\mathbf{B}$), and the following properties hold:

(E1) for each pair of integers i, j with $1 \leq i, j \leq k$ and with $i + j \leq k$, and for each pair of elements $x \in A^i$ and $y \in A^j$, the product xy is defined in $\mathbf{B}$ and lies in A^{i+j};

(E2) if i, j with $1 \leq i, j \leq k$ have $i + j > k$, then for each pair of elements $x \in A^i \setminus (\bigcup_{\ell=1}^{i-1} A^\ell)$ and $y \in A^j \setminus (\bigcup_{\ell=1}^{j-1} A^\ell)$ the product xy is undefined in $\mathbf{B}$ unless $1 \in \{x, y\}$;

(E3) if i, j, ℓ have $1 \leq i, j, \ell$ and $i + j + \ell \leq k$, then for each triple of elements $x \in A^i$, $y \in A^j$ and $z \in A^\ell$ the products $(xy)z$ and $x(yz)$ are both defined in $\mathbf{B}$ and equal;

(E4) $B = \bigcup_{i=1}^k A^i$.

Again, it is clear that for any k, there are only finitely many different rank k extensions of a finite partial group, and all may be effectively listed. Furthermore for any k, the partial group $\mathbf{A}$ embeds into a group if and only if some extension of rank k of $\mathbf{A}$ embeds into the same group. Condition (E3) in this definition actually follows from condition (E1) of the definition and our assumption on associativity. (This assumption is not required in [5], however it does no harm to include it, and makes a subtle appearance at one point in this article.)

Finally, we will make essential use of a further concept, this time a generalisation of the quite widely used notion of an isotopy (in the sense of quasigroups). Following Albert [1], we say that a triple of maps (α, β, γ), each mapping from a semigroup (or partial group) $\mathbf{A}$ to a semigroup (or group) $\mathbf{B}$ is a *homotopy* if whenever $a \cdot b$ is defined in $\mathbf{A}$ we have $\alpha(a)\beta(b) = \gamma(a \cdot b)$. In the case where $\mathbf{A}$ and $\mathbf{B}$ are both groups and α, β, γ are bijections, this is called an isotopy.[2]

[2] While we acknowledge that the word *homotopy* is already used for a different concept from topology, it also has wide usage in the present setting.

Albert [1] showed that isotopic groups are isomorphic (and moreover, that any loop isotopic to a group is itself isomorphic to that group). The following lemma shows that the essence of this fact is already present when **A** is a partial group (since associativity in **A** is not used in this proof, one could equally well refer to **A** as a partial loop). The argument is already present in [5], however we give the full details here for completeness.

Lemma 1.2 ([5]). *Let* **A** *be a partial group and* **G** *be a group. If* (α, β, γ) *is a homotopy from* **A** *to* **G**, *then* $\eta : a \mapsto \gamma(a)\beta(1)^{-1}\alpha(1)^{-1}$ *is a homomorphism from* **A** *to* **G**. *Moreover, if* $\delta \in \{\alpha, \beta, \gamma\}$ *has* $\delta(a) \neq \delta(b)$ *then* $\eta(a) \neq \eta(b)$.

Proof. We have that $\alpha(a)\beta(1) = \gamma(a1) = \gamma(a)$, so that $\alpha(a) = \gamma(a)\beta(1)^{-1}$. Similarly $\beta(b) = \alpha^{-1}(1)\gamma(b)$. Hence if the product ab is defined we have

$$\begin{aligned}\eta(ab) &= \gamma(ab)\beta(1)^{-1}\alpha(1)^{-1} = \alpha(a)\beta(a)\beta(1)^{-1}\alpha(1)^{-1}\\ &= \gamma(a)\beta(1)^{-1}\alpha(1)^{-1}\gamma(b)\beta(1)^{-1}\alpha(1)^{-1} = \eta(a)\eta(b).\end{aligned}$$

Thus η is a homomorphism.

For the last claim if $\gamma(a) \neq \gamma(b)$, then certainly cancellation ensures that $\eta(a) \neq \eta(b)$. But if $\alpha(a) \neq \alpha(b)$ then $\gamma(a) = \alpha(a)\beta(1) \neq \alpha(b)\beta(1) = \gamma(b)$, and similarly for $\beta(a) \neq \beta(b)$. □

Hence we only require one of the maps in a homotopy to be an injective map (or indeed, the intersection of the kernels of the maps to be the diagonal relation) for there to be an embedding of **A** into **G**. Of course Lemma 1.2 is true in the particular case when **A** is itself a totally defined group.

2. Decidable membership

Throughout the remainder of the article, we use the fact – established in [5] – that if $\mathscr{V}$ is a pseudovariety of groups with *decidable* membership problem, then the finite membership problem is decidable in any of the classes $\mathbb{S}(\mathsf{CS}^0(\mathscr{V}))$, $\mathbb{S}(\mathsf{Br}(\mathscr{V}))$, $\mathbb{S}(\mathsf{CS}^0_{m,n}(\mathscr{V}))$, $\mathbb{S}(\mathsf{Br}_m(\mathscr{V}))$, $\mathbb{Q}(\mathsf{CS}^0(\mathscr{V}))$, $\mathbb{Q}(\mathsf{Br}(\mathscr{V}))$, $\mathbb{Q}(\mathsf{CS}^0_{m,n}(\mathscr{V}))$, $\mathbb{Q}(\mathsf{Br}_m(\mathscr{V}))$. This gives the forward implication (A1) $\Rightarrow$ (Ai) for each $i = 2, \ldots, 15$ of Theorems A and A$^+$. Thus we need only prove the reverse direction in the cases discussed above (obviously not including (A7) $\Rightarrow$ (A1), which we instead prove is false).

3. Proof of Theorem A

As explained in the introductory section, the equivalence of condition (A1) with each of conditions (A4)–(A6) and (A10)–(A12) in Theorem A are deserving of a complete proof. We give such a proof in this section using the same construction and technique used in [5].

We first prove the equivalence of condition (A1) with each of the conditions (A5) and (A11).

Let **A** be a finite symmetric partial group, let $\mathbf{A}'$ be an extension of rank 2 of **A** and $\mathbf{A}''$ an extension of rank 3 of **A**. Following [5], we define semigroups $S_1(\mathbf{A}, \mathbf{A}')$, $S_2(\mathbf{A}, \mathbf{A}')$ and $S_3(\mathbf{A}, \mathbf{A}'')$ by:

$$\begin{aligned}S_1(\mathbf{A}, \mathbf{A}') \;=\; &\{(1,1,1), (2,1,2), (3,1,3)\} \cup \big(\{1\} \times A \times \{2\}\big)\\ &\cup \big(\{2\} \times A \times \{3\}\big) \cup \big(\{1\} \times A' \times \{3\}\big) \cup \{0\};\end{aligned}$$

$$S_2(\mathbf{A}, \mathbf{A}') = \big(\{1\} \times A \times \{2\}\big) \cup \big(\{2\} \times A \times \{3\}\big) \cup \big(\{1\} \times A' \times \{3\}\big) \cup \{0\};$$

$$\begin{aligned}S_3(\mathbf{A}, \mathbf{A}'') \;=\; &\big(\{1\} \times A \times \{2\}\big) \cup \big(\{2\} \times A \times \{3\}\big) \cup \big(\{3\} \times A \times \{4\}\big)\\ &\cup \big(\{1\} \times A^2 \times \{3\}\big) \cup \big(\{2\} \times A^2 \times \{4\}\big)\\ &\cup \big(\{1\} \times A'' \times \{4\}\big) \cup \{0\},\end{aligned}$$

and where multiplication is defined as in a Brandt semigroup.

Proposition 3.1. *Let* $\mathbf{A}'$ *be an extension of rank 2 of a finite symmetric partial group* **A**, *let* $\mathscr{V}$ *be a pseudovariety of groups and* $m \geq 3$ *an integer. The following are equivalent:*

(1) $\mathbf{A}'$ *embeds into a group from* $\mathscr{V}$;
(2) $S_2(\mathbf{A}, \mathbf{A}')$ *embeds into a* $m \times m$ *Brandt semigroup over a group from* $\mathscr{V}$;
(3) $S_2(\mathbf{A}, \mathbf{A}')$ *embeds into a Brandt semigroup over a group from* $\mathscr{V}$;
(4) $S_2(\mathbf{A}, \mathbf{A}')$ *embeds into a pure completely 0-simple semigroup over a group from* $\mathscr{V}$.

Proof. The implication (1) $\Rightarrow$ (2) is easy, and is already explained in [5]. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are all trivial. Now suppose that (4) holds, with $\iota : S_2(\mathbf{A}, \mathbf{A}') \to M^0[\mathbf{G}, P]$ the embedding and where P is a matrix over $\{0, 1\}$.

As ι is an embedding it is clear that $\iota(0) = 0$, and that every other element of $S_2(\mathbf{A}, \mathbf{A}')$ has a left and a right coordinate in $M^0[\mathbf{G}, P]$. Let ℓ, r be the left and right coordinate respectively of $\iota(1, 1, 3)$. For each $a \in A$ we have $\iota(1, 1, 3) = \iota(1, a, 2)\iota(2, a', 3)$, so that the left coordinate of each $\iota(1, a, 2)$ is ℓ. Similarly, we find that the right coordinate of each

$\iota(2, a, 3)$ is r. Then as every $c \in A'$ is of the form ab for some $a, b \in A$, we have that all elements of $\iota(\{1\} \times A' \times \{3\})$ have left and right coordinates ℓ and r respectively.

Define maps $\alpha, \beta : \mathbf{A} \to \mathbf{G}$ by letting $\alpha(a)$ and $\beta(a)$ be the group coordinates of $\iota(1, a, 2)$ and of $\iota(2, a, 3)$ respectively. For $b \in A'$, define $\gamma(b)$ to be the group coordinate of $\iota(1, b, 3)$. (Strictly, we have taken the domain of γ to be A', which is possibly larger than A—the domain of α and β. We will use this slight abuse of definition at the end of the proof.) Note that γ is injective, since all elements of $\iota(\{1\} \times A' \times \{3\})$ have left and right coordinates ℓ and r, and ι is injective. For $a \in A$, let r_a denote the right coordinate of $\iota(1, a, 2)$ and ℓ_a denote the left coordinate of $\iota(2, a, 3)$. For any $a, b \in A$ we have

$$\begin{aligned}(\ell, \gamma(ab), r) &= \iota(1, ab, 3) = \iota(1, a, 2)\iota(2, b, 3)\\ &= (\ell, \alpha(a), r_a)(\ell_b, \beta(b), r) = (\ell, \alpha(a)P_{r_a,\ell_b}\beta(b), r) = (\ell, \alpha(a)\beta(b), r),\end{aligned}$$

since $P_{r_a,\ell_b} = 1$. Hence $\alpha(a)\beta(b) = \gamma(ab)$ and the triple (α, β, γ) forms a homotopy, for which the associated homomorphism $\eta : c \mapsto \gamma(c)\beta(1)^{-1}\alpha(1)^{-1}$ from $\mathbf{A}$ to $\mathbf{G}$ is an embedding (by Lemma 1.2). Now observe that in fact η also is an embedding of $\mathbf{A}'$ into $\mathbf{G}$ since if $c \in A' \setminus A$, then the only products to preserve are $c \cdot 1$ and $1 \cdot c$ and we have $\eta(1 \cdot c) = \eta(c \cdot 1) = \eta(c) = \eta(c)\eta(1) = \eta(1)\eta(c)$. Hence condition (1) holds. □

Theorem 3.2. *Let $\mathscr{V}$ be a pseudovariety of groups with undecidable uniform word problem and $\mathscr{U}$ be any class with*

$$\mathbb{S}(\mathsf{Br}_3(\mathscr{V})) \cap \mathsf{Nil}_3 \subseteq \mathscr{U} \subseteq \mathbb{S}(\mathsf{PCS}^0(\mathscr{V})).$$

Then $\mathscr{U}$ has undecidable finite membership problem.

