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One-to-one disjoint path covers on k-ary n-cubes[☆]

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A B S T R A C T

The k-ary n-cube, Q_n^k, is one of the most popular interconnection networks. Let $n \geq 2$ and $k \geq 3$. It is known that Q_n^k is a nonbipartite (resp. bipartite) graph when k is odd (resp. even). In this paper, we prove that there exist r vertex disjoint paths $\{P_i \mid 0 \leq i \leq r-1\}$ between any two distinct vertices u and v of Q_n^k when k is odd, and there exist r vertex disjoint paths $\{R_i \mid 0 \leq i \leq r-1\}$ between any pair of vertices w and b from different partite sets of Q_n^k when k is even, such that $\bigcup_{i=0}^{r-1} P_i$ or $\bigcup_{i=0}^{r-1} R_i$ covers all vertices of Q_n^k for $1 \leq r \leq 2n$. In other words, we construct the one-to-one r-disjoint path cover of Q_n^k for any r with $1 \leq r \leq 2n$. The result is optimal since any vertex in Q_n^k has exactly $2n$ neighbors.

1. Introduction

In today's telecommunication networks, the construction of node-disjoint paths between a pair of distinct nodes in any network has been an important subject [9,21]. The node-disjoint paths are used to speed up the transfer of a large amount of data by splitting the data over several node-disjoint communication paths [6]. Additional benefits of adopting such a node-disjoint routing scheme are the enhanced robustness to node failures and congestion, and the enhanced capability of load balancing [21]. Recently, studies of disjoint paths in a variety of networks can be found in the literature [8,32]. In this article, we further request that the set of these node-disjoint paths between any given pair of distinct nodes is a cover of the network. Namely, the union of the node-disjoint paths must cover all nodes of the network, which we term as a "one-to-one disjoint path cover". One of the well-known applications of multiple disjoint path covers is software testing [23]. For example, if the graph G represents all possible execution sequences of a computer program, then a path cover is a set of test runs that covers each program statement at least once. In pipeline computation, an embedding of multiple disjoint path covers in a network implies that every node can participate. Studies about disjoint path covers of some networks or graphs can be found in the literature [5,13,19,20,25]. Among them, one-to-one disjoint path covers are also named spanning containers.

The k-ary n-cube, denoted by Q_n^k, has been proposed as an alternative to the hypercube Q_n, which is one of the most well-known interconnection networks in parallel computers due to its many attractive properties such as vertex/edge symmetry, recursive structure, easy routing, high degree of fault tolerance, and so on. See [7,10,18,28–30], for example. It is known that the hypercube network has been used as the interconnection topology of many distributed memory multiprocessors such as the Cosmic Cube, the Ametek S/14, the iPSC, the Ncube, and the CM-200. Besides, the properties of hypercubes relevant

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to parallel computing have been well studied. Readers can refer to [27] and its references. The k-ary n-cube, Q_n^k, shares many nice properties of Q_n such as regular degrees, vertex symmetry, edge symmetry, recursive structure etc. A number of distributed memory multiprocessors have been built with a k-ary n-cube forming the underlying topology, such as the Cray T3E, the iWARP, the Cray T3D and so on. Please see [1,3,17,22]. Many researchers have been working on k-ary n-cubes [4,6,11,12,14,26,27,31,33].

In this paper, we construct one-to-one node-disjoint path covers of k-ary n-cubes for any integer $k \geq 3$ and $n \geq 2$. More precisely, we show that given any two distinct vertices u, v of a k-ary n-cube Q_n^k, there exist(s) m vertex/node-disjoint path(s) between u and v whose union covers all vertices of Q_n^k for $1 \leq m \leq 2n$ when k is odd, and given any pair of vertices w and b from the different partite sets of a k-ary n-cube Q_n^k, there exist(s) m internally disjoint path(s) between w, b whose union covers all vertices of Q_n^k for $1 \leq m \leq 2n$ when k is even. The result is optimal since any vertex of Q_n^k has exactly $2n$ neighbors. Note that a network is conveniently represented by a graph, in which vertices represent the nodes (processors) of the network and edges represent the communication links of the network. Therefore, throughout this paper, we use networks and graph, node and vertex and, link and edge interchangeably.

2. Preliminaries

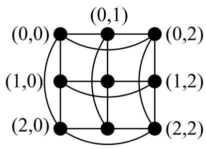
In what follows, we follow [2] for the graph definitions and notations. The sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. If u, v are vertices of a graph G such that there is an edge $e = (u, v) \in E(G)$ between u and v, then we say that the vertices u and v are *adjacent* in G. The *degree* of any vertex x is the number of distinct vertices adjacent to x. We use $N(x)$ to denote the set of vertices which are adjacent to x. A *path* P between two vertices v_0 and v_k is represented by $P = \langle v_0, v_1, \ldots, v_k\rangle$, where each pair of consecutive vertices are connected by an edge. We use P^{-1} to denote the path $\langle v_k, v_{k-1}, v_{k-2}, \ldots, v_0\rangle$. We also write the path $P = \langle v_0, v_1, \ldots, v_k\rangle$ as $\langle v_0, v_1, \ldots, v_i, Q, v_j, v_{j+1}, \ldots, v_k\rangle$, where Q denotes the path $\langle v_i, v_{i+1}, \ldots, v_j\rangle$. The *length* of a path P is the number of edges in P. We use $d_G(u, v)$ to denote the length of the shortest path between the two vertices u and v in G. A *hamiltonian path* between u and v, where u and v are two distinct vertices of G, is a path joining u to v that visits every vertex of G exactly once. A *cycle* is a path of at least three vertices such that the first vertex is the same as the last vertex. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. A *hamiltonian graph* is a graph with a hamiltonian cycle. A graph G is *connected* if there is a path between any two distinct vertices in G and is *hamiltonian connected* if there is a hamiltonian path between any two distinct vertices in G [24]. A graph $H = (W \cup B, E)$ is *bipartite* if $V(H) = W \cup B$ and $E(H)$ is a subset of $\{(w, b) \mid w \in W,\ b \in B\}$. We will call any vertex $w \in W$ a "white" vertex, and any vertex $b \in B$ a "black" vertex, respectively. A bipartite graph H is balanced if $|W| = |B|$. It is easy to see that any bipartite graph with at least three vertices is not hamiltonian connected. For example, let $H = (W \cup B, E)$ be a bipartite graph with $|W| \geq |B|$. Obviously, there exists no hamiltonian path in H that joins two black vertices. On the other hand, a balanced bipartite graph is *hamiltonian laceable* if there exists a hamiltonian path between any two vertices w, b with $w \in W$ and $b \in B$.

