Some $\mathcal{I}$-convergent sequence spaces defined by using sequence of moduli and $n$-normed space

S.K. Sharma, Ayhan Esi *

School of Mathematics, Shri Mata Vaishno Devi University, Katra 182 320, J&K, India
Department of Mathematics, Adiyaman University, Adiyaman 02040, Turkey

Received 7 November 2012; revised 25 December 2012; accepted 21 January 2013
Available online 7 March 2013

Abstract In the present paper we study some $\mathcal{I}$-convergent sequence spaces defined by a sequence of modulus functions over $n$-normed spaces. We also examine some topological properties and prove some inclusion relations between these spaces.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 40A05, 40C05, 46A45

© 2013 Egyptian Mathematical Society. Production and hosting by Elsevier B.V.
Open access under CC BY-NC-ND license.

1. Introduction and preliminaries

The concept of 2-normed spaces was initially developed by Gähler [1] in the mid of 1960s, while that of $n$-normed spaces one can see in Misiak [2]. Since then, many others have studied this concept and obtained various results, see Gunawan [3,4], Gunawan and Mashadi [5], Mursaleen and Mohiuddine [26], Mohiuddine et al. [27] and Mohiuddine and Aiyup [28]. Let $n \in \mathbb{N}$ and $X$ be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension $d$, where $d \geq n \geq 2$. A real valued function $\| \ldots \|$ on $X^n$ satisfying the following four conditions:

1. $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent in $X$;
2. $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutation;
3. $\|x_1 x_2, x_3, \ldots, x_n\| = \|x_1, x_2, \ldots, x_n\|$ for any $x \in \mathbb{K}$, and
4. $\|x + x', x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|

is called a $n$-norm on $X$ and the pair $(X, \| \ldots \|)$ is called a $n$-normed space over the field $\mathbb{K}$.

For example, we may take $X = \mathbb{R}^d$ being equipped with the Euclidean $n$-norm $\|x_1, x_2, \ldots, x_n\|_E = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ which may be given explicitly by the formula $\|x_1, x_2, \ldots, x_n\|_E = |\det(x_n)|,$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$. Let $(X, \| \ldots \|)$ be an $n$-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ be linearly independent set in $X$. Then the following function $\| \ldots \|_n$ on $X^{n-1}$ defined by $\|x_1, x_2, \ldots, x_n\|_n = \max\{|x_1, x_2, \ldots, x_{n-1}, a_i| : i = 1, 2, \ldots, n\}$ defines an $(n - 1)$-norm on $X$ with respect to $\{a_1, a_2, \ldots, a_n\}$.

The standard $n$-norm on $X$, a real inner product space of dimension $d \geq n$ is as follows:

...
where \( \langle \cdot , \cdot \rangle \) denotes the inner product on \( X \). If \( X = \mathbb{R}^d \), then this \( n \)-norm is exactly the same as the Euclidean \( n \)-norm \( \| x_1, x_2, \ldots, x_d \|_n \) mentioned earlier. For \( n = 1 \), this \( n \)-norm is the usual norm \( \| x \| = \langle x, x \rangle^{1/2} \).

A sequence \( (x_k) \) in a \( n \)-normed space \( (X, \| \cdot \|_n) \) is said to converge to some \( x \in X \) if

\[
\lim_{k \to \infty} \| x_k - x \|_n = 0 \quad \text{for every } x_1, x_2, \ldots, x_{n-1} \in X.
\]

A sequence \( (x_k) \) in a \( n \)-normed space \( (X, \| \cdot \|_n) \) is said to be Cauchy if

\[
\lim_{k \to \infty} \| x_k - x_\nu, z_1, \ldots, z_{n-1} \|_n = 0 \quad \text{for every } x_1, x_2, \ldots, x_{n-1} \in X.
\]

If every Cauchy sequence in \( X \) converges to some \( x \in X \), then \( X \) is said to be complete with respect to the \( n \)-norm. Any complete \( n \)-normed space is said to be \( n \)-Banach space.

The notion of ideal convergence was introduced first by Kostyrko et al. [6] as a generalization of statistical convergence which was further studied in topological spaces by Das et al. [7]. More applications of ideals can be seen in [7,8,24,25].

Let \( (X, \| \cdot \|) \) be a normed space. A sequence \( (x_n)_{n \in \mathbb{N}} \) of elements of \( X \) is called statistically convergent to \( x \in X \) if the set \( A(\varepsilon) = \{ n \in \mathbb{N} : \| x_n - x \| > \varepsilon \} \) has natural density zero for each \( \varepsilon > 0 \).