Proof. For any finite symmetric partial group $\mathbf{A}$, Proposition 3.1 shows that there is an extension of rank 2 $\mathbf{A}'$ of $\mathbf{A}$ with $S_2(\mathbf{A}, \mathbf{A}') \in \mathscr{U}$ if and only if $\mathbf{A}$ embeds into a member of $\mathscr{V}$. □

The implications $\neg$(A1) $\Rightarrow$ $\neg$(A5) and $\neg$(A1) $\Rightarrow$ $\neg$(A11) of Theorem A follow from Theorem 3.2 using $\mathscr{U} = \mathsf{Br}(\mathscr{V})$ and $\mathscr{U} = \mathsf{Br}_m(\mathscr{V})$ respectively. The reverse directions are explained in Section 2.

Now we use the $S_3(\mathbf{A}, \mathbf{A}'')$ construction to prove the equivalence of condition (A1) with conditions (A4), (A6), (A10), (A12) in Theorem A. Our proof is in line with those parts of the proof presented in [5].

Proposition 3.3. *Let $\mathbf{A}$ be a finite symmetric partial group, $\mathscr{V}$ a pseudovariety of groups. The following are equivalent:*

(1) *$\mathbf{A}$ embeds into a group from $\mathscr{V}$;*
(2) *there is an extension $\mathbf{A}''$ of rank 3 of $\mathbf{A}$ for which $S_3(\mathbf{A}, \mathbf{A}'')$ embeds into a 4×4 Brandt semigroup over a group from $\mathscr{V}$;*
(3) *there is an extension $\mathbf{A}''$ of rank 3 of $\mathbf{A}$ for which $S_3(\mathbf{A}, \mathbf{A}'')$ embeds into a direct product of completely 0-simple semigroups whose subgroups are from $\mathscr{V}$.*

Proof. The implication (1) $\Rightarrow$ (2) follows easily from the definition of the semigroup $S_3(\mathbf{A}, \mathbf{A}'')$, and is explained in [5]. The implication (2) $\Rightarrow$ (3) is trivial. Now suppose that (3) holds. Property (3) is equivalent to the property that each pair of distinct elements from $S_3(\mathbf{A}, \mathbf{A}'')$ can be separated by a homomorphism into a completely 0-simple semigroup with maximal subgroups from $\mathscr{V}$. We will prove the following claim.

Claim. *Each pair of distinct elements from $\mathbf{A}$ can be separated by a homomorphism into a group from $\mathscr{V}$.*

This will show that (1) holds, since this property is equivalent to the claim that $\mathbf{A}$ embeds into a direct product of groups from $\mathscr{V}$, indexed by the pairs of elements of $\mathbf{A}$. Because $\mathbf{A}$ is finite, this direct product is a finite direct product, and hence lies in $\mathscr{V}$. So, we prove the above Claim.

Let a, b be distinct elements of A and let ϕ be a homomorphism from $S_3(\mathbf{A}, \mathbf{A}'')$ into a Rees matrix semigroup $M^0[\mathbf{G}, P]$ with $\mathbf{G} \in \mathscr{V}$ and with $\phi(1, a, 4) \neq \phi(1, b, 4)$. Without loss of generality we may assume that $\phi(1, a, 4) \neq 0$. We now prove our Claim by a series of easy subclaims.

Subclaim 1. *$\phi(2, 1, 4)$ and $\phi(1, 1, 3)$ are nonzero.*

This is because $0 \neq \phi(1, a, 4) = \phi(1, a, 2)\phi(2, 1, 4) = \phi(1, 1, 3)\phi(3, a, 4)$.

Subclaim 2. *For any $c, d \in A$ we have $\phi(2, c, 3)$ and $\phi(2, d, 3)$ are nonzero and have the same left and right coordinates.*

This is because $\phi(1, c', 2)\phi(2, c, 3) = \phi(1, 1, 3)$ while $\phi(2, c, 3)\phi(3, c', 4) = \phi(2, 1, 4)$, and both are nonzero by Subclaim 1. Thus the arbitrary element $c \in A$ gives $\phi(2, c, 3)$ the same right coordinate as $\phi(1, 1, 3)$ and the same left coordinate as $\phi(2, 1, 4)$.

Subclaim 3. *For each $c \in A$ we have $\phi(1, c, 4)\ \mathscr{H}\ \phi(1, a, 4)$.*

First, by Subclaim 2 we have that $\phi(2, a, 3)\ \mathscr{H}\ \phi(2, c, 3)$. Then we have that

$$\begin{aligned}0 \neq \phi(1, a, 4) &= \phi(1, 1, 2)\phi(2, a, 3)\phi(3, 1, 4)\\ &\quad \mathscr{H}\ \phi(1, 1, 2)\phi(2, c, 3)\phi(3, 1, 4)\\ &= \phi(1, c, 4).\end{aligned}$$

Subclaim 4. *For each $c \in A^2$ we have that $\phi(1, c, 3) \neq 0$.*

Say that $c = pq$ for $p, q \in A$. Then the product $pqq' = p$ holds in $\mathbf{A}''$, so that $\phi(1, c, 3)\phi(3, q', 4) = \phi(1, p, 4) \neq 0$, by Subclaim 3.

Subclaim 5. *For each $c \in A^2$ we have that $\phi(1, c, 3) \mathscr{H} \phi(1, 1, 3)$.*

Again, say that $c = pq$ for $p, q \in A$. By Subclaim 4, we have that $0 \neq \phi(1, c, 3) = \phi(1, p, 2)\phi(2, q, 3)$ and $0 \neq \phi(1, 1, 3) = \phi(1, p, 2)\phi(2, p', 3)$. The subclaim follows since $\phi(2, q, 3) \mathscr{H} \phi(2, p', 3)$, by Subclaim 2.

Let ℓ_1, r_1 be the left and right coordinates of the elements in $\phi(\{2\} \times A \times \{3\})$. Let ℓ be the left coordinate of the element $\phi(1, a, 4)$. We may assume that the ℓ_1^{th} column of the matrix P contains entries only from $\{0, 1\}$.

Subclaim 6. *For each $c \in A$, the element $\phi(1, c, 2)$ is nonzero and has the left coordinate equal to ℓ.*

This is because $\phi(1, c, 2)\phi(2, 1, 4) = \phi(1, c, 4) \mathscr{H} \phi(1, a, 4)$, by Subclaim 3.

For each element $c \in A$, let $r(c)$ denote the right coordinate of $\phi(1, c, 2)$. For each $c \in A$, let $\alpha(a)$ be the group coordinate from $\phi(1, c, 2)$ and $\beta(c)$ be the group coordinate from $\phi(2, c, 3)$. For each $c \in A^2$, let $\gamma(c)$ be the group coordinate of $\phi(1, c, 3)$.

Subclaim 7. *For every $c, d \in A$ we have $\alpha(c)\beta(d) = \gamma(cd)$.*

We have that

$$\begin{aligned}(\ell, \gamma(cd), r_1) &= \phi(1, cd, 3)\\ &= \phi(1, c, 2)\phi(2, d, 3)\\ &= (\ell, \alpha(c), r(c))(\ell_1, \beta(d), r_1)\\ &= (\ell, \alpha(c)P_{r(c),\ell_1}\beta(d), r_1)\\ &= (\ell, \alpha(c)\beta(d), r_1).\end{aligned}$$

Subclaim 8. $\beta(a) \neq \beta(b)$.

This is because

$$\begin{aligned}&\phi(1, 1, 2)\phi(2, a, 3)\phi(3, 1, 4) = \phi(1, a, 4) \neq\\ &\phi(1, b, 4) = \phi(1, 1, 2)\phi(2, b, 3)\phi(3, 1, 4),\end{aligned}$$

yet $\phi(2, a, 3) \mathscr{H} \phi(2, b, 3)$ by Subclaim 2.

Our Claim (whence the proposition) now follows by Lemma 1.2. □

Theorem 3.4. *Let $\mathscr{V}$ be a pseudovariety of groups with undecidable uniform word problem and $\mathscr{U}$ with*

$$\mathbb{S}(\mathsf{Br}_4(\mathscr{V})) \cap \mathsf{Nil}_4 \subseteq \mathscr{U} \subseteq \mathbb{SP}(\mathsf{CS}^0(\mathscr{V})).$$

Then $\mathscr{U}$ has undecidable finite membership problem.

4. Proof of Theorem A$^+$

In this section we prove Theorem A$^+$. The general approach is similar to the previous section, however we need a more technical construction.

Let $\mathbf{S}$ be a 3-nilpotent semigroup. An element $c \in S$ is *composite* if there are $a, b \in S$ with $ab = c$. An element is *prime* if it is not composite.

Let $m, n \geq 3$ be integers and P be any $(m-2) \times (n-2)$ matrix over the alphabet $\{0, 1\}$ with the property that no row nor column consists entirely of 0's. Let $\mathbf{B}_1$ be an extension of rank 2 of a partial group $\mathbf{B}$. We now define a 3-nilpotent semigroup $S_P(\mathbf{B}, \mathbf{B}_1)$ in the following way. Let $\Lambda_L = \{\lambda_0, \lambda_1, \ldots, \lambda_{m-2}, \lambda_\infty\}$ and $\Lambda_R = \{\lambda_0, \lambda_1, \ldots, \lambda_{n-2}, \lambda_\infty\}$. The universe of $S_P(\mathbf{B}, \mathbf{B}_1)$ is the set

$$\{0\} \cup \bigcup_{1 \leq i \leq m-2} \big(\{\lambda_0\} \times B \times \{\lambda_i\}\big) \cup \bigcup_{1 \leq i \leq n-2} \big(\{\lambda_i\} \times B \times \{\lambda_\infty\}\big) \cup \{\lambda_0\} \times B_1 \times \{\lambda_\infty\}.$$

Define a matrix $\Lambda_L \times \Lambda_R$ matrix Q over $\{0, 1\}$ by letting $Q_{\lambda_0,\lambda_i} = 1$ and $Q_{\lambda_j,\lambda_0} = 1$ if and only if $i = j = 0$, and $Q_{\lambda_\infty,\lambda_i} = 1$ and $Q_{\lambda_j,\lambda_\infty} = 1$ if and only if $i = j = \infty$. For $1 \leq i \leq m-2$ and $1 \leq j \leq n-2$, we define $Q_{\lambda_i,\lambda_j} = P_{i,j}$. The multiplication in $S_P(\mathbf{B}, \mathbf{B}_1)$ is now defined as in any Rees matrix semigroup with sandwich matrix Q. We mention that when P is the 1×1 matrix with entry 1, the constructions $S_P(\mathbf{B}, \mathbf{B}_1)$ and $S_2(\mathbf{B}, \mathbf{B}_1)$ coincide.