Suppose that u and v are two vertices of a graph G. We say a set of m paths between u and v, denoted by $C(u, v)$, is an *m-disjoint path cover* in G if the m paths do not contain the same vertex besides u and v and their union covers all vertices of G. An m-disjoint path cover is abbreviated as an *m-DPC* for simplicity. A nonbipartite graph G is *one-to-one m-disjoint path coverable* (*m-DPC-able* for short) if there is an m-DPC between any two vertices of G. Moreover, let H be a bipartite graph with $V(H) = W \cup B$. A bipartite graph H is *one-to-one bi-m-disjoint path coverable* (*bi-m-DPC-able* for short) if there is an m-DPC between any pair of vertices $\{u, v \mid u \in B \text{ and } v \in W\}$. Obviously, a nonbipartite (resp. bipartite) graph G is hamiltonian connected (resp. hamiltonian laceable) if and only if G is 1-DPC-able (resp. bi-1-DPC-able). Furthermore, a nonbipartite (resp. bipartite) graph is hamiltonian if and only if the graph is 2-DPC-able (resp. bi-2-DPC-able). It is worth mentioning that "G is r-DPC-able" and "G is $(r + 1)$-DPC-able" do not imply each other. For example, C_n (the cycle with n vertices) is 2-DPC-able (resp. bi-2-DPC-able) but not 1-DPC-able (resp. bi-1-DPC-able) for $n \geq 5$ being an odd integer (resp. an even integer). Besides, in [15] (resp. [16]), examples of 2-DPC-able nonbipartite graphs (resp. bi-2-DPC-able bipartite graphs) that are not 3-DPC-able (resp. bi-3-DPC-able) are given.

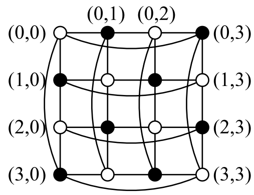
The k-ary n-cube, Q_n^k, is defined for all integers $k \geq 2$ and $n \geq 1$. The subclass Q_n^2 is the well-studied hypercube family. The subclass Q_1^k with $k \geq 3$ is defined as the cycle of length k. The k-ary n-cube, Q_n^k, for $k \geq 3$ and $n \geq 2$ is defined as follows. Let $u \in V(Q_n^k)$ be represented by $(u(0), u(1), \ldots, u(n-1))$, where $0 \leq u(i) \leq k-1$. Two vertices u and v are adjacent if and only if $|u(i) - v(i)| = 1$ or $k - 1$ for some i and $u(j) = v(j)$ for any $0 \leq j \leq n-1$ with $j \neq i$. It is shown that Q_n^k is bipartite if k is even [14]. See Fig. 1 for an illustration. Here we mention some properties of Q_n^k that will be used in this paper.

Q_n^k is *vertex-symmetric* (and *edge-symmetric*) [14]. It means that given any two distinct vertices v and v' of Q_n^k, there is an automorphism of Q_n^k mapping v to v'. Note that each vertex of Q_n^k is represented by a n-bit tuple. We will call the dth-bit *the dth dimension*. We can partition Q_n^k over dimension d by fixing the dth element of any vertex tuple at some value a for every $a \in \{0, 1, \ldots, k-1\}$. This results in k copies of Q_{n-1}^k, denoted by $Q_{n-1}^{k,0}, Q_{n-1}^{k,1}, \ldots, Q_{n-1}^{k,k-1}$, with corresponding vertices in $Q_{n-1}^{k,0}, Q_{n-1}^{k,1}, \ldots, Q_{n-1}^{k,k-1}$ joined in a cycle of length k (in dimension d) [27].

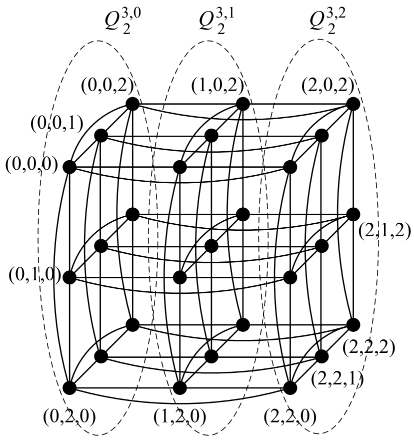
In this article, we always partition Q_n^k over the 0-th dimension by letting $V(Q_{n-1}^{k,i}) = \{((i), v(1), v(2), \ldots, v(n-1)) \mid 0 \leq v(j) \leq k-1, \forall 1 \leq j \leq n-1\}$ for $0 \leq i \leq k-1$. See Fig. 1(c) for an illustration. Given a vertex $x = (x(0), x(1), \ldots, x(n-1)) \in V(Q_n^k)$, the symbol $x^j = ((j), x(1), x(2), \ldots, x(n-1))$, where $0 \leq j \leq k-1$, is defined to be the vertex corresponding to x



(a) Q_2^3.



(b) Q_2^4.



(c) Q_3^3.

Fig. 1. Three graphs, Q_2^3, Q_2^4 and Q_3^3.

in $Q_{n-1}^{k,j}$ for simplicity. If $P = \langle x_0, x_1, \ldots, x_{n-1}\rangle$, P^j is represented by $\langle x_0^j, x_1^j, \ldots, x_{n-1}^j\rangle$. Throughout this paper, let $n \geq 2$ be an integer and $k \geq 3$ an integer.

Theorem 1 ([31]). *For any odd integer $k \geq 3$, Q_n^k is hamiltonian connected for $n \geq 2$. In other words, Q_n^k is 1-DPC-able.*

Theorem 2 ([14]). *For any even integer $k \geq 4$, Q_n^k is hamiltonian laceable for $n \geq 2$. In other words, Q_n^k is bi-1-DPC-able.*

Theorem 3 ([4]). *The graph Q_n^k is hamiltonian. In other words, Q_n^k is 2-DPC-able when k is odd and bi-2-DPC-able when k is even.*

3. Main results

In this section, we will derive our main theorem, Theorems 4 and 5, using mathematical induction on n. For this purpose, two lemmas are presented in Section 3.1 for the following construction schemes. In Section 3.2, the disjoint path covers of Q_2^k are specifically constructed for $k \in \{3, 4, 5, 6\}$, and then a step-by-step algorithm is given to obtain the disjoint path covers of Q_2^k for any integer k with $k \geq 5$. In Section 3.3, with the induction base derived in Section 3.2, we prove the main theorems by mathematical induction on n.

3.1. Two lemmas

Lemma 1. *Given Q_n^k and its k subcubes, $Q_{n-1}^{k,i}$, where $0 \leq i \leq k-1$. Let j and j' be two integers satisfying $0 \leq j \leq j' \leq k-1$. When k is odd, let $u \in V(Q_{n-1}^{k,j})$ and $v \in V(Q_{n-1}^{k,j'})$ be arbitrary. Then there exists a path between u and v that visits each vertex in $Q_{n-1}^{k,j}$, $Q_{n-1}^{k,j+1}$, ..., and $Q_{n-1}^{k,j'}$ exactly once. On the other hand, when k is even, let $w \in V(Q_{n-1}^{k,j})$ be an arbitrary white vertex, and $b \in V(Q_{n-1}^{k,j'})$ an arbitrary black vertex. Then there exists a path between w and b that visits each vertex in $Q_{n-1}^{k,j}$, $Q_{n-1}^{k,j+1}$, ..., and $Q_{n-1}^{k,j'}$ exactly once.*

Proof. We have the following two cases.

Case 1. When k is odd, we construct the required path in the following three cases.

Case 1.1. $j = j'$. W.L.O.G., let $j = j' = 0$. By Theorem 1, $Q_{n-1}^{k,0}$ is hamiltonian connected. Thus there is a hamiltonian path between u and v that visits every vertex of $Q_{n-1}^{k,0}$ exactly once.