A family \( I \subset 2^X \) of subsets of a non empty set \( Y \) is said to be an ideal in \( Y \) if

1. \( \emptyset \in I \)
2. \( A, B \in I \) imply \( A \cup B \in I \)
3. \( A \in I, B \subset A \) imply \( B \in I \),

while an admissible ideal \( I \) of \( Y \) further satisfies \( \{ x \} \in I \) for each \( x \in Y \) see [9]. Given \( I \subset 2^X \) be a non trivial ideal in \( \mathbb{N} \). A sequence \( (x_n)_{n \in \mathbb{N}} \) in \( X \) is said to be \( I \)-convergent to \( x \in X \), if for each \( \varepsilon > 0 \) the set \( A(\varepsilon) = \{ n \in \mathbb{N} : \| x_n - x \| > \varepsilon \} \) belongs to \( I \) see [6].

A sequence of positive integers \( \theta = (k_j) \) is called lacunary if \( k_0 = 0, 0 < k_1 < k_2 - 1 \) and \( h_j = k_j - k_{j-1} \to \infty \) as \( r \to \infty \). The intervals determined by \( \theta \) will be denoted by \( I_r = (k_{j-1}, k_j) \) and \( q_r = \frac{k_j}{h_j} \). The space of lacunary strongly convergent sequences \( N_0 \) was defined by Freedman et al. [10] as

\[
N_0 = \left\{ x \in X : \lim_{r \to \infty} \frac{1}{h_j} \sum_{k \in I_r} \| x_k - x \| = 0 \quad \text{for some } J \right\}.
\]

Strongly almost convergent sequence was introduced and studied by Maddox [11] and Freedman et al. [10]. Parashar and Choudhary [12] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function \( M \), which generalized the well-known Orlicz sequence spaces \( [C, 1, p] \), \( [C, 1, p]_0 \) and \( [C, 1, p]_\infty \). It may be noted here that the space of strongly summable sequences were discussed by Maddox [13].

Mursaleen and Noman [14] introduced the notion of \( \lambda \)-convergent and \( \lambda \)-bounded sequences as follows:

Let \( \lambda = (\lambda_k)_{k=1}^{\infty} \) be a strictly increasing sequence of positive real numbers tending to infinity i.e.

\[
0 < \lambda_0 < \lambda_1 < \cdots \quad \text{and} \quad \lambda_k \to \infty \quad \text{as} \quad k \to \infty
\]

and said that a sequence \( x = (x_n) \in w \) is \( \lambda \)-convergent to the number \( L \), called the \( \lambda \)-limit of \( x \) if \( A_m(x) \to L \) as \( m \to \infty \), where

\[
\lambda_m(x) = \frac{1}{m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) x_k.
\]

The sequence \( x = (x_k) \in w \) is \( \lambda \)-bounded if \( \sup_{m} \| A_m(x) \| < \infty \).

It is well known [14] that if \( \lim_{m} x_m = a \) in the ordinary sense of convergence, then

\[
\lim_{m} \frac{1}{\lambda_m} \left( \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) |x_k - a| \right) = 0.
\]

This implies that

\[
\lim_{m} |A_m(x) - a| = \lim_{m} \frac{1}{\lambda_m} \left( \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) |x_k - a| \right) = 0
\]

which yields that \( \lim_{m} A_m(x) = a \) and hence \( x = (x_k) \in w \) is \( \lambda \)-convergent to \( a \).

A modulus function is a function \( f : [0, \infty) \to [0, \infty) \) such that

1. \( f(x) = 0 \) if and only if \( x = 0 \),
2. \( f(x + y) \leq f(x) + f(y) \) for all \( x, y \geq 0, y \geq 0 \),
3. \( f \) is increasing
4. \( f \) is continuous from right at 0.1t follows from (i) and (iv) that \( f \) must be continuous everywhere on \( [0, \infty) \).

A sequence of modulus function \( F = (f_m) \), we give the following conditions:

5. \( \sup_{x > 0} f(x) < \infty \) for all \( x > 0 \),
6. \( \lim_{x \to 0} f(x) = 0 \) uniformly in \( k = \infty \).

We remark that in case \( f = (f_k) \) for all \( k \), where \( f \) is a modulus, the conditions (v) and (vi) are automatically fulfilled. The modulus function may be bounded or unbounded. For example, if we take \( f(x) = \frac{1}{x^p} \), then \( f(x) \) is bounded. If \( f(x) = x^p \), \( 0 < p < 1 \), then the modulus \( f(x) \) is unbounded. Subsequently, modulus function has been discussed in [12,15–23] and references therein.