The following lemma follows easily from the definitions.

Lemma 4.1. *Let* $\mathbf{A}$ *be a symmetric partial group and* $\mathscr{V}$ *be a pseudovariety of groups. Let* P *be an* $(m-2)\times(n-2)$ *matrix over* $\{0,1\}$ *with no row nor column consisting entirely of 0's. If* $\mathbf{A}$ *embeds into a group* $\mathbf{G}\in\mathscr{V}$*, then there is an extension* $\mathbf{A}_1$ *of rank 2 of* $\mathbf{A}$ *such that* $S_P(\mathbf{A},\mathbf{A}_1)$ *embeds into a pure completely 0-simple semigroup* $M^0[\mathbf{G},P']$ *where* P' *has dimension* $m\times n$*. If the matrix* P *is the* $(m-2)\times(m-2)$ *identity matrix, then* P' *can be chosen to be the* $m\times m$ *identity matrix, so that* $M^0[\mathbf{G},P']$ *is the Brandt semigroup* $\mathbf{B}_m(\mathbf{G})$*.*

For integers $m,n\geq 3$, let us say that a matrix P has the property $\heartsuit_{m,n}$ if:

> P is over the alphabet $\{0,1\}$; has dimension $(m-2)\times(n-2)$; no column (row) of P consists entirely of 0's; no two rows of P are the same.

Let P be a matrix satisfying $\heartsuit_{m,n}$. The 3-nilpotent semigroup $\mathbf{S}$ will be called a *P-shell* if the following properties hold.

(P1) the set of prime elements of $\mathbf{S}$ can be written as a disjoint union of some nonempty sets $L_1,\dots,L_{m-2},R_1,\dots,R_{n-2}$;
(P2) the only non-zero products in $\mathbf{S}$ are of the form xy, where $x\in L_i$ and $y\in R_j$ for $P_{i,j}=1$;
(P3) there is at least one nonzero element $c\in S\setminus\{0\}$ (a *fulcrum*) such that whenever $P_{i,j}=1$ and $x\in L_i$, $y\in R_j$, there are $y_x\in R_j$ and $x_y\in L_i$ with $xy_x=c=x_yy$.

We note that there need not be a unique fulcrum element in a P-shell. We also observe that the construction $S_P(\mathbf{A},\mathbf{A}_1)$ used above is a P-shell with fulcrum $(\lambda_0,1,\lambda_\infty)$.

Lemma 4.2. *Let* P *be a matrix with property* $\heartsuit_{m,n}$*, for some* $m,n\geq 3$*. Let* $\mathbf{S}$ *be a* P*-shell and* $c\in S$ *be a fulcrum. Assume that there is a homomorphism* ϕ *from* $\mathbf{S}$ *into a completely 0-simple semigroup* $M^0[\mathbf{G},P']$*, where* P' *has dimension* $m\times n$ *and* $\phi(c)\neq 0$*. Then*

(1) *for each* $1\leq i\leq m-2$*, the set* $\phi(L_i)$ *lies in an* $\mathscr{H}$*-class of* $M^0[\mathbf{G},P']$*;*
(2) *for every* i *with* $1\leq i\leq m-2$*,* j *with* $1\leq j\leq n-2$ *and every* $x\in L_i$ *and* $y\in R_j$ *we have* $\phi(x)\phi(y)\neq 0$ *and* $xy\neq 0$*.*
(3) *the union of the sets* $\phi(L_iR_j)$ *over all* i,j *for which* $P_{i,j}=1$ *lie in a single* $\mathscr{H}$*-class.*

Proof. We prove (1) first. For notational convenience, we index P' as a

$$\{0,1,\dots,m-2,\infty\}\times\{0,1,\dots,n-2,\infty\}$$

matrix. Note also that $\phi(0)=0$.

Let c be a fulcrum element and let $L=\cup_{1\leq i\leq m-2}L_i$ and $R=\cup_{1\leq i\leq n-2}R_i$. The set of all divisors of c in $\mathbf{S}$ is $L\cup R$, and as $\phi(c)\neq 0$, we have that $0\notin\phi(L\cup R)$. So each element of $\phi(L\cup R)$ has a left and right coordinate in $M^0[\mathbf{G},P']$. By switching rows and columns of P', we can assume that the left coordinate of $\phi(c)$ is 0 and the right coordinate is ∞. Since for every element in $x\in L$ there is an element $y\in R$ with $xy=c$, we have for every $x\in L$, that the left coordinate of $\phi(x)$ is 0; dually we have that when $y\in R$, the right coordinate of $\phi(y)$ is ∞. Now for each $k\leq m-2$ and $\ell\leq n-2$, let I_k be the set of right coordinates of elements of $\phi(L_k)$, and J_ℓ be the set of left coordinates of elements in $\phi(R_\ell)$. These sets are obviously nonempty. Let $I=\bigcup_{1\leq k\leq m-2}I_k$ and $J=\bigcup_{1\leq\ell\leq n-2}J_\ell$.

Claim 1. $0\notin J$ *and* $\infty\notin I$*.*

Proof. Assume, for a contradiction, that there is $y\in R_\ell$ for which the left coordinate of $\phi(y)$ equals 0. There is a number k such that $P_{k,\ell}=1$, hence we may find an $x\in L_k$ with $xy=c$; let i be the right coordinate of $\phi(x)$. As $\phi(x)\phi(y)=\phi(xy)\neq 0$ we have that $P'_{i,0}\neq 0$. But then $\phi(0)=\phi(xx)=\phi(x)\phi(x)\neq 0$, a contradiction. The fact that $\infty\notin I$ is by a dual argument. □

Choose some ℓ with $P'_{\infty,\ell}\neq 0$; such an ℓ exists, since every row and every column of P' has a nonzero entry. Now we have that $\ell\neq 0$, since $RL=\{0\}$ (by property (P2) in the definition of a P-shell), and the right coordinate of every element in $\phi(R)$ is ∞ and the left coordinate of every element of $\phi(L)$ is 0. Hence, by switching columns if necessary, we can assume that $\ell=\infty$ (note that we have already fixed the 0^{th} column of P', however we can still arrange $\ell=\infty$ because $\ell\neq 0$). Likewise, there is some k with $Q_{k,0}\neq 0$, and we can similarly assume that $k=0$. So we have now fixed $P'_{0,0}\neq 0$ and $P'_{\infty,\infty}\neq 0$.

Claim 2. $\infty\notin J$ *and* $0\notin I$*.*

Proof. If $\infty\in J$ then there is $y\in R$ with $\phi(y)\phi(y)\neq 0$ (since we have negotiated the property $P'_{\infty,\infty}\neq 0$), contradicting the fact that the only nonzero products in $\mathbf{S}$ are between elements of L with elements of R. The $0\notin I$ case is by symmetry. □

Claim 3. *If* $1\leq r,s\leq m-2$ *and* $r\neq s$*, then* $I_r\cap I_s=\varnothing$*.*

Proof. For a contradiction, assume that $r\neq s$ but there is $i\in I_r\cap I_s$. Hence there are $x_r\in L_r$ and $x_s\in L_s$ for which $\phi(x_r)$ and $\phi(x_s)$ have right coordinate equal to i. Now using property $\heartsuit_{m,n}$ we may find a number ℓ such that $P_{r,\ell}\neq P_{s,\ell}$; say $P_{r,\ell}=1\neq 0=P_{s,\ell}$. By property (P3) of the definition of a P-shell, there is some $y\in R_\ell$ with $x_ry=c$. Note that $x_sy_\ell=0$ since $P_{s,\ell}=0$. But then $\phi(x_r)\phi(y_\ell)=\phi(c)\neq 0$ implies that $\phi(0)=\phi(x_sy)=\phi(x_s)\phi(y)\neq 0$, which contradicts the fact that $\phi(0)=0$. □

Now, as $0, \infty \notin I$ by Claims 1 and 2, we have by Claim 3 that the sets $I_1, \ldots, I_{m-2}$ form a partition of the $m-2$ element set $\{1, 2, \ldots, m-2\}$. In other words, they are singletons. Hence $\phi(L_1), \ldots, \phi(L_{m-2})$ each lie within $\mathscr{H}$-classes of $M^0[\mathbf{G}, P']$. This proves (1).

For condition (2) of the lemma, let $P_{i,j} = 1$, and $x \in L_i, y \in R_j$. Let $x_y \in L_i$ be such that $x_y y = c$. Then as $\phi(x_y)$ has the same right coordinate as $\phi(x)$, and as $\phi(x_y)\phi(y) = \phi(c) \neq 0$, we must have $\phi(x)\phi(y) \neq 0$. Condition (3) follows from condition (2), since we showed that nonzero elements of $\phi(L_iR_j)$ have left coordinate λ_0 and right coordinate λ_∞. $\square$

Proposition 4.3. *Let $\mathscr{V}$ be a pseudovariety of groups, and $m, n \geq 3$ be integers for which there is a matrix P satisfying property $\heartsuit_{m,n}$. If there is an extension $\mathbf{A}_2$ of rank 2 of an extension $\mathbf{A}_1$ of rank 2 of $\mathbf{A}$ such that $S_P(\mathbf{A}_1, \mathbf{A}_2)$ embeds into a direct product of completely 0-simple semigroups from $\mathsf{CS}^0_{m,n}(\mathscr{V})$, then $\mathbf{A}$ embeds into a group from $\mathscr{V}$.*

Proof. Say that ι is an embedding of $S_P(\mathbf{A}_1, \mathbf{A}_2)$ into a direct product of $m \times n$ completely 0-simple semigroups with groups from $\mathscr{V}$. For any $u, v \in A$ we have $\iota(\lambda_0, u, \lambda_\infty) \neq \iota(\lambda_0, v, \lambda_\infty)$, so by following ι by a suitable projection map we can find a homomorphism ϕ from $S_P(\mathbf{A}_1, \mathbf{A}_2)$ into a single $m \times n$ completely 0-simple semigroup $M^0[\mathbf{G}, P']$ (where $\mathbf{G} \in \mathscr{V}$) with $\phi(\lambda_0, u, \lambda_\infty) \neq \phi(\lambda_0, v, \lambda_\infty)$.