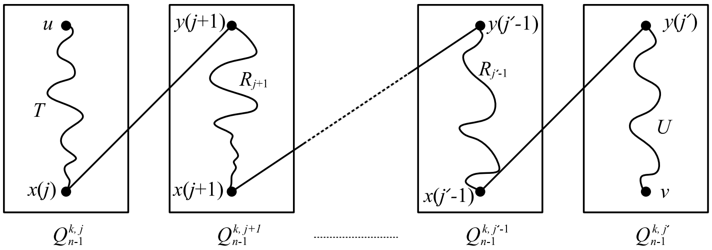


Fig. 2. An illustration for Case 1.3 of Lemma 1.

Case 1.2. $j' - j = 1$. W.L.O.G., let $j = 0$ and $j' = 1$. We can find a vertex $x \in V(Q^{k,0}_{n-1})$ such that $x = x^0 \neq u$ and $x^1 \neq v$. By Theorem 1, there exists a hamiltonian path P_0 of $Q^{k,0}_{n-1}$ between u and x^0, and a hamiltonian path P_1 of $Q^{k,1}_{n-1}$ between x^1 and v. Let $P = \langle u, P_0, x^0, x^1, P_1, v\rangle$. Hence P is the path between u and v that visits every vertex of $Q^{k,0}_{n-1}$ and $Q^{k,1}_{n-1}$ exactly once.

Case 1.3. For $j' - j \geq 2$, there are $j' - j + 1$ k-ary $(n-1)$-cubes, $Q^{k,j}_{n-1}$, $Q^{k,j+1}_{n-1}$, $\ldots$, $Q^{k,j'-1}_{n-1}$ and $Q^{k,j'}_{n-1}$. There are $j' - j$ pairs of adjacent vertices $x(r) \in Q^{k,r}_{n-1}$, and $y(r+1) \in Q^{k,r+1}_{n-1}$ for $j \leq r \leq j' - 1$ such that $x(j) \neq u$ and $y(j') \neq v$. By Theorem 1, there is a hamiltonian path R_r of $Q^{k,r}_{n-1}$ joining $y(r)$ to $x(r)$, where $j + 1 \leq r \leq j' - 1$. Again, with Theorem 1, there exists a hamiltonian path T of $Q^{k,j}_{n-1}$ joining u to $x(j)$, and a hamiltonian path U of $Q^{k,j'}_{n-1}$ joining $y(j')$ to v. Let $P = \langle u, T, x(j), y(j+1), R_{j+1}, x(j+1), y(j+2), R_{j+2}, x(j+2), \ldots, y(j'-1), R_{j'-1}, x(j'-1), y(j'), U, v\rangle$. Therefore, P is a path covering all the vertices of $Q^{k,j}_{n-1}, Q^{k,j+1}_{n-1}, \ldots, Q^{k,j'}_{n-1}$ between u and v. Please see Fig. 2 for an illustration.

By *Case* 1.1, *Case* 1.2 and *Case* 1.3, this lemma is proved when k is odd.

Case 2. When k is even, the proof is similar to *Case* 1 and is omitted. □

Lemma 2. *Given Q^k_n and its k subcubes $Q^{k,i}_{n-1}$ for $0 \leq i \leq k-1$. Let j be an integer with $0 \leq i \leq j \leq k-1$. When k is odd, let u and v be any pair of vertices in $Q^{k,i}_{n-1}$. There exists a path between u and v that covers all the vertices of $Q^{k,i}_{n-1}$, $Q^{k,i+1}_{n-1}$, …, and $Q^{k,j}_{n-1}$. On the other hand, when k is even, let w be a white vertex and b a black vertex in $Q^{k,i}_{n-1}$. There exists a path between w and b that covers all the vertices of $Q^{k,i}_{n-1}$, $Q^{k,i+1}_{n-1}$, …, and $Q^{k,j}_{n-1}$.*

Proof. We consider the following two cases.

Case 1. When k is odd.

Case 1.1. If $j = i$, there is only one k-ary $(n-1)$-cube $Q^{k,i}_{n-1}$. By Theorem 1, the lemma holds in this case.

Case 1.2. If $j \neq i$, there are $j - i + 1$ k-ary $(n-1)$-cubes. According to Theorem 1, there is a hamiltonian path P_i that covers all the vertices of $Q^{k,i}_{n-1}$ between u and v of the form $\langle u, S_i, x^i, y^i, T_i, v\rangle$, where $\{x^i, y^i\}$ is an edge of $Q^{k,i}_{n-1}$ with $\{x^i, y^i\} \cap \{u, v\} = \emptyset$. Notice that by Theorem 1, $Q^{k,r}_{n-1}$ is hamiltonian connected and hence there exists a hamiltonian path P_r between x^r and y^r of the form: $\langle x^r, S_r, z^r, w^r, T_r, y^r\rangle$ for $i + 1 \leq r \leq j$. Let the required path between u and v be R.

Case 1.2.1. If $j-i+1$ is even, then $R = \langle u, S_i, x^i, x^{i+1}, S_{i+1}, z^{i+1}, z^{i+2}, (S_{i+2})^{-1}, x^{i+2}, x^{i+3}, S_{i+3}, z^{i+3}, z^{i+4}, (S_{i+4})^{-1}, x^{i+4}, \ldots, x^j, S_j, z^j, w^j, T_j, y^j, y^{j-1}, (T_{j-1})^{-1}, w^{j-1}, w^{j-2}, T_{j-2}, y^{j-2}, y^{j-3}, (T_{j-3})^{-1}, w^{j-3}, \ldots, y^{i+1}, y^i, T_i, v\rangle$. Please see Fig. 3(a) for an illustration.

Case 1.2.2. If $j-i+1$ is odd, then $R = \langle u, S_i, x^i, x^{i+1}, S_{i+1}, z^{i+1}, z^{i+2}, (S_{i+2})^{-1}, x^{i+2}, x^{i+3}, S_{i+3}, z^{i+3}, z^{i+4}, (S_{i+4})^{-1}, x^{i+4}, \ldots, z^j, (S_j)^{-1}, x^j, y^j, (T_j)^{-1}, w^j, w^{j-1}, T_{j-1}, y^{j-1}, y^{j-2}, (T_{j-2})^{-1}, w^{j-2}, w^{j-3}, T_{j-3}, y^{j-3}, \ldots, y^{i+1}, y^i, T_i, v\rangle$. Please see Fig. 3(b) for an illustration.

By *Case* 1.1 and *Case* 1.2, the lemma holds when k is odd.

Case 2. When k is even, the required path can be derived by the same approach as in *Case* 1, so we skip it. □

3.2. The disjoint path covers of Q^k_2

Lemma 3. *The graph Q^3_2 is 3-DPC-able and 4-DPC-able.*

Proof. To prove that Q^3_2 is m-DPC-able, where $m \in \{3, 4\}$, we need to construct an m-DPC between u and v for any pair of vertices $\{u, v\} \in V(Q^3_2)$. Since Q^3_2 is vertex-symmetric, W.L.O.G., let $u = (0, 0)$. Then we must consider the cases when $v \in \{(0, 1), (1, 1)\}$.