Let \( I \) be an admissible ideal, \( F = (f_k) \) be a sequence of modulus functions, \( (X, \| \cdot \|) \) be an \( n \)-normed space, \( p = (p_k) \) be a sequence of positive real numbers and \( u = (u_k) \) be a sequence of strictly positive real numbers. By \( w(n - X) \) we denote the space of all sequences defined over \( n \)-normed space \( (X, \| \cdot \|) \). In the present paper, we define the following classes of sequences:

\[
\{ x = (x_k) \in w(n - X) : \left\{ \sum_{k \in I_r} f_k(\| A_k(x) - L_z, z_1, \ldots, z_{n-1} \|) > \varepsilon, \text{ for some } I_r \text{ and for every } z_1, z_2, \ldots, z_{n-1} \in X \right\} \in I \}
\]

\[
\{ x = (x_k) \in w(n - X) : \left\{ \sum_{k \in I_r} f_k(\| A_k(x) \|, z_1, z_2, \ldots, z_{n-1} \|) > \varepsilon, \text{ for every } z_1, z_2, \ldots, z_{n-1} \in X \right\} \in I \}
\]
If \( F(x) = x \), we get

\[
[N_0, A, u, p, \ldots, ||X]] = \left\{ x = (x_k) \in \mathbb{R}^{n(X)} : \left[ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in \mathbb{Z}} u_k(||A_k(x) - L, z_1, z_2, \ldots, z_{n-1}||)^p \right] \geq \epsilon \right\}
\]

If \( p = (p_k) = 1 \), we get

\[
[N_0, F, A, u, ||X]] = \left\{ x = (x_k) \in \mathbb{R}^{n(X)} : \left[ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in \mathbb{Z}} f_k(||A_k(x) - L, z_1, z_2, \ldots, z_{n-1}||) \geq \epsilon \right] \right\}
\]

If \( p = (p_k) = 1 \) and \( u = (u_k) = 1 \), we get

\[
[N_0, F, A, u, ||X]] = \left\{ x = (x_k) \in \mathbb{R}^{n(X)} : \left[ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in \mathbb{Z}} f_k(||A_k(x) - L, z_1, z_2, \ldots, z_{n-1}||) \geq \epsilon \right] \right\}
\]

The following inequality will be used throughout the paper. If

\[
0 \leq p_k \leq \text{supp} \leq H, D = \max(1, 2^{2\delta - 1})
\]

then

\[
|a_k + b_k|^p \leq D(|a_k|^p + |b_k|^p)
\]

for all \( k \) and \( a_k, b_k \in \mathbb{C} \). Also \( |a|^p \leq \max(1, |a|^p) \) for all \( a \in \mathbb{C} \).

The aim of this paper is to study \( \mathcal{I} \)-convergent sequence spaces defined by a sequence of modulus functions in \( \mathbb{N} \)-normed spaces and examine some topological properties and inclusion relations between the spaces \([N_0, F, A, u, p, ||X]] \) and \([N_0, F, A, u, p, ||X]]\).

2. Main results

Theorem 2.1. Let \( F = (f_k) \) be a sequence of modulus functions, \( p = (p_k) \) be a bounded sequence of positive real numbers and \( u = (u_k) \) be any sequence of strictly positive real numbers, the spaces \([N_0, F, A, u, p, ||X]]\) and \([N_0, F, A, u, p, ||X]]\) are linear over the field of complex numbers \( \mathbb{C} \).

Proof. Let \( x = (x_k), y = (y_k) \in [N_0, F, A, u, p, ||X]]\). Then, there exist positive integers \( M_n \) and \( N_p \) such that \( 0 \leq M_n \leq M \) and \( 0 \leq N_p \leq N. \) Since \( ||X, \ldots, ||X, \ldots \) is a \( n \)-norm and \((f_k)\) is a sequence of modulus functions for all \( k \) and also by using (1.1), the following inequality holds

\[
\sum_{k \in \mathbb{Z}} u_k f_k(||A_k(x + y) - L, z_1, z_2, \ldots, z_{n-1}||)^p \leq D(M_n)^p \sum_{k \in \mathbb{Z}} u_k f_k(||A_k(x) - L, z_1, z_2, \ldots, z_{n-1}||)^p + D(N_p)
\]

Thus, \( \sum_{k \in \mathbb{Z}} u_k f_k(||A_k(x) - L, z_1, z_2, \ldots, z_{n-1}||)^p \geq \epsilon \) on the other hand from the above inequality, we get

\[
\sum_{k \in \mathbb{Z}} u_k f_k(||A_k(x) - L, z_1, z_2, \ldots, z_{n-1}||)^p \geq \epsilon.
\]