Let $\mathbf{T}_u$ be the subsemigroup of $S_P(\mathbf{A}_1, \mathbf{A}_2)$ generated by the set of all elements dividing $(\lambda_0, u, \lambda_\infty)$. Let L_i denote the elements of T_u with the right coordinate equal to λ_i, and R_j denote the elements of T_u with the left coordinate equal to λ_j $(1 \leq i \leq m-2$ and $1 \leq j \leq n-2)$. It is easy to see that $\mathbf{T}_u$ is a P-shell with $(\lambda_0, u, \lambda_\infty)$ as a fulcrum element. We are going to prove that $\mathbf{T}_u$ contains $S_P(\mathbf{A}, \mathbf{A}_1)$ as a subsemigroup. The same statements will be true of $\mathbf{T}_v$, defined in the obvious way.

Consider any $p \in A$. As $p', u \in A$ we have $p'u, up' \in A_1$. Hence we can perform the calculation $p(p'u) = (pp')u = u = u(p'p) = (up')p$ in A_2. Then, whenever $P_{i,j} = 1$ we have

$$(\lambda_0, p, \lambda_i)(\lambda_j, p'u, \lambda_\infty) = (\lambda_0, u, \lambda_\infty) = (\lambda_0, up', \lambda_i)(\lambda_j, p, \lambda_\infty)$$

in $\mathbf{T}_u$. It is easy to see that $S_P(\mathbf{A}, \mathbf{A}_1)$ is exactly the subsemigroup of $\mathbf{T}_u$ generated by the elements $(\lambda_0, p, \lambda_i)$ and $(\lambda_j, p, \lambda_\infty)$, ranging over all $i = 1, \ldots, m-2, j = 1, \ldots, n-2$ and $p \in A$. In other words, $S_P(\mathbf{A}, \mathbf{A}_1)$ is a subsemigroup of $\mathbf{T}_u$ (however $(\lambda_0, u, \lambda_\infty)$ is not necessarily a fulcrum for $S_P(\mathbf{A}, \mathbf{A}_1)$).

By Lemma 4.2 (applied to T_u), we have for each i, j and each $p, q \in A$ (so that $pq \in A_1$), that the elements $\phi(\lambda_0, p, \lambda_i)$, $\phi(\lambda_i, p, \lambda_\infty)$ and $\phi(\lambda_0, pq, \lambda_\infty)$ are nonzero, and the set $\phi(\{\lambda_0\} \times A \times \{\lambda_i\})$ lies in an $\mathscr{H}$-class. Let r_1 be the right coordinate of the elements in $\phi(L_1)$, and let j be some number such that $P_{1,j} = 1$. By Lemma 4.2 we can also let ℓ, r denote the left and right coordinates respectively of the elements $\phi(\{\lambda_0\} \times A_1 \times \{\lambda_\infty\})$. For each $a \in A$, let ℓ_a denote the left coordinate of $\phi(\lambda_j, a, \lambda_\infty)$. The isomorphism theory of completely 0-simple semigroups ensures that we can assume that the r_1^{th} row of the matrix P' has all entries in $\{0, 1\}$. In particular, we may assume that $P'_{r_1,\ell_a} = 1$, for any $a \in A$. Hence we can define a triple of maps (α, β, γ) from $\mathbf{A}_1$ into the group $\mathbf{G}$ by letting $\alpha(p)$, $\beta(q)$ and $\gamma(pq)$ be the group coordinate of $\phi(\lambda_0, p, \lambda_1)$, $\phi(\lambda_j, q, \lambda_\infty)$ and $\phi(\lambda_0, pq, \lambda_\infty)$ respectively.

We have for $a, b \in A$ that

$$\begin{aligned}(\ell, \gamma(ab), r) &= \phi(\lambda_0, ab, \lambda_\infty)\\ &= \phi(\lambda_0, a, \lambda_1)\phi(\lambda_j, b, \lambda_\infty)\\ &= (\ell, \alpha(a), r_1)(\ell_b, \beta(b), r)\\ &= (\ell, \alpha(a)P'_{r_1,\ell_b}\beta(b), r)\\ &= (\ell, \alpha(a)\beta(b), r).\end{aligned}$$

Hence the triple (α, β, γ) is a homotopy from $\mathbf{A}$ into $\mathbf{G}$. Note that by assumption we have $\iota(\lambda_0, u, \lambda_\infty) \neq \iota(\lambda_0, v, \lambda_\infty)$ and we proved that both are $\mathscr{H}$-related. Hence $\gamma(u) \neq \gamma(v)$. By Lemma 1.2 we have that there is a homomorphism of $\mathbf{A}$ into the group $\mathbf{G}$ under which u is separated from v.

This can be done for each of the finitely many (unordered) pairs u, v of elements of A. In each case we obtain a homomorphism $\phi_{u,v}$ from $\mathbf{A}$ into a group $\mathbf{G}_{u,v}$ from $\mathscr{V}$ separating the corresponding pair u, v. Hence we obtain an embedding of $\mathbf{A}$ into the finite direct product $\prod_{u \neq v \in A} \mathbf{G}_{u,v}$ of groups from $\mathscr{V}$. Since $\mathscr{V}$ is closed under taking finitary direct products, we have proved that $\mathbf{A}$ embeds into a group from $\mathscr{V}$. $\square$

Now we prove Theorem A$^+$.

Corollary 4.4. *Let $\mathscr{V}$ be a pseudovariety of groups, let $n_1, n_2, m_2 \geq 3$ be integers and $\mathscr{U}_1, \mathscr{U}_2$ any classes with:*

(1) $\mathbb{S}(\mathsf{Br}_{n_1}(\mathscr{V})) \cap \mathsf{Nil}_3 \subseteq \mathscr{U}_1 \subseteq \mathbb{SP}(\mathsf{CS}^0_{n_1,n_1}(\mathscr{V}))$;

(2) $\mathbb{S}(\mathsf{PCS}^0_{n_2,m_2}(\mathscr{V})) \cap \mathsf{Nil}_3 \subseteq \mathscr{U}_2 \subseteq \mathbb{SP}(\mathsf{CS}^0_{n_2,m_2}(\mathscr{V}))$.

If $\mathscr{V}$ has undecidable uniform word problem, then $\mathscr{U}_1, \mathscr{U}_2$ both have undecidable finite membership problems.

Proof. Let $\mathbf{A}$ be a symmetric partial group. If $\mathbf{A}$ is embeddable in a group $\mathbf{G} \in \mathscr{V}$, then there is an extension $\mathbf{A}_1$ of rank 2 of $\mathbf{A}$ and an extension $\mathbf{A}_2$ of rank 2 of $\mathbf{A}_1$ with $\mathbf{A}_2$ embedding into $\mathbf{G}$. Up to symmetry, we may assume that $m_2 \leq n_2$. Now let P_1 be the $(n_1-2) \times (n_1-2)$ identity matrix and choose P_2 to be any $(m_2-2) \times (n_2-2)$ matrix over $\{0, 1\}$ with condition $\heartsuit_{m_2,n_2}$. By Lemma 4.1, we have that the 3-nilpotent semigroups $S_{P_1}(\mathbf{A}_1, \mathbf{A}_2)$, $S_{P_2}(\mathbf{A}_1, \mathbf{A}_2)$ are in $\mathscr{U}_1$ and $\mathscr{U}_2$, respectively.

Conversely, if $S_{P_i}(\mathbf{A}_1, \mathbf{A}_2)$, is contained in $\mathscr{U}_i$, then Proposition 4.3 shows that $\mathbf{A}$ embeds into a group $\mathbf{G} \in \mathscr{V}$. Hence the embeddability of symmetric partial groups in groups from $\mathscr{V}$ has been reduced to the finite membership problem for each of the classes $\mathscr{U}_1$ and $\mathscr{U}_2$. $\square$

This proves one direction of the equivalence of condition (A1) of Theorem A with conditions (A13)–(A15) of Theorem A$^+$. The reverse directions are explained in Section 2.

5. Proof of Theorem D

For any semigroup $\mathbf{S}$ without identity element, let $\mathbf{S}^{(0)} := \mathbf{S}$ and $\mathbf{S}^{(i+1)}$ be the result of adjoining a new identity element to $\mathbf{S}^{(i)}$. If $\mathbf{S}$ embeds into an inverse semigroup $\mathbf{T}$ of height k, then $\mathbf{S}^{(n)}$ embeds into an inverse semigroup of height $k+n$; namely $\mathbf{T}^{(n)}$. Conversely, if $\mathbf{S}^{(n)}$ is a subsemigroup of an inverse semigroup $\mathbf{U}$ of height $k+n$, then the inverse subsemigroup of $\mathbf{U}$ generated by S cannot have height more than k.

Lemma 5.1. *Let* $\mathbf{A}$ *be a symmetric partial group and* $\mathbf{A}'$ *be an extension of rank 2 of* $\mathbf{A}$*. Assume that* $S_1(\mathbf{A}, \mathbf{A}')$ *embeds into a finite semigroup* $\mathbf{R}$ *in which the ordered set of* $\mathscr{J}^{\mathbf{R}}$*-classes of the nonzero elements of* $S_1(\mathbf{A}, \mathbf{A}')$ *is of height* 0*. Then* $\mathbf{A}'$ *embeds into a subgroup of* $\mathbf{R}$*.*

Proof. First observe that if J_1 and J_2 are incomparable $\mathscr{J}$-classes of $\mathbf{R}$, with $a \in J_1$ and $b \in J_2$, then as both a and b divide the product ab, this product lies in a $\mathscr{J}$-class that is strictly lower than J_1 and J_2. In the contrapositive, this shows that if a, b are distinct elements, and the product ab is not in a strictly lower $\mathscr{J}$-class than that of a and b, then all three of a, b, ab are $\mathscr{J}$-related. Note that the element 0 of $S_1(\mathbf{A}, \mathbf{A}')$ cannot divide any nonzero element of $S_1(\mathbf{A}, \mathbf{A}')$ in $\mathbf{R}$.