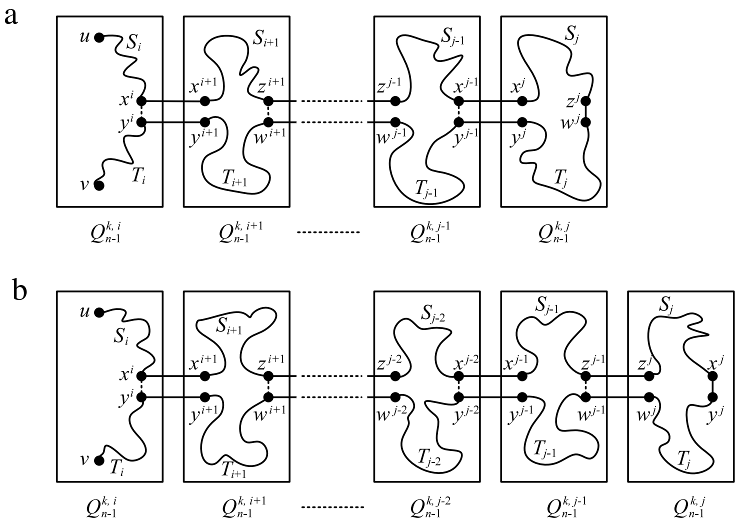


Fig. 3. An illustration for Case 1.2 of Lemma 2.

Case 1. The 3-DPC $\{P_1, P_2, P_3\}$ (resp. $\{R_1, R_2, R_3\}$) from $(0, 0)$ to $(0, 1)$ (resp. $(1, 1)$) whose union covers $V(Q_2^3)$ are constructed in the following table.

$v = (0, 1)$	$P_1 = \langle(0, 0), (0, 1)\rangle$ $P_2 = \langle(0, 0), (1, 0), (1, 1), (0, 1)\rangle$ $P_3 = \langle(0, 0), (2, 0), (2, 1), (2, 2), (1, 2), (0, 2), (0, 1)\rangle$
$v = (1, 1)$	$R_1 = \langle(0, 0), (0, 1), (1, 1)\rangle$ $R_2 = \langle(0, 0), (1, 0), (1, 1)\rangle$ $R_3 = \langle(0, 0), (2, 0), (2, 1), (2, 2), (0, 2), (1, 2), (1, 1)\rangle$

Case 2. The 4-DPC $\{P_1, P_2, P_3, P_4\}$ (resp. $\{R_1, R_2, R_3, R_4\}$) from $(0, 0)$ to $(0, 1)$ (resp. $(1, 1)$) whose union covers $V(Q_2^3)$ are constructed in the following table.

$v = (0, 1)$	$P_1 = \langle(0, 0), (0, 1)\rangle$ $P_2 = \langle(0, 0), (0, 2), (0, 1)\rangle$ $P_3 = \langle(0, 0), (1, 0), (1, 2), (1, 1), (0, 1)\rangle$ $P_4 = \langle(0, 0), (2, 0), (2, 2), (2, 1), (0, 1)\rangle$
$v = (1, 1)$	$R_1 = \langle(0, 0), (0, 1), (1, 1)\rangle$ $R_2 = \langle(0, 0), (1, 0), (1, 1)\rangle$ $R_3 = \langle(0, 0), (0, 2), (1, 2), (1, 1)\rangle$ $R_4 = \langle(0, 0), (2, 0), (2, 2), (2, 1), (1, 1)\rangle$

□

Lemma 4. *The graph Q_2^4 is bi-3-DPC-able and bi-4-DPC-able.*

Proof. To prove that Q_2^4 is bi-m-DPC-able, where $m \in \{3, 4\}$, we need to construct an m-DPC between any pair of vertices w and b from different partite sets in $V(Q_2^4)$. Since Q_2^4 is vertex-symmetric, W.L.O.G., let $w = (0, 0)$. Then we must consider the cases when $b \in \{(1, 0), (2, 1)\}$.

Case 1. The 3-DPC $\{P_1, P_2, P_3\}$ (resp. $\{R_1, R_2, R_3\}$) from $(0, 0)$ to $(1, 0)$ (resp. $(2, 1)$) whose union covers $V(Q_2^4)$ are constructed in the following table.

$b = (1, 0)$	$P_1 = \langle(0, 0), (1, 0)\rangle$ $P_2 = \langle(0, 0), (0, 1), (1, 1), (1, 0)\rangle$ $P_3 = \langle(0, 0), (3, 0), (3, 1), (3, 2), (3, 3), (2, 3), (1, 3), (0, 3), (0, 2), (1, 2), (2, 2), (2, 1), (2, 0), (1, 0)\rangle$
$b = (2, 1)$	$R_1 = \langle(0, 0), (1, 0), (2, 0), (2, 1)\rangle$ $R_2 = \langle(0, 0), (0, 1), (1, 1), (2, 1)\rangle$ $R_3 = \langle(0, 0), (3, 0), (3, 1), (3, 2), (3, 3), (2, 3), (1, 3), (0, 3), (0, 2), (1, 2), (2, 2), (2, 1)\rangle$

Case 2. The 4-DPC $\{P_1, P_2, P_3, P_4\}$ (resp. $\{R_1, R_2, R_3, R_4\}$) from $(0, 0)$ to $(1, 0)$ (resp. $(2, 1)$) whose union covers $V(Q_2^4)$ are constructed in the following table.

$b = (1, 0)$	$P_1 = \langle(0, 0), (1, 0)\rangle$ $P_2 = \langle(0, 0), (0, 1), (1, 1), (1, 0)\rangle$ $P_3 = \langle(0, 0), (0, 3), (0, 2), (1, 2), (1, 3), (1, 0)\rangle$ $P_4 = \langle(0, 0), (3, 0), (3, 1), (3, 2), (3, 3), (2, 3), (2, 2), (2, 1), (2, 0), (1, 0)\rangle$
$b = (2, 1)$	$R_1 = \langle(0, 0), (3, 0), (3, 1), (2, 1)\rangle$ $R_2 = \langle(0, 0), (1, 0), (2, 0), (2, 1)\rangle$ $R_3 = \langle(0, 0), (0, 1), (1, 1), (2, 1)\rangle$ $R_4 = \langle(0, 0), (0, 3), (0, 2), (1, 2), (1, 3), (2, 3), (3, 3), (3, 2), (2, 2), (2, 1)\rangle$

□

Lemma 5. *The graph Q_2^5 is 3-DPC-able and 4-DPC-able.*

Proof. To prove that Q_2^5 is m-DPC-able, where $m \in \{3, 4\}$, we need to construct an m-DPC between u and v for any pair of vertices $\{u, v\} \in V(Q_2^5)$. Since Q_2^5 is vertex-symmetric, W.L.O.G., let $u = (0, 0)$. We must consider the cases when $v \in \{(0, 1), (1, 1), (0, 2), (1, 2), (2, 2)\}$.

Case 1. The 3-DPC $\{P_1, P_2, P_3\}$ (resp. $\{R_1, R_2, R_3\}$, $\{S_1, S_2, S_3\}$, $\{T_1, T_2, T_3\}$, $\{U_1, U_2, U_3\}$) whose union covers $V(Q_2^5)$ between $(0, 0)$ and $(0, 1)$ (resp. $(1, 1)$, $(0, 2)$, $(1, 2)$, $(2, 2)$) are listed below.