This completes the proof of (i). Similarly we can prove (ii).
Theorem 2.4. Let $F = (f_k)$ be a sequence of modulus functions. If $\lim \sup \frac{2^{n_k}}{f_n} = A > 0$ for all $k$, then
\[ [N_0, F, A, u, p, \|\cdot\|, \ldots, \|X\|, XS_{10}^T] = [N_0, A, u, p, \|\cdot\|, \ldots, \|X\|, XS_{10}^T] \]
and
\[ [N_0, A, u, p, \|\cdot\|, \ldots, \|X\|, XS_{10}^T] = [N_0, F, A, u, p, \|\cdot\|, \ldots, \|X\|, XS_{10}^T]. \]

Proof. To prove $[N_0, F, A, u, p, \|\cdot\|, \ldots, \|X\|, XS_{10}^T] = [N_0, F, A, u, p, \|\cdot\|, \ldots, \|X\|, XS_{10}^T]$, it is sufficient to show that $[N_0, F, A, u, p, \|\cdot\|, \ldots, \|X\|, XS_{10}^T] \subset [N_0, A, u, p, \|\cdot\|, \ldots, \|X\|, XS_{10}^T]$. Let $x \in [N_0, F, A, u, p, \|\cdot\|, \ldots, \|X\|, XS_{10}^T]$. Since $A > 0$, for every $t > 0$ we write $f(t) \geq A t$ for all $k$. From this inequality
\[ \frac{1}{h_k} \sum_{k \in \mathbb{Z}} u_k f_k(A_k(x), z_1, z_2, \ldots, z_{n_k-1}))) |(*) | \]
we get the result. Similarly we can prove the other part. \( \square \)

Corollary 2.5. Let $F = (f_k)$ and $F' = (f'_k)$ be sequences of modulus functions. If $\lim \sup \frac{2^{n_k}}{f'_n} < \infty$ implies
\[ [N_0, F, A, u, p, \|\ldots\|, \|X\|, XS_{10}^T] \subset [N_0, F', A, u, p, \|\ldots\|, \|X\|, XS_{10}^T]. \]
and
\[ [N_0, F, A, u, p, \|\ldots\|, \|X\|, XS_{10}^T] \subset [N_0, F', A, u, p, \|\ldots\|, \|X\|, XS_{10}^T]. \]
Proof. It is trivial. \( \square \)

Theorem 2.6. Let $(X, \|\cdot\|, \ldots, \|X\|)$ and $(X, \|\cdot\|, \ldots, \|X\|)$ be standard and Euclidian $n$-normed spaces, respectively. Then
\[ [N_0, F, A, u, p, \|\ldots\|, \|X\|, XS_{10}^T] \cap [N_0, F, A, u, p, \|\ldots\|, \|X\|, XS_{10}^T] \]
and
\[ [N_0, F, A, u, p, \|\ldots\|, \|X\|, XS_{10}^T] \cap [N_0, F, A, u, p, \|\ldots\|, \|X\|, XS_{10}^T]. \]
Proof. We have the following inclusion:
\[ \{x + \sum_{k \in \mathbb{Z}} u_k f_k(A_k(x), z_1, z_2, \ldots, z_{n_k-1}))) \} \subset \{x + \sum_{k \in \mathbb{Z}} u_k f_k(A_k(x), z_1, z_2, \ldots, z_{n_k-1}))) \}, \]
by using inequality (1.1). This completes the proof. \( \square \)

Theorem 2.7. Let $F = (f_k)$ and $F' = (f'_k)$ be two sequences of modulus functions. Then

(i) 
\[ [N_0, F, A, u, p, \|\ldots\|, \|X\|, XS_{10}^T] \subset [N_0, F'] \]
and
\[ [N_0, F', A, u, p, \|\ldots\|, \|X\|, XS_{10}^T]. \]
(ii) 
\[ [N_0, F, A, u, p, \|\ldots\|, \|X\|, XS_{10}^T] \cap [N_0, F, A, u, p, \|\ldots\|, \|X\|, XS_{10}^T] \]
\[ \subset [N_0, F + F', A, u, p, \|\ldots\|, \|X\|, XS_{10}^T]. \]

Proof. Let $x = (x_k) \in [N_0, F, A, u, p, \|\ldots\|, \|X\|, XS_{10}^T]$ and this completes the proof of (i).

The authors would like to thank the referees for giving useful comments and suggestions for the improvement of this paper.

References