Hence $(1, 1, 1)$ lies in the same $\mathscr{J}$-class (of $\mathbf{R}$) as all elements of the form $(1, a, 2)$ and $(1, a, 3)$. Likewise, $(2, 1, 2)$ lies in the $\mathscr{J}$-class of $(1, a, 2)$ and $(2, a, 3)$. And $(3, 1, 3)$ lies in the $\mathscr{J}$-class of $(1, a, 3)$ and $(2, a, 3)$. Hence all lie in the same $\mathscr{J}$-class, J say. Let $\mathbf{S}$ be the subsemigroup of $\mathbf{R}$ generated by J, and let I be the ideal of all elements of $\mathbf{S}$ that do not divide an element from J. Now $S_1(\mathbf{A}, \mathbf{A}')$ still embeds into $\mathbf{S}/I$, which is completely 0-simple. It is proved in [5] that this implies that $\mathbf{A}$ embeds into a subgroup of $\mathbf{S}/I$, whence into a subgroup of $\mathbf{R}$ (essentially it is the proof of $\neg$(A1) $\Rightarrow$ $\neg$(A2)). However, as in the proof of Proposition 3.1, one can extend this embedding to an embedding of $\mathbf{A}'$ into a $\mathbf{R}$. $\square$

Proposition 5.2. *The following are equivalent for an extension* $\mathbf{A}'$ *of rank* 2 *of a symmetric partial group* $\mathbf{A}$*:*

(1) $\mathbf{A}'$ *embeds into a finite group (a group);*
(2) $S_1(\mathbf{A}, \mathbf{A}')^{(n)}$ *is a subsemigroup of a finite inverse semigroup of height at most* $n+1$*;*
(3) $S_1(\mathbf{A}, \mathbf{A}')^{(n)}$ *is a subsemigroup of a finite regular semigroup of height most* $n+1$*;*
(4) $S_1(\mathbf{A}, \mathbf{A}')^{(n)}$ *is a subsemigroup of a finite inverse semigroup with at most* $n+2$ *distinct* $\mathscr{J}$*-classes;*
(5) $S_1(\mathbf{A}, \mathbf{A}')^{(n)}$ *is a subsemigroup of a finite regular semigroup with at most* $n+2$ *distinct* $\mathscr{J}$*-classes.*

Proof. If $\mathbf{A}'$ embeds into a finite group $\mathbf{G}$ then $S_1(\mathbf{A}, \mathbf{A}')$ embeds as a full subsemigroup of the Brandt semigroup $\mathbf{B}_3(\mathbf{G})$, while $S_1(\mathbf{A}, \mathbf{A}')^{(n)}$ embeds into $\mathbf{B}_3(\mathbf{G})^{(n)}$. Now $\mathbf{B}_3(\mathbf{G})^{(n)}$ is a finite inverse (whence regular) semigroup and $\mathbf{B}_3(\mathbf{G})^{(n)}$ has precisely $n+2$ distinct $\mathscr{J}$-classes, and is of height precisely $n+1$. This shows that conditions (2)–(5) hold.

The reverse directions are all similar. We use the fact that if $S_1(\mathbf{A}, \mathbf{A}')$ embeds into a completely 0-simple with subgroups from some pseudovariety $\mathscr{V}$, then $\mathbf{A}'$ embeds into a group from $\mathscr{V}$; this is established in [5].

For $i \in \{2, 3, 4, 5\}$, say that condition (i) holds. Now in a finite semigroup, an idempotent e cannot be $\mathscr{J}$-related to an element $a \neq e$ for which $ea = ae = a$ holds. Hence, in case (i), the nonzero elements of $S_1(\mathbf{A}, \mathbf{A}')$ must lie in $\mathscr{J}$-classes of height 1. By Lemma 5.1 we have that $\mathbf{A}'$ embeds into a subgroup of the embedding semigroup; hence (1) holds. $\square$

Now we can prove the undecidability of the finite membership problem for the classes in (D2) and (D3) of Theorem D. Let $\mathscr{V}$ be a class of finite regular semigroups containing the finite inverse semigroups. The undecidability of the uniform word problem for finite groups and Evans' Connection for symmetric partial groups show that it is undecidable as to whether an extension $\mathbf{A}'$ of rank 2 of a finite symmetric partial group $\mathbf{A}$ embeds into a finite group. If $\mathbf{A}'$ does embed into a finite group, then $S_1^{(n)}(\mathbf{A}, \mathbf{A}')$ embeds into a height $n+1$ member of $\mathscr{V}$ with at most $n+2$ distinct $\mathscr{J}$-classes, by Proposition 5.2 parts (2) and (4). Conversely, if $S_1^{(n)}(\mathbf{A}, \mathbf{A}')$ embeds into member of $\mathscr{V}$ with either height at most $n+1$ or with at most $n+2$ distinct $\mathscr{J}$-classes, then $\mathbf{A}'$ embeds into a finite group, by Proposition 5.2 parts (3) and (5).

Finally, we prove the undecidability of the finite membership problem for the class in (D1) of Theorem D. Consider the function $n \mapsto \mathscr{P}_n$ described in Lemma 1.1. Let $K \subseteq \mathbb{N}$ be the set of all numbers for which some member $\mathbf{A}$ of $\mathscr{P}_n$ has an extension $\mathbf{A}'$ of rank 2 for which $S_1(\mathbf{A}, \mathbf{A}_1)$ embeds as a full subsemigroup of a member of $\mathscr{U}$. We prove that $I \subseteq K$ and $J \cap K = \varnothing$, showing that K is not recursive. Note that if we can decide membership of finite subsemigroups in the class of full subsemigroups of members of $\mathscr{U}$, then we can decide membership of numbers in K. That is, there is a reduction of the membership problem for K to the finite subsemigroups in the class of full subsemigroups of members of $\mathscr{U}$. Hence, once K is proved to be nonrecursive, the proof of case (D1) in Theorem D is complete.

If $i \in I$, then there is $\mathbf{A} \in \mathscr{P}_i$ embedding into a finite group $\mathbf{G}$, whence $S_1(\mathbf{A}, \mathbf{A}_1)$ embeds (as a full subsemigroup) into $\mathbf{B}_3(\mathbf{G}) \in \mathsf{Br}_3(\mathscr{G}_{\mathrm{fin}}) \subseteq \mathscr{U}$. So $I \subseteq K$. Now consider any $n \in K$: that is, there is $\mathbf{A} \in \mathscr{P}_n$ and an extension $\mathbf{A}'$ of rank 2 of $\mathbf{A}$ for which $S_1(\mathbf{A}, \mathbf{A}')$ embeds as a full subsemigroup of a member of $\mathscr{U}$. Then as $\mathscr{U}$ consists of regular semigroups, we have that $S_1(\mathbf{A}, \mathbf{A}')$ is a full subsemigroup of a regular semigroup, $\mathbf{R}$ say.

In a regular semigroup, each element x is $\mathscr{R}$-related to an idempotent which acts as a left identity for x. In the case of $S_1(\mathbf{A}, \mathbf{A}')$, considered within the regular semigroup $\mathbf{R}$, we can deduce that all elements of the form $(1, a, 2)$ and $(1, a, 3)$ are $\mathscr{R}$-related to $(1, 1, 1)$, while a dual argument shows that elements of the form $(2, a, 3)$ are and $(1, a, 3)$ are $\mathscr{L}$-related to $(3, 1, 3)$. Hence all nonzero elements are related by $\mathscr{R} \vee \mathscr{L} = \mathscr{D}$, and that $\mathbf{R}$ contains a single $\mathscr{D}$-class not containing the element 0, and in which there are primitive idempotents (indeed, there are only four idempotents in $\mathbf{R}$). By factoring out a minimal ideal if necessary, we can assume that 0 is multiplicative zero element for $\mathbf{R}$. This shows that $\mathbf{R}$ is a completely 0-simple semigroup. As $S_1(\mathbf{A}, \mathbf{A}')$ embeds into a completely 0-simple semigroup, it follows from the arguments in [5] that $\mathbf{A}$ (even $\mathbf{A}'$) embeds into a group. Hence we must have $n \notin J$.

Hence K is not recursive, and the finite membership problem for the class in (D1) of Theorem D is undecidable. This completes the proof of Theorem D. $\square$

There of course many further variations of Theorem D. We mention in particular, that Kublanovsky has shown (the result is stated in an extended form in [12] for example) that the problem of deciding which semigroups embed into a finite regular semigroup from the pseudovariety generated by finite completely 0-simple semigroups (or Brandt semigroups) is undecidable. This follows using the $S_1(\mathbf{A}, \mathbf{A}')$ construction, since it is shown in [5, Lemma 3.2] that embedding in a regular semigroup from this pseudovariety is equivalent to embedding into a direct product of finite completely 0-simple semigroups.

Remark 5.3. One can construct a finite semigroup that is a full subsemigroup of an infinite regular semigroup, but not of any finite regular semigroup.

Proof. As is explained by Evans in [3], the construction of a finitely presented infinite group $\mathbf{G}$ with no nontrivial finite quotients (as is done by Higman [6] for example) enables the construction of a partial group $\mathbf{P}$ that is embeddable in $\mathbf{G}$, but in no finite group. It is clear that $\mathbf{P}$ can be chosen to be symmetric, and that we can find a rank 2 extension $\mathbf{P}'$ of $\mathbf{P}$, also embeddable into $\mathbf{G}$. Then the semigroup $S_1(\mathbf{P}, \mathbf{P}')$ is a full subsemigroup of $\mathbf{B}_3(\mathbf{G})$, but is not a full subsemigroup of any regular semigroup whose subgroups are all finite. $\square$

Again, many other similar examples can be constructed in this way.

6. Subsemigroups of combinatorial completely 0-simple semigroups: An axiomatic characterisation

Here we give an axiomatic characterisation of the class $\mathbb{S}(\mathsf{CS}^0(\mathsf{Triv}))$ of subsemigroups of combinatorial completely 0-simple semigroups, in the language of semigroups with zero element. We use this to establish Theorem B part (B3′). We mention that the restriction to semigroups with zero is not all that artificial: a finite subsemigroup $\mathbf{S}$ of a completely 0-simple semigroup $\mathbf{C}$ either shares the multiplicative zero element of $\mathbf{C}$, or is itself a completely simple semigroup (and in the combinatorial case, a completely simple semigroup is simply a rectangular band). We also adopt a simplified notation for Rees matrix semigroups with trivial subgroups. In our definition, the nonzero elements of a Rees matrix semigroup are triples, with the central entry of each triple coming from a fixed group. In the combinatorial case, the group is trivial, and hence we can more simply consider just pairs, consisting of the left and right entries (coordinates).

6.1. Characterisation of subsemigroups

Define an equivalence $\lambda^\flat$ on a semigroup $\mathbf{S}$ with 0 to be the relation

$$\{(0,0)\} \cup \{(x,y) \mid (\exists u \in S)(\exists v, w \in S^1)\, x = uv \neq 0 \quad \& \quad y = uw \neq 0\}$$

and $\rho^\flat$ to be the relation

$$\{(0,0)\} \cup \{(x,y) \mid (\exists v \in S)(\exists u, w \in S^1)\, x = uv \neq 0 \quad \& \quad y = wv \neq 0\}.$$

Let λ and ρ be the transitive closure of these relations. Let $\gamma = \lambda \cap \rho$. The equivalence relations λ and ρ are easily seen to be the smallest choices of the relations by the same name in [5]. We let $\Delta_{\mathbf{S}}$ denote the diagonal relation on a semigroup $\mathbf{S}$; the subscript is dropped if the choice of $\mathbf{S}$ is obvious.