$v = (0, 1)$	$P_1 = \langle(0, 0), (0, 1)\rangle$ $P_2 = \langle(0, 0), (1, 0), (2, 0), (3, 0), (3, 1), (2, 1), (1, 1), (0, 1)\rangle$ $P_3 = \langle(0, 0), (4, 0), (4, 1), (4, 2), (3, 2), (2, 2), (1, 2), (1, 3), (2, 3), (3, 3), (4, 3), (4, 4), (3, 4), (2, 4), (1, 4), (0, 4),$ $(0, 3), (0, 2), (0, 1)\rangle$
$v = (1, 1)$	$R_1 = \langle(0, 0), (1, 0), (1, 1)\rangle$ $R_2 = \langle(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (1, 4), (1, 3), (1, 2), (1, 1)\rangle$ $R_3 = \langle(0, 0), (4, 0), (3, 0), (2, 0), (2, 4), (3, 4), (4, 4), (4, 3), (3, 3), (2, 3), (2, 2), (3, 2), (4, 2), (4, 1), (3, 1), (2, 1), (1, 1)\rangle$
$v = (0, 2)$	$S_1 = \langle(0, 0), (0, 1), (0, 2)\rangle$ $S_2 = \langle(0, 0), (0, 4), (1, 4), (2, 4), (3, 4), (4, 4), (4, 3), (3, 3), (2, 3), (1, 3), (0, 3), (0, 2)\rangle$ $S_3 = \langle(0, 0), (4, 0), (3, 0), (2, 0), (1, 0), (1, 1), (2, 1), (3, 1), (4, 1), (4, 2), (3, 2), (2, 2), (1, 2), (0, 2)\rangle$
$v = (1, 2)$	$T_1 = \langle(0, 0), (0, 1), (0, 2), (1, 2)\rangle$ $T_2 = \langle(0, 0), (1, 0), (1, 1), (1, 2)\rangle$ $T_3 = \langle(0, 0), (4, 0), (4, 1), (4, 2), (4, 3), (4, 4), (3, 4), (3, 3), (3, 2), (3, 1), (3, 0), (2, 0), (2, 1), (2, 2), (2, 3), (2, 4),$ $(1, 4), (0, 4), (0, 3), (1, 3), (1, 2)\rangle$
$v = (2, 2)$	$U_1 = \langle(0, 0), (1, 0), (2, 0), (2, 1), (2, 2)\rangle$ $U_2 = \langle(0, 0), (0, 4), (1, 4), (2, 4), (2, 3), (1, 3), (0, 3), (0, 2), (0, 1), (1, 1), (1, 2), (2, 2)\rangle$ $U_3 = \langle(0, 0), (4, 0), (3, 0), (3, 1), (4, 1), (4, 2), (4, 3), (4, 4), (3, 4), (3, 3), (3, 2), (2, 2)\rangle$

Case 2. The 4-DPC $\{P_1, P_2, P_3, P_4\}$ (resp. $\{R_1, R_2, R_3, R_4\}$, $\{S_1, S_2, S_3, S_4\}$, $\{T_1, T_2, T_3, T_4\}$, $\{U_1, U_2, U_3, U_4\}$) whose union covers $V(Q_2^5)$ between $(0, 0)$ and $(0, 1)$ (resp. $(1, 1)$, $(0, 2)$, $(1, 2)$, $(2, 2)$) are listed below.

$v = (0, 1)$	$P_1 = \langle(0, 0), (0, 1)\rangle$ $P_2 = \langle(0, 0), (1, 0), (1, 1), (0, 1)\rangle$ $P_3 = \langle(0, 0), (4, 0), (3, 0), (2, 0), (2, 1), (3, 1), (4, 1), (0, 1)\rangle$ $P_4 = \langle(0, 0), (0, 4), (0, 3), (1, 3), (1, 4), (2, 4), (2, 3), (3, 3), (3, 4), (4, 4), (4, 3), (4, 2), (3, 2), (2, 2), (1, 2), (0, 2), (0, 1)\rangle$
$v = (1, 1)$	$R_1 = \langle(0, 0), (0, 1), (1, 1)\rangle$ $R_2 = \langle(0, 0), (1, 0), (1, 1)\rangle$ $R_3 = \langle(0, 0), (0, 4), (1, 4), (1, 3), (0, 3), (0, 2), (1, 2), (1, 1)\rangle$ $R_4 = \langle(0, 0), (4, 0), (4, 4), (4, 3), (4, 2), (4, 1), (3, 1), (3, 2), (3, 3), (3, 4), (3, 0), (2, 0), (2, 4), (2, 3), (2, 2), (2, 1), (1, 1)\rangle$
$v = (0, 2)$	$S_1 = \langle(0, 0), (0, 1), (0, 2)\rangle$ $S_2 = \langle(0, 0), (4, 0), (4, 1), (4, 2), (0, 2)\rangle$ $S_3 = \langle(0, 0), (1, 0), (1, 1), (2, 1), (2, 0), (3, 0), (3, 1), (3, 2), (2, 2), (1, 2), (0, 2)\rangle$ $S_4 = \langle(0, 0), (0, 4), (1, 4), (2, 4), (3, 4), (4, 4), (4, 3), (3, 3), (2, 3), (1, 3), (0, 3), (0, 2)\rangle$
$v = (1, 2)$	$T_1 = \langle(0, 0), (0, 1), (0, 2), (1, 2)\rangle$ $T_2 = \langle(0, 0), (1, 0), (1, 1), (1, 2)\rangle$ $T_3 = \langle(0, 0), (4, 0), (4, 1), (4, 2), (3, 2), (3, 1), (3, 0), (2, 0), (2, 1), (2, 2), (1, 2)\rangle$ $T_4 = \langle(0, 0), (0, 4), (0, 3), (4, 3), (4, 4), (3, 4), (3, 3), (2, 3), (2, 4), (1, 4), (1, 3), (1, 2)\rangle$
$v = (2, 2)$	$U_1 = \langle(0, 0), (1, 0), (1, 1), (2, 1), (2, 2)\rangle$ $U_2 = \langle(0, 0), (0, 1), (0, 2), (0, 3), (1, 3), (1, 2), (2, 2)\rangle$ $U_3 = \langle(0, 0), (0, 4), (1, 4), (2, 4), (2, 0), (3, 0), (3, 1), (3, 2), (2, 2)\rangle$ $U_4 = \langle(0, 0), (4, 0), (4, 1), (4, 2), (4, 3), (4, 4), (3, 4), (3, 3), (2, 3), (2, 2)\rangle$

□

Lemma 6. *The graph Q_2^6 is bi-3-DPC-able and bi-4-DPC-able.*

Proof. To prove that Q_2^6 is bi-m-DPC-able, where $m \in \{3, 4\}$, we need to construct an m-DPC between any pair of vertices w and b from different partite sets in $V(Q_2^6)$. Since Q_2^6 is vertex-symmetric, W.L.O.G., let $w = (0, 0)$. Then we must consider the cases when $b \in \{(1, 0), (2, 1), (3, 0), (3, 2)\}$.

Case 1. The 3-DPC $\{P_1, P_2, P_3\}$ (resp. $\{R_1, R_2, R_3\}$, $\{S_1, S_2, S_3\}$, $\{T_1, T_2, T_3\}$) whose union covers $V(Q_2^6)$ between $(0, 0)$ and $(1, 0)$ (resp. $(2, 1)$, $(3, 0)$, $(3, 2)$) are constructed below.