Theorem 6.1. *A semigroup $\mathbf{S}$ is embeddable in a combinatorial completely 0-simple semigroup if and only if $\gamma = \Delta_{\mathbf{S}}$ and the following law holds:*

$$\big(a\,\lambda\, b \quad \& \quad c\,\rho\, d \quad \& \quad ca \neq 0\big) \to db \neq 0. \tag{1}$$

These conditions are verifiable in polynomial time on finite semigroups.

Proof. Say that **S** is a subsemigroup of a combinatorial completely 0-simple semigroup **C**. We argue that $\lambda^\flat$ is contained within the $\mathscr{R}$-relation of **C**, and that $\rho^\flat$ is contained within the $\mathscr{L}$-relation of **C**. Let (x, y) with $x = uv \neq 0$ and $y = uw \neq 0$ for some $u \in S$ with $v, w \in S^1$. Then both x and y share the same left coordinate as u, hence they are $\mathscr{R}$-related. So $\lambda^\flat \subseteq \mathscr{R}$. By taking the transitive closure of both sides we obtain $\lambda \subseteq \mathscr{R}$ as required. The $\rho \subseteq \mathscr{L}$ case is by symmetry. Now we have $\Delta_{\mathbf{S}} \subseteq \gamma \subseteq \mathscr{H} = \Delta_{\mathbf{C}}$. The implication

$$(a\,\mathscr{R}\,b \quad \& \quad c\,\mathscr{L}\,d \quad \& \quad ca \neq 0) \rightarrow db \neq 0$$

obviously holds in every completely 0-simple semigroup, and since $a\,\lambda\,b$ implies $a\,\mathscr{R}\,b$ and $c\,\rho\,d$ implies $c\,\mathscr{L}\,d$ we have that implication (1) holds. This establishes necessity.

Now say the semigroup **S** satisfies both $\gamma = \Delta_{\mathbf{S}}$ and implication (1). Let U be the family of non-zero λ-classes and V the family of ρ-classes. We embed **S** into a combinatorial Rees matrix semigroup over a $V \times U$ sandwich matrix P (note that every such semigroup embeds into a completely 0-simple semigroup). We define $P_{v,u} = 1$ if there are $x \in v$ and $y \in u$ with $xy \neq 0$. Note that the value of P is independent of the choice of x, y in this definition since if $xy \neq 0$ and $x' \in x/\rho$ and $y' \in y/\lambda$ then implication (1) shows that $x'y' \neq 0$.

We now define the embedding ι. For $x \in S \setminus \{0\}$, define $\iota : x \mapsto (x/\lambda, x/\rho)$ and $\iota : 0 \mapsto 0$. As $\gamma = \Delta$ this map, ϕ is injective. Now suppose that $xy = z$ in **S**. If $z \neq 0$, then $P_{x/\rho,y/\lambda} = 1$ and z is λ-related to x because $z = xy$ and $x = x1$. Similarly, z is ρ-related to xy. Hence $\iota(z) = \iota(x)\iota(y)$. If $z = 0$, then $xy = 0$, so $P_{x/\rho,y/\lambda} = 0$ and $\iota(x)\iota(y) = 0$.

Clearly, the relations $\lambda^\flat$ and $\rho^\flat$ can be constructed in polynomial time from the Cayley table of a finite semigroup. It is well known the transitive closure of a relation computable in polynomial time is also computable in polynomial time. Therefore the relations λ and ρ on a finite semigroup can be constructed in polynomial time whence both the condition $\lambda \cap \rho = \Delta$ and implication (1) are verifiable in polynomial time on finite semigroups. □

The observation that the conditions in Theorem 6.1 can be verified on a finite semigroup in polynomial time justifies the corresponding claim in Theorem B part (B3′).

We now discuss how the conditions of Theorem 6.1 can be turned into universal sentences.

Write $L_n(u, v)$ to abbreviate the following expression (u, v, the x_i, u_i^L and v_i^L are variables):

$$\begin{aligned} & u \approx x_1 v_1^L \not\approx 0 \\ \& \quad & x_1 u_1^L \approx x_2 v_2^L \not\approx 0 \\ & \quad\vdots \\ \& \quad & x_{n-1} u_{n-1}^L \approx x_n v_n^L \not\approx 0 \\ \& \quad & x_n u_n^L \approx v \not\approx 0. \end{aligned}$$

Similarly $R_n(u, v)$ abbreviates

$$\begin{aligned} & u \approx v_1^R y_1 \not\approx 0 \\ \& \quad & u_1^R y_1 \approx v_2^R y_2 \not\approx 0 \\ & \quad\vdots \\ \& \quad & u_{n-1}^R y_{n-1} \approx v_n^R y_n \not\approx 0 \\ \& \quad & u_n^R y_n \approx v \not\approx 0. \end{aligned}$$

Let $\mathsf{L}_n^\flat(x, y)$ denote the set of all formulas obtained from $L_n(x, y)$ by deleting a subset (possibly empty) of the variables u_i^L, v_i^L for $i = 1, \ldots, n$ throughout $L_n(x, y)$. The notation $\bigvee \mathsf{L}_n^\flat(x, y)$ is the disjunction of all formulas in $\mathsf{L}_n^\flat(x, y)$. It is easy to see that in a semigroup **S**, we have $a\lambda b$ in **S** if and only if $\bigvee \mathsf{L}_n^\flat(a, b)$ holds in **S** for some n. We can dually define $\mathsf{R}_n^\flat(x, y)$ and $\bigvee \mathsf{R}_n^\flat(x, y)$, and obtain a corresponding statement for ρ.

Proposition 6.2. *The class of subsemigroups of combinatorial completely simple semigroups is axiomatised by the closure of the following sentences (universal quantifiers have been omitted):*

$$\left\{\left(\bigvee \mathsf{L}_n^\flat(a, b) \quad \& \quad \bigvee \mathsf{R}_n^\flat(a, b)\right) \rightarrow a \approx b \mid n \in \omega\right\} \bigcup$$
$$\left\{\left(\bigvee \mathsf{L}_n^\flat(a, b) \quad \& \quad \bigvee \mathsf{R}_n^\flat(c, d) \quad \& \quad ca \not\approx 0\right) \rightarrow db \not\approx 0 \mid n \in \omega\right\}.$$

Proof. This is because $a\,\lambda\,b$ if and only if there is a true evaluation $\bigvee \mathsf{L}_n^\flat(a, b)$ in **S** for some n, and similarly for ρ and $\bigvee \mathsf{R}_n^\flat(a, b)$. Also $\bigvee \mathsf{L}_n^\flat(a, b)$ has a true evaluation implies $\bigvee \mathsf{L}_m^\flat(a, b)$ has a true evaluation whenever $m \geq n$ (this allows us to use the same subscript n in the hypotheses of the implications). The result now follows from Theorem 6.1. □

Now define two new relations extending λ and ρ as follows. We let λ_{Br} be the equivalence relation generated by $\lambda^\flat \cup \{(x, y) \mid (\exists z \in S)\ zx \neq 0 \ \ \& \ \ zy \neq 0\}$. Similarly, let ρ_{Br} be the equivalence relation generated by $\rho^\flat \cup \{(x, y) \mid (\exists z \in S)\ xz \neq 0 \ \ \& \ \ yz \neq 0\}$. We let $\gamma_{\mathrm{Br}} := \lambda_{\mathrm{Br}} \cap \rho_{\mathrm{Br}}$. (We mention that these relations are also particular cases of a definition given on pages 86 and 87 in [5]. Note that in items 4 and 5 on page 87 of [5], the ρ and λ should be switched.)

Theorem 6.3. *A semigroup is embeddable in a combinatorial Brandt semigroup if and only if it is embeddable in a completely simple semigroup and $\gamma_{\mathrm{Br}} = \Delta$ and the following law holds:*

$$(a \,\lambda_{\mathrm{Br}}\, b \quad \& \quad c \,\rho_{\mathrm{Br}}\, d \quad \& \quad ca \neq 0) \rightarrow db \neq 0. \tag{2}$$

This property is verifiable in polynomial time on finite semigroups.

Proof. The conditions are obviously necessary since if $zx \neq 0$ and $zy \neq 0$ for some elements x, y, z of a Brandt semigroup, then $x \mathscr{R} y$, and dually for $\mathscr{L}$. Now for sufficiency. As in the proof of Theorem 6.1, we let U denote the set of λ-classes (except for $\{0\}$) of $\mathbf{S}$ and V denote the ρ-classes. We construct the same $V \times U$ sandwich matrix P of entries from 0, 1 by setting $P_{v,u} = 1$ if and only if $xy \neq 0$ for some $x \in u$ and $y \in v$. As before, this is an injective homomorphism into a Rees matrix semigroup. To prove that this Rees matrix semigroup embeds in a combinatorial Brandt semigroup, we just have to prove that each row and column of P contains at most one 1. Say $P_{u,v} = 1$ and $P_{u,w} = 1$. So we have $z \in u$, $x \in v$ and $y \in w$ with $zx \neq 0$ and $zy \neq 0$. By the definition of λ_{Br} we have $x\lambda_{\mathrm{Br}}y$ and then $v = w$. The column case is by symmetry.

The polynomial time complexity of finite membership in this class is due to the fact that the relations λ_{Br} and ρ_{Br} on a finite semigroup can be constructed in a polynomial number of steps. □

One can construct an infinite family of universal sentences from this result in much the same way as in Proposition 6.2, however because λ_{Br} is the closure of a relation defined by two different conditions, this system becomes somewhat cumbersome.

Recall that a regular semigroup is *orthodox* if its idempotent elements form a subsemigroup. One can also give a version of Theorem 6.3 for the class of combinatorial orthodox completely 0-simple semigroups. Here we use the original λ and ρ, but adjoin the extra law $(xv \not\approx 0 \quad \& \quad uv \not\approx 0 \quad \& \quad uy \not\approx 0) \rightarrow xy \not\approx 0$. We leave the details to the reader (it is very similar to the second half of the proof of Theorem 6.3).

6.2. *Proof of* Theorem C

The systems of axioms so far obtained from Proposition 6.2 and Theorem 6.3 are infinite in character. We now show that this is necessary. In fact, we show that a universal class has no finite axiomatisation in first order logic, provided it contains the class of 3-nilpotent subsemigroups of combinatorial Brandt semigroups, and is contained within (and possibly equal to) the universal class generated by the combinatorial completely 0-simple semigroups. To achieve this it suffices to find a family $\mathscr{S} := \{\mathbf{S}_i \mid i \in \mathbb{N}\}$ of semigroups, with the property that no member of $\mathscr{S}$ is a subsemigroup of a combinatorial completely 0-simple semigroup, yet for each $n \in \mathbb{N}$ there exists $i \in \mathbb{N}$ such that the n-generated subsemigroups of $\mathbf{S}_i$ are 3-nilpotent and lie in the universal class generated by the combinatorial Brandt semigroups.