$v = (1, 0)$	$P_1 = \langle(0, 0), (1, 0)\rangle$ $P_2 = \langle(0, 0), (0, 1), (1, 1), (1, 0)\rangle$ $P_3 = \langle(0, 0), (5, 0), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (4, 5), (3, 5), (2, 5), (1, 5), (0, 5), (0, 4), (1, 4), (2, 4), (3, 4), (4, 4), (4, 3), (4, 2), (4, 1), (4, 0), (3, 0), (3, 1), (3, 2), (3, 3), (2, 3), (1, 3), (0, 3), (0, 2), (1, 2), (2, 2), (2, 1), (2, 0), (1, 0)\rangle$
$v = (2, 1)$	$R_1 = \langle(0, 0), (1, 0), (2, 0), (2, 1)\rangle$ $R_2 = \langle(0, 0), (0, 1), (1, 1), (2, 1)\rangle$ $R_3 = \langle(0, 0), (5, 0), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (4, 5), (3, 5), (2, 5), (1, 5), (0, 5), (0, 4), (1, 4), (2, 4), (3, 4), (4, 4), (4, 3), (4, 2), (4, 1), (4, 0), (3, 0), (3, 1), (3, 2), (3, 3), (2, 3), (1, 3), (0, 3), (0, 2), (1, 2), (2, 2), (2, 1)\rangle$
$v = (3, 0)$	$S_1 = \langle(0, 0), (1, 0), (2, 0), (3, 0)\rangle$ $S_2 = \langle(0, 0), (5, 0), (4, 0), (3, 0)\rangle$ $S_3 = \langle(0, 0), (0, 5), (1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (5, 4), (4, 4), (3, 4), (2, 4), (1, 4), (0, 4), (0, 3), (1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (5, 2), (5, 1), (4, 1), (4, 2), (3, 2), (2, 2), (1, 2), (0, 2), (0, 1), (1, 1), (2, 1), (3, 1), (3, 0)\rangle$
$v = (3, 2)$	$T_1 = \langle(0, 0), (1, 0), (2, 0), (3, 0), (3, 1), (3, 2)\rangle$ $T_2 = \langle(0, 0), (0, 1), (0, 2), (1, 2), (1, 1), (2, 1), (2, 2), (3, 2)\rangle$ $T_3 = \langle(0, 0), (5, 0), (4, 0), (4, 1), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (4, 5), (3, 5), (2, 5), (1, 5), (0, 5), (0, 4), (0, 3), (1, 3), (1, 4), (2, 4), (2, 3), (3, 3), (3, 4), (4, 4), (4, 3), (4, 2), (3, 2)\rangle$

Case 2. The 4-DPC $\{P_1, P_2, P_3, P_4\}$ (resp. $\{R_1, R_2, R_3, R_4\}$, $\{S_1, S_2, S_3, S_4\}$, $\{T_1, T_2, T_3, T_4\}$) whose union covers $V(Q_2^6)$ between $(0, 0)$ and $(1, 0)$ (resp. $(2, 1)$, $(3, 0)$, $(3, 2)$) are constructed below.

$v = (1, 0)$	$P_1 = \langle(0, 0), (1, 0)\rangle$ $P_2 = \langle(0, 0), (0, 1), (1, 1), (1, 0)\rangle$ $P_3 = \langle(0, 0), (0, 5), (0, 4), (0, 3), (0, 2), (1, 2), (1, 3), (1, 4), (1, 5), (1, 0)\rangle$ $P_4 = \langle(0, 0), (5, 0), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (4, 5), (4, 4), (4, 3), (4, 2), (4, 1), (4, 0), (3, 0), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (2, 5), (2, 4), (2, 3), (2, 2), (2, 1), (2, 0), (1, 0)\rangle$
$v = (2, 1)$	$R_1 = \langle(0, 0), (1, 0), (2, 0), (2, 1)\rangle$ $R_2 = \langle(0, 0), (0, 1), (1, 1), (2, 1)\rangle$ $R_3 = \langle(0, 0), (5, 0), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (4, 5), (4, 4), (4, 3), (4, 2), (4, 1), (4, 0), (3, 0), (3, 1), (2, 1)\rangle$ $R_4 = \langle(0, 0), (0, 5), (1, 5), (2, 5), (3, 5), (3, 4), (2, 4), (1, 4), (0, 4), (0, 3), (0, 2), (1, 2), (1, 3), (2, 3), (3, 3), (3, 2), (2, 2), (2, 1)\rangle$
$v = (3, 0)$	$S_1 = \langle(0, 0), (1, 0), (2, 0), (3, 0)\rangle$ $S_2 = \langle(0, 0), (0, 1), (1, 1), (2, 1), (3, 1), (3, 0)\rangle$ $S_3 = \langle(0, 0), (5, 0), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (4, 5), (4, 4), (4, 3), (4, 2), (4, 1), (4, 0), (3, 0)\rangle$ $S_4 = \langle(0, 0), (0, 5), (0, 4), (0, 3), (0, 2), (1, 2), (1, 3), (1, 4), (1, 5), (2, 5), (2, 4), (2, 3), (2, 2), (3, 2), (3, 3), (3, 4), (3, 5), (3, 0)\rangle$
$v = (3, 2)$	$T_1 = \langle(0, 0), (1, 0), (2, 0), (3, 0), (3, 1), (3, 2)\rangle$ $T_2 = \langle(0, 0), (0, 1), (0, 2), (1, 2), (1, 1), (2, 1), (2, 2), (3, 2)\rangle$ $T_3 = \langle(0, 0), (5, 0), (4, 0), (4, 1), (5, 1), (5, 2), (4, 2), (3, 2)\rangle$ $T_4 = \langle(0, 0), (0, 5), (1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (5, 4), (5, 3), (4, 3), (4, 4), (3, 4), (2, 4), (1, 4), (0, 4), (0, 3), (1, 3), (2, 3), (3, 3), (3, 2)\rangle$

□

Lemma 7. *For any odd integer $k \geq 5$, Q_2^k is 3-DPC-able and 4-DPC-able.*

Proof. With Lemma 5, we have shown that Q_2^5 is 3-DPC-able and 4-DPC-able. Now we will present a recursive algorithm that uses a 3-DPC (resp. 4-DPC) of Q_2^k to construct a 3-DPC (resp. 4-DPC) of Q_2^{k+2}. Let R be a subset of $V(Q_2^k) \cup E(Q_2^k)$. We define a function, f, which maps R from Q_2^k into Q_2^{k+2} in the following way:
(1) If $(i, j) \in R \cap V(Q_2^k)$, where $0 \leq i, j \leq k-1$, then

$$f((i, j)) = \begin{cases} (i, j) & \text{if } 0 \leq i, j \leq k-2; \\ (i+2, j) & \text{if } i = k-1, 0 \leq j \leq k-2; \\ (i, j+2) & \text{if } j = k-1, 0 \leq i \leq k-2; \\ (i+2, j+2) & \text{if } i = k-1 = j. \end{cases}$$

(2) If $((i, j), (i', j')) \in R \cap E(Q_2^k)$, where $i \leq i'$, $j \leq j'$, then

$$f(((i, j), (i', j'))) = \begin{cases} ((i, j), (i', j')) & \text{if } 0 \leq i, j \leq k-3, 1 \leq i', j' \leq k-2; \\ ((i+2, j), (i'+2, j)) & \text{if } i = i' = k-1, 0 \leq j \leq k-3, 1 \leq j' \leq k-2; \\ ((i, j+2), (i', j'+2)) & \text{if } j = j' = k-1, 0 \leq i \leq k-3, 1 \leq i' \leq k-2; \\ ((i, j), (i', j'+2)) & \text{if } 0 \leq i = i' \leq k-2, j = 0, j' = k-1; \\ ((i, j), (i'+2, j')) & \text{if } 0 \leq j = j' \leq k-2, i = 0, i' = k-1; \\ ((i, j+2), (i'+2, j'+2)) & \text{if } i = 0, i' = k-1, j = j' = k-1; \\ ((i+2, j), (i'+2, j'+2)) & \text{if } j = 0, j' = k-1, i = i' = k-1. \end{cases}$$

Please see Fig. 4 for an illustration.

Let u, v be a pair of distinct vertices of Q_2^k. We say that a 3-DPC (resp. 4-DPC) $C(u, v)$ of Q_2^k is *regular* if $C(u, v)$ contains some edges in $\{((\alpha, k-2), (\alpha, k-1)) \mid 0 \leq \alpha \leq k-1\}$ and $\{((k-2, \beta), (k-1, \beta)) \mid 0 \leq \beta \leq k-1\}$. For example,

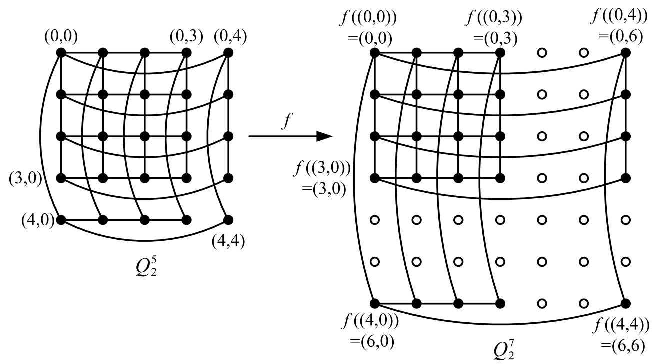


Fig. 4. Using function f to map a subset of edges and vertices of Q_2^5 into Q_2^7.

all 3-DPC and 4-DPC of Q_2^5 constructed in Lemma 5 are regular. Assume that k is an odd integer and $k \geq 5$. Let $C(u, v)$ be a regular 3-DPC (resp. 4-DPC) of Q_2^k with the endvertex set $P = \{u = (0, 0), v = (x, y)\}$. We construct a regular 3-DPC (resp. 4-DPC) of Q_2^{k+2} with the endvertex set $f(P)$ using the following algorithm.

Step 1. In Q_2^k, let $\{v_0, v_1, \ldots, v_{t-1}\}$ and $\{h_0, h_1, \ldots, h_{s-1}\}$ be finite sequences of indices satisfying the following requirements:
(1) $0 \leq v_0 < v_1 < \cdots < v_{t-1} \leq k-1$ and $k-1 \geq h_0 > h_1 > \cdots > h_{s-1} \geq 0$;
(2) for $0 \leq i \leq k-1$, $((v_i, k-2), (v_i, k-1))$ is an edge of $C(u, v)$, and for $0 \leq j \leq k-1$, $((k-2, h_j), (k-1, h_j))$ is an edge of $C(u, v)$.

Step 2. Let $\overline{C}(u, v)$ be the image in Q_2^{k+2} of $C(u, v) - (\{((v_i, k-2), (v_i, k-1)) \mid 0 \leq i \leq k-1\} \cup \{((k-2, h_j), (k-1, h_j)) \mid 0 \leq j \leq k-1\})$ under the function f.

Step 3. For any two positive integers r and d, we use $[r]_d$ to denote $r(\textbf{mod}\ d)$. In Q_2^{k+2}, define the following path patterns, where r_1, r_2 are integers:

$$I_\alpha(r_1, r_2) = \langle (r_1, \alpha), ([r_1+1]_{k+2}, \alpha), ([r_1+2]_{k+2}, \alpha), \ldots, (r_2, \alpha)\rangle;$$
$$I_\alpha^{-1}(r_2, r_1) = \langle (r_2, \alpha), ([r_2-1]_{k+2}, \alpha), ([r_2-2]_{k+2}, \alpha), \ldots, (r_1, \alpha)\rangle;$$
$$H_\beta(r_1, r_2) = \langle (\beta, r_1), (\beta, [r_1+1]_{k+2}), (\beta, [r_1+2]_{k+2}), \ldots, (\beta, r_2)\rangle;$$
$$H_\beta^{-1}(r_2, r_1) = \langle (\beta, r_2), (\beta, [r_2-1]_{k+2}), (\beta, [r_2-2]_{k+2}), \ldots, (\beta, r_1)\rangle.$$

Let $\overline{v}_i = v_i + 2$ if $v_i = k-1$ and $\overline{v}_i = v_i$ if $0 \leq v_i \leq k-2$, and $\overline{h}_j = h_j + 2$ if $h_j = k-1$ and $\overline{h}_j = h_j$ if $0 \leq h_j \leq k-2$.

Case 1. $v_0 = k-1$.
Let $P_0 = \langle (k+1, k-2), (k+1, k-1), (0, k-1), I_{k-1}(0, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, 0), (0, k), (k+1, k), (k+1, k+1)\rangle$.

Case 1.1. $s = 1$.
Let $\overline{P}_0 = \langle (k-2, \overline{h}_0), (k-1, \overline{h}_0), H_{k-1}^{-1}(\overline{h}_0, [\overline{h}_0+1]_{k+2}), (k-1, [\overline{h}_0+1]_{k+2}), (k, [\overline{h}_0+1]_{k+2}), H_k([\overline{h}_0+1]_{k+2}, \overline{h}_0), (k, \overline{h}_0), (k+1, \overline{h}_0)\rangle$. Then $\overline{C}(u, v) \cup P_0 \cup \overline{P}_0$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}.

Case 1.2. $s \geq 2$.
Let $\overline{P}_i = \langle (k-2, \overline{h}_i), (k-1, \overline{h}_i), H_{k-1}^{-1}(\overline{h}_i, \overline{h}_{i+1}+1), (k-1, \overline{h}_{i+1}+1), (k, \overline{h}_{i+1}+1), H_k(\overline{h}_{i+1}+1, \overline{h}_i), (k, \overline{h}_i), (k+1, \overline{h}_i)\rangle$ for $0 \leq i \leq s-2$, and $\overline{P}_{s-1} = \langle (k-2, \overline{h}_{s-1}), (k-1, \overline{h}_{s-1}), H_{k-1}^{-1}(\overline{h}_{s-1}, [\overline{h}_0+1]_{k+2}), (k-1, [\overline{h}_0+1]_{k+2}), (k, [\overline{h}_0+1]_{k+2}), H_k([\overline{h}_0+1]_{k+2}, \overline{h}_{s-1}), (k, \overline{h}_{s-1}), (k+1, \overline{h}_{s-1})\rangle$. Then $\overline{C}(u, v) \cup P_0 \cup \{\overline{P}_i \mid 0 \leq i \leq s-1\}$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}. Please see Fig. 5 for an illustration.

Case 2. $v_{t-1} \leq k-2$ and $((k-2, k-1), (k-1, k-1)) \in E(C(u, v))$ in Q_2^k.

Case 2.1. $t = 1$.
Let $P_0 = \langle (\overline{v}_0, k-2), (\overline{v}_0, k-1), I_{k-1}(\overline{v}_0, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \overline{v}_0), (\overline{v}_0, k), (\overline{v}_0, k+1)\rangle$.