Let X_n denote the set

$$\{a_i, b_i, c_i^L, c_i^R, d_i^L, d_i^R \mid 1 \leq i \leq n\}.$$

Let $\mathbf{X}_n^\sharp$ denote the 3-nilpotent semigroup generated by X_n with all products equal to zero, except for the following products which observe the stated equalities:

$$\begin{aligned}
&a_1 d_1^L = d_1^R b_1,\\
&a_i c_i^L = a_{i+1} d_{i+1}^L \quad \text{for } i = 1, \dots, n-1,\\
&c_i^R b_i = d_{i+1}^R b_{i+1} \quad \text{for } i = 1, \dots, n-1,\\
&a_n c_n^L = d_n^R b_n.
\end{aligned}$$

It is obvious that the definition of $\mathbf{X}_n^\sharp$ is based around the equalities used in the axiomatisation of subsemigroups of combinatorial completely 0-simple semigroups in Proposition 6.2. Indeed if we let g and h denote the elements $a_1 d_1^L$ and $a_n d_n^L$ respectively, we see that $\mathbf{X}_n^\sharp$ has the property $L_n(g, h) \quad \& \quad R_n(g, h)$ but has $g \neq h$. So $\mathbf{X}_n^\sharp$ is not a subsemigroup of a combinatorial completely 0-simple semigroup. For i strictly between 1 and n, let $\mathbf{Y}_{n,i}$ denote the subsemigroup of $\mathbf{X}_n^\sharp$ generated by the generators in X_n whose numerical subscript is not equal to i. We now show that any m-generated subsemigroup of $\mathbf{X}_{2m+3}^\sharp$ is embeddable in a Brandt semigroup; so our family $\mathscr{S}$ is $\{\mathbf{X}_{2n+3}^\sharp \mid n \in \mathbb{N}\}$.

Let A be any m-element subset of $\mathbf{X}_{2m+3}^\sharp$ and let B be the set of elements of the generator set X_{2m+3} that divide a nonzero element of A in $\mathbf{X}_{2m+3}^\sharp$. Each prime element of $\mathbf{X}_{2m+3}^\sharp$ has a single numerical subscript, while each nonzero composite element of $\mathbf{X}_{2m+3}^\sharp$ can be written as a product of two generators, and hence involving at most two numerical subscripts. Since there are only m elements of A, and $2m+3$ numerical subscripts in the elements of X_{2m+3}, there are at least three numbers amongst $1, \dots, 2m+3$ that are not subscripts of any element of B. One of these numbers must be strictly between 1 and $2m+3$, say i. So the subsemigroup of $\mathbf{X}_{2m+3}^\sharp$ generated by A lies inside the subsemigroup $\mathbf{Y}_{2m+3,i}$. Hence it will suffice to show that, for any n and any i with $1 < i < n$, the semigroup $\mathbf{Y}_{n,i}$ is embeddable in a combinatorial Brandt semigroup.

Let I_i be the set

$$\{\ell_j, r_j \mid 1 \le j \le i\} \cup \{p_j, q_j \mid 1 \le j < i\} \cup \{\ell'_j, r'_j \mid i \le j \le n\} \cup \{p'_j, q'_j \mid i < j \le n\}.$$

We make the following assignments into the combinatorial Brandt semigroup $\mathbf{B}$ with the $I_i \times I_i$ identity matrix: for $j < i$ let

$$\begin{array}{ll} a_j \mapsto (\ell_1, p_j) & b_j \mapsto (q_j, r_1), \\ c_j^L \mapsto (p_j, r_{j+1}) & c_j^R \mapsto (\ell_{j+1}, q_j), \\ d_j^L \mapsto (p_j, r_j) & d_j^R \mapsto (\ell_j, q_j), \end{array}$$

and for $j > i$ let

$$\begin{array}{ll} a_j \mapsto (\ell'_n, p'_j) & b_j \mapsto (q'_j, r'_n), \\ c_j^L \mapsto (p'_j, r'_j) & c_j^R \mapsto (\ell'_j, q'_j), \\ d_j^L \mapsto (p'_j, r'_{j-1}) & d_j^R \mapsto (\ell'_{j-1}, q'_j). \end{array}$$

This, with $0 \mapsto 0$, gives an isomorphism from $\mathbf{Y}_{n,i}$ into the Brandt semigroup $\mathbf{B}$; we leave the verification to the reader. This completes the proof of Theorem C, as well as the nonfinite axiomatisability claim of Theorem B part (B3′).

7. 3-nilpotent semigroups in quasivarieties generated by completely 0-simple semigroups

Recall that Theorem B part (B3) is established in [5, Theorem 2.5]. In fact [5, Theorem 2.5] uses the property

$$xyz \approx 0 \to (xy \approx 0 \vee yz \approx 0) \tag{3}$$

instead of our formula $xy \approx 0 \vee yz \approx 0$, but the two are easily seen to be equivalent since every product xyz is equal to 0 in a 3-nilpotent semigroup. Also, the statement of [5, Theorem 2.5] is not phrased in terms of pseudovarieties at all, however the proof shows that a 3-nilpotent semigroup satisfying implication (3), which is necessary for embeddability into a completely 0-simple semigroup, embeds into a completely 0-simple semigroup over *any* sufficiently large group. The ultraproduct closure of any nontrivial pseudovariety of groups contains groups of arbitrary cardinality, which is why the statement in Theorem B part (B3) holds.

7.1. Theorem B *part (B2)*

Let $\mathbf{S}$ be a 3-nilpotent semigroup. Let us define four subsets of S:

- $L = L_{\mathbf{S}} := \{x \mid (\exists y)\, xy \ne 0\}$;
- $R = R_{\mathbf{S}} := \{y \mid (\exists x)\, xy \ne 0\}$;
- $C = C_{\mathbf{S}} := \{z \mid (\exists x, y)\, xy = z \ne 0\}$; and
- $N = N_{\mathbf{S}} := S \setminus (L \cup R \cup M \cup \{0\})$.

Consider a product $xy = z$, where z is non-zero. The set N (null elements) corresponds to non-zero elements that cannot be any of x, y or z. On the other hand x must lie in L, y must lie in R and z must lie in C (non-zero composite elements).

The four sets $L \cup R$, C, N, $\{0\}$ partition $\mathbf{S}$, but L and R need not be disjoint. If they are, then we say that $\mathbf{S}$ is *split*. (This class of 3-nilpotent semigroups first introduced by Sapir [15] has played a significant role in several of his papers, see, e.g., [16].) It is easy to see that $\mathbf{S}$ is split if and only if $\mathbf{S}$ satisfies the property $xy \approx 0 \vee yz \approx 0$, appearing in Theorem B part (B3), and so the split 3-nilpotent semigroups are exactly the class $\mathsf{Nil}_3 \cap \mathbb{S}(\mathsf{CS}_0(\mathscr{G}))$.

The law characterising split 3-nilpotent semigroups is obviously equivalent to

$$(xy \approx u \quad \& \quad yz \approx v) \to (u \approx 0 \vee v \approx 0)\,.$$

Let us say that a 3-nilpotent semigroup is *weakly split* if it satisfies

$$(xy \approx w \quad \& \quad yz \approx w) \to w \approx 0,$$

or equivalently, if $xy \approx yz \to xy \approx 0$ holds. A weakly split 3-nilpotent semigroup need not be split. The weakly split property is obviously polynomial time verifiable, so Theorem B part (B2) will be proved once the following proposition is proved.

Proposition 7.1. *For any non-empty class of groups $\mathscr{K}$, the quasivariety $\mathsf{Nil}_3 \cap \mathbb{Q}(\mathsf{CS}_0(\mathscr{K}))$ is precisely the class of weakly split 3-nilpotent semigroups.*

Proof. Recall that $\mathscr{G}$ denotes the class of all groups. We first show that all semigroups in $\mathsf{Nil}_3 \cap \mathbb{SP}(\mathsf{CS}^0(\mathscr{G}))$ are weakly split, and then show that any weakly split 3-nilpotent semigroup is contained in $\mathsf{Nil}_3 \cap \mathbb{SP}(\mathsf{CS}^0(\mathsf{Triv}))$. This will complete the proof since we will have $\mathsf{Nil}_3 \cap \mathbb{SP}(\mathsf{CS}^0(\mathsf{Triv})) \subseteq \mathsf{Nil}_3 \cap \mathbb{SP}(\mathsf{CS}^0(\mathscr{K})) \subseteq \mathsf{Nil}_3 \cap \mathbb{SP}(\mathsf{CS}^0(\mathscr{G})) \subseteq \mathsf{Nil}_3 \cap \mathbb{SP}(\mathsf{CS}^0(\mathsf{Triv}))$.

Consider a 3-nilpotent semigroup $\mathbf{T}$ that is not weakly split. So we have elements $x, y, z, w \in T$ with $xy = yz = w \neq 0$. Consider an arbitrary homomorphism ϕ from $\mathbf{T}$ into a completely 0-simple semigroup. We have $\phi(x)\phi(y) = \phi(y)\phi(z) = \phi(w)$ which in a completely 0-simple semigroup implies $\phi(w) = \phi(x)\phi(y)\phi(z) = \phi(xyz) = \phi(0) = 0$. Hence, every homomorphism from $\mathbf{T}$ to a member of $\mathbf{CS}_0(\mathscr{G})$ identifies w with 0, showing that $\mathbf{T} \notin \mathbb{SP}(\mathbf{CS}_0(\mathscr{G}))$. This completes the proof of necessity.

Now say that $\mathbf{T}$ is weakly split. We need to show that we can separate arbitrary elements of T by way of homomorphisms into combinatorial completely 0-simple semigroups. If a is a prime element in T then we can separate a from all other elements by way of a homomorphism into the two element null semigroup (a subsemigroup of any combinatorial completely 0-simple semigroup with zero divisors). Now we show that we can do the same when a is a non-zero composite element, which will complete the proof.

Let I_a be the ideal of $\mathbf{T}$ consisting of all elements that do not divide a, and let $\mathbf{T}_a := \mathbf{T}/I_a$. Let $L = L_{\mathbf{T}_a}$ and $R = R_{\mathbf{T}_a}$. Let ∞ be a symbol not in T_a. We embed $\mathbf{T}_a$ into a combinatorial Rees matrix semigroup with $(L \cup \{\infty\}) \times (R \cup \{\infty\})$ sandwich matrix (which in turn embeds into a completely 0-simple semigroup). We map $x \in L$ to (∞, x) and $y \in R$ to (y, ∞). The element a is mapped to (∞, ∞) and all other elements are mapped to 0. Now we define the sandwich matrix P by $P_{xy} = 1$ if $xy = a$ and 0 otherwise. This is obviously an embedding. The corresponding homomorphism ϕ from $\mathbf{T}$ to the just constructed semigroup has $\phi^{-1}(\infty, \infty) = \{a\}$ as required. □

7.2. *Brandt semigroups: Theorem B part (B1)*

In this subsection we prove part (B1) of Theorem B, which will demonstrate the inequivalence of condition (A7) with condition (A1) in Theorem A. The situation for Brandt semigroups is certainly different to that of completely 0-simple semigroups, since a result from Jackson and Volkov [10] establishes the nonfinite axiomatisability of the quasivariety $\mathscr{Q}$ generated by any class of inverse semigroups (in the type $\langle 2 \rangle$), provided only that $\mathscr{Q}$ contains a proper 3-nilpotent semigroup (here "proper" means "not 2-nilpotent"). Since the 3×3 Brandt semigroup contains a 3-nilpotent subsemigroup, this yields the nonfinite axiomatisability claim in Theorem B part (B1).

We now prove the remaining claims from Theorem B part (B1). Here is an outline of the proof: we first describe some necessary conditions for membership in $\mathsf{Nil}_3 \cap \mathbb{Q}(\mathsf{Br}(\mathscr{G}))$, and observe that these conditions can be tested in polynomial time; then we verify that the conditions are sufficient for membership of a 3-nilpotent semigroup in $\mathsf{Nil}_3 \cap \mathbb{Q}(\mathsf{Br}(\mathsf{Triv}))$. Along the way, we observe how to turn the three conditions into an actual quasiequational axiomatisation for the quasivariety in question.

We are going to consider a 3-nilpotent semigroup $\mathbf{T}$ lying in $\mathbb{Q}(\mathbf{Br}(\mathscr{G}))$, but we first work more generally and construct a family of congruences on an arbitrary 3-nilpotent semigroup. To motivate the construction of these congruences, observe that as $\mathbf{T} \in \mathbb{SPP}_u(\mathsf{Br}(\mathscr{G})) = \mathbb{SP}(\mathsf{Br}(\mathscr{G}))$, for each pair of distinct elements $b, c \in T$ there is a homomorphism $\phi : \mathbf{T} \to \mathbf{B}_\kappa(\mathbf{G})$ (for some Brandt semigroup $\mathbf{B}_\kappa(\mathbf{G})$) with $\phi(b) \neq \phi(c)$. In particular, we may assume without loss of generality that $\phi(c) \neq 0$. We attempt to approximate the kernel of this homomorphism in the case when c is a composite element. We inductively construct a large congruence θ_c with $\theta_c \subseteq \ker(\phi)$. The actual construction will take place in an arbitrary 3-nilpotent semigroup $\mathbf{S}$ and with an arbitrary element $a \in S$, but we observe some consequences of the assumption that a is the composite element c of $\mathbf{T} \in \mathbb{Q}(\mathsf{Br}(\mathscr{G}))$.

So, let $\mathbf{S}$ be an arbitrary 3-nilpotent semigroup and $a \in S$. We begin with $\theta_a^0 := \Delta$. Now say that we have defined a congruence θ_a^i. Create a new relation ξ_a^i by adjoining the pair (x, y) to θ_a^i whenever there is $z \in S$ such that either both $xz/\theta_a^i = a/\theta_a^i$ and $yz/\theta_a^i = a/\theta_a^i$ or both $zx/\theta_a^i = a/\theta_a^i$ and $zy/\theta_a^i = a/\theta_a^i$. Define the relation θ_a^{i+1} to be the congruence generated by ξ_a^i. Now let θ_a denote the congruence $\cup_{i \in \omega} \theta_a^i$.

Remark 7.2. If $\mathbf{S}$ is finite (say $|S| = n$), the congruence θ_a^i can be constructed in polynomial time from the Cayley table of $\mathbf{S}$. So can the congruence θ_a, since if $\theta_a^i = \theta_a^{i+1}$ then $\theta_a^i = \theta_a$ (so that $\theta_a^{n^2} = \theta_a$).

So far the definition of θ_a makes sense in any 3-nilpotent semigroup $\mathbf{S}$ (and for any element $a \in S$; not necessarily composite nor nonzero). When a is prime, the relation θ_a is just the diagonal relation, while if $a = 0$ the relation θ_a is the universal relation. Now we look at the particular choice $\mathbf{S} := \mathbf{T}$, and the nonzero composite element $c \in T$ sent by the homomorphism ϕ to a non-zero element of a Brandt semigroup.

Claim 1. $\theta_c \subseteq \ker(\phi)$.

Proof. We prove by induction that this is true for the θ_c^i, from which the claim will follow.

It is certainly true for $i = 0$. Now say that $\theta_c^i \subseteq \ker(\phi)$ and $0 \notin c/\theta_c^i$. Let $(x, y) \in \rho_c^{i+1} \setminus \theta_c^i$. Up to symmetry we assume that there is $z \in T$ such that $xz/\theta_c^i = c/\theta_c^i = yz/\theta_c^i$. So $\phi(x)\phi(z) = \phi(y)\phi(z) = \phi(c) \neq 0$. In a Brandt semigroup this implies that $\phi(x) = \phi(y)$. So $(x, y) \in \ker(\phi)$. This shows that $\xi_c^i \subseteq \ker(\phi)$. So the congruence generated by ξ_c^i, that is, θ_c^{i+1}, also lies inside $\ker(\phi)$. □

So far we have found the following necessary conditions for lying in $\mathsf{Br}(\mathscr{G})$ (here b and c are distinct composite elements, where $c \neq 0$):

(Br1) at least one of $b \notin c/\theta_c$ or $c \notin b/\theta_b$ are true (because ϕ separates b from c);
(Br2) for each composite $b \in T$ we have $0/\theta_b \neq b/\theta_b$ (because $\phi(b) \neq \phi(0)$);
(Br3) for each $x, y, z \in T$ at least one of the following equalities fails: $x/\theta_b y/\theta_b = b/\theta_b$ and $y/\theta_b z/\theta_b = b/\theta_b$ (because otherwise we would have $\phi(x)\phi(y) = \phi(y)\phi(z) = \phi(b) \neq \phi(0)$, contradicting the fact that $\phi(\mathbf{T})$ is a split 3-nilpotent semigroup).

These conditions can be tested in polynomial time if $\mathbf{T}$ is finite. Note that if a is a prime element then θ_a is the diagonal, while θ_0 is the universal relation if a is 0. As we now explain, this enables us to write each of the 3 conditions as quasiequations in the language of 3-nilpotent semigroups (we can use the symbol 0 since it is a term operation for 3-nilpotent semigroups). First, we may write property (Br1) as $(a\,\theta_a\, b \quad \& \quad a\,\theta_b\, b) \to a = b$, property (Br2) as $a\,\theta_a\, 0 \to a = 0$, and property (Br3) as $(xy\,\theta_a\, a \quad \& \quad yz\,\theta_a\, a) \to a = 0$. Statements of the form $u\,\theta_w\, v$ in these expression correspond to certain conjunctions of semigroup equalities of some finite (but in general unbounded) length. In this way, the three conditions can in principle be written as an infinite set of abstract quasiequations, which are necessarily satisfied by any 3-nilpotent semigroup in the quasivariety generated by $\mathsf{Br}(\mathscr{G})$.

These quasiequations along with the 3-nilpotent axiom $x_1y_1z_1 \approx x_2y_2z_2$ (which allows us to write 0 in place of any word xyz) actually fully axiomatise the 3-nilpotent semigroups in $\mathbb{Q}(\mathsf{Br}(\mathsf{Triv}))$, since we now prove that the three conditions are sufficient for membership in $\mathbb{Q}(\mathsf{Br}(\mathsf{Triv}))$. To this end, we now let $\mathbf{S}$ be a 3-nilpotent semigroup satisfying (Br1)–(Br3).

Let a and b be distinct elements of S. If one of a or b is prime, then we can separate a from b using a homomorphism into a null semigroup (so certainly into a combinatorial Brandt semigroup). So we may assume that a is a non-zero composite element and without loss of generality we may assume that $b \notin a/\theta_a$, by (Br1).

Let η_a denote the congruence extending θ_a obtained by including the ideal

$$I_a := \{x/\theta_a \mid x/\theta_a \text{ does not divide } a/\theta_a\}$$

in the block $0/\theta_a$. Note that $b \notin a/\eta_a$. Let $\mathbf{S}_a$ denote $\mathbf{S}/\eta_a$.

Claim 2. *$\mathbf{S}_a$ is split and $S_a \cdot S_a = \{a/\eta_a, 0/\eta_a\}$ and $L := L_{\mathbf{S}_a}, R := R_{\mathbf{S}_a}$ are disjoint.*

Proof. By the definition of I_a (and since $\mathbf{S}$ is 3-nilpotent), there is at most one non-zero composite element, namely a/η_a. By (Br2) we have $a/\eta_a \neq 0/\eta_a$ (so there is precisely one non-zero composite element).

Assume that $x/\eta_a y/\eta_a = a/\xi_a$ and $y/\eta_a z/\eta_a = a/\eta_a$. So $x/\theta_a y/\theta_a = a/\theta_a$ and $y/\theta_a z/\theta_a = a/\theta_a$, contradicting (Br3). Hence $\mathbf{S}_a$ is split. □

Claim 3. *If $x/\eta_a y/\eta_a = a/\eta_a$ then $x/\eta_a u/\eta_a = 0/\eta_a = v/\eta_a y/\eta_a$ for every $u/\eta_a \in S_a \setminus \{y/\eta_a\}$ and $v/\eta_a \in S_a \setminus \{x/\eta_a\}$.*

Proof. Say that $x/\eta_a y/\eta_a = a/\eta_a$ and $x/\eta_a u/\eta_a = a/\eta_a$. So $x/\theta_a y/\theta_a = a/\theta_a$ and $x/\theta_a u/\theta_a = a/\theta_a$. So there is $i \in \omega$ such that $x/\theta_a^i y/\theta_a^i = a/\theta_a^i$ and $x/\theta_a^i u/\theta_a^i = a/\theta_a^i$. So $u/\theta_a^{i+1} = y/\theta_a^{i+1}$ and then $y/\eta_a = u/\eta_a$. The other case is the same up to symmetry. □

Claims 2 and 3 show that the sets L and R as constructed for $\mathbf{S}_a$ in Claim 2 have a very special form: there is a bijection $\iota : L \to R$ with the property that each $c, d \in \mathbf{S}_a$ have $cd = a/\eta_a$ if and only if $d = \iota(c)$. Let ℓ, r be two distinct symbols not appearing in L. We now represent $\mathbf{S}_a$ as a subsemigroup of the combinatorial Brandt semigroup of dimension $L \cup \{\ell, r\}$. The map is defined as follows: for $x \in L$ we map $x \mapsto (\ell, x)$. For $y = \iota(x) \in R$ we map $y \mapsto (x, r)$. We map a/η_a to (ℓ, r) and $0/\eta_a$ to 0. This is clearly an injective homomorphism. As $b \notin a/\eta_a$ we have separated a from b by an injective homomorphism into a combinatorial Brandt semigroup. The pair a, b was arbitrary, and so we have shown that $\mathbf{S} \in \mathbb{Q}(\mathsf{Br}(\mathsf{Triv}))$, which completes the proof of Theorem B part (B1).

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