Case 2.1.1. $s = 1$.
Let $\overline{P}_0 = \langle (k-2, \overline{h}_0), (k-1, \overline{h}_0), H_{k-1}^{-1}(\overline{h}_0, 0), (k-1, 0), (k, 0), H_k(0, k-1), (k, k-1), (k+1, k-1), I_{k-1}(k+1, [\overline{v}_0-1]_{k+2}), ([\overline{v}_0-1]_{k+2}, k-1), ([\overline{v}_0-1]_{k+2}, k), I_k^{-1}([\overline{v}_0-1]_{k+2}, k+1), (k+1, k), (k, k), (k, \overline{h}_0), (k+1, \overline{h}_0)\rangle$. Then $\overline{C}(u, v) \cup P_0 \cup \overline{P}_0$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}.

Case 2.1.2. $s = 2$.
Let $\overline{P}_0 = \langle (k-2, \overline{h}_0), (k-1, \overline{h}_0), H_{k-1}^{-1}(\overline{h}_0, \overline{h}_1+1), (k-1, \overline{h}_1+1), (k, \overline{h}_1+1), H_k(\overline{h}_1+1, k-1), (k, k-1), (k+1, k-1), I_{k-1}(k+1, [\overline{v}_0-1]_{k+2}), ([\overline{v}_0-1]_{k+2}, k-1), ([\overline{v}_0-1]_{k+2}, k), I_k^{-1}([\overline{v}_0-1]_{k+2}, k+1), (k+1, k), (k, k), (k, \overline{h}_0), (k+1, \overline{h}_0)\rangle$, and $\overline{P}_1 = \langle (k-2, \overline{h}_1), (k-1, \overline{h}_1), H_{k-1}^{-1}(\overline{h}_1, 0), (k-1, 0), (k, 0), H_k(0, \overline{h}_1), (k, \overline{h}_1), (k+1, \overline{h}_1)\rangle$. Then $\overline{C}(u, v) \cup P_0 \cup \overline{P}_0 \cup \overline{P}_1$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}.

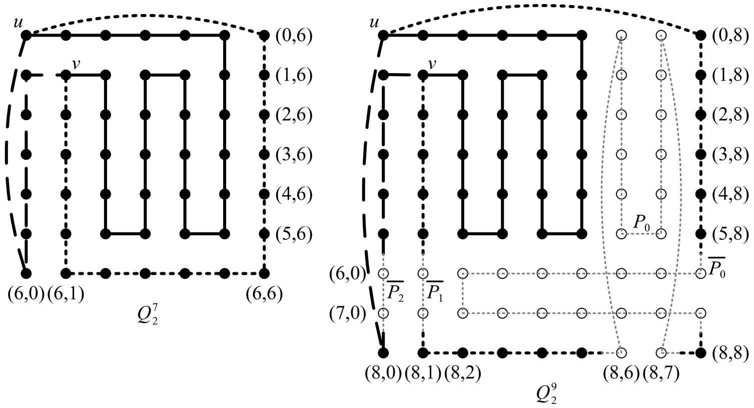


Fig. 5. An illustration for Case 1.2 of Lemma 7. Use the 3-DPC of Q_2^7 to construct the 3-DPC of Q_2^9, where $s = 3, t = 1, h_0 = 6, h_1 = 1, h_2 = 0, v_0 = 6$.

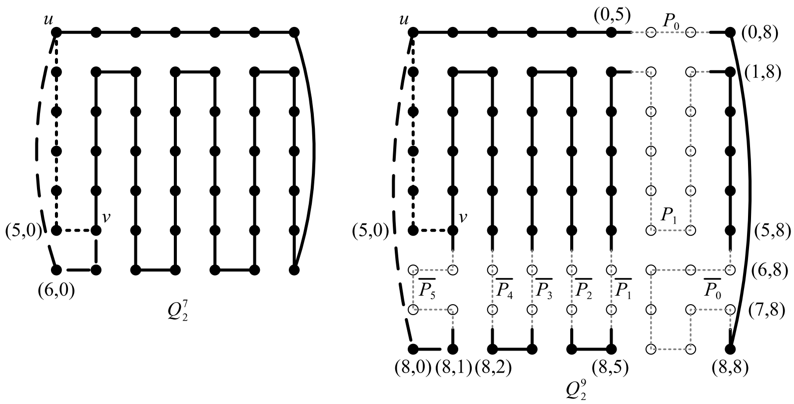


Fig. 6. An illustration for Case 2.2.3 of Lemma 7. Use the 3-DPC of Q_2^7 to construct the 3-DPC of Q_2^9, where $s = 6, t = 2, h_0 = 6, h_1 = 5, h_2 = 4, h_3 = 3, h_4 = 2, h_5 = 1, v_0 = 0, v_1 = 1$.

Case 2.1.3. $s \geq 3$.
Let $\overline{P}_0 = \langle (k-2, \overline{h}_0), (k-1, \overline{h}_0), H_{k-1}^{-1}(\overline{h}_0, \overline{h}_1 + 1), (k-1, \overline{h}_1 + 1), (k, \overline{h}_1 + 1), H_k(\overline{h}_1 + 1, k-1), (k, k-1), (k+1, k-1), I_{k-1}(k+1, [\overline{v}_0 - 1]_{k+2}), ([\overline{v}_0 - 1]_{k+2}, k-1), ([\overline{v}_0 - 1]_{k+2}, k), I_k^{-1}([\overline{v}_0 - 1]_{k+2}, k+1), (k+1, k), (k, k), (k, \overline{h}_0), (k+1, \overline{h}_0) \rangle$, $\overline{P}_i = \langle (k-2, \overline{h}_i), (k-1, \overline{h}_i), H_{k-1}^{-1}(\overline{h}_i, \overline{h}_{i+1} + 1), (k-1, \overline{h}_{i+1} + 1), (k, \overline{h}_{i+1} + 1), H_k(\overline{h}_{i+1} + 1, \overline{h}_i), (k, \overline{h}_i), (k+1, \overline{h}_i) \rangle$ for $1 \leq i \leq s-2$, and $\overline{P}_{s-1} = \langle (k-2, \overline{h}_{s-1}), (k-1, \overline{h}_{s-1}), H_{k-1}^{-1}(\overline{h}_{s-1}, 0), (k-1, 0), (k, 0), H_k(0, \overline{h}_{s-1}), (k, \overline{h}_{s-1}), (k+1, \overline{h}_{s-1}) \rangle$. Then $\overline{C}(u, v) \cup P_0 \cup \{\overline{P}_i \mid 0 \leq i \leq s-1\}$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}.

Case 2.2. $t \geq 2$.
Let $P_i = \langle (\overline{v}_i, k-2), (\overline{v}_i, k-1), I_{k-1}(\overline{v}_i, \overline{v}_{i+1} - 1), (\overline{v}_{i+1} - 1, k-1), (\overline{v}_{i+1} - 1, k), I_k^{-1}(\overline{v}_{i+1} - 1, \overline{v}_i), (\overline{v}_i, k), (\overline{v}_i, k+1) \rangle$ for $0 \leq i \leq t-2$, and $P_{t-1} = \langle (\overline{v}_{t-1}, k-2), (\overline{v}_{t-1}, k-1), I_{k-1}(\overline{v}_{t-1}, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \overline{v}_{t-1}), (\overline{v}_{t-1}, k), (\overline{v}_{t-1}, k+1) \rangle$.

Case 2.2.1. $s = 1$.
Using the same $\overline{P}_0$ as in Case 2.1.1, then $\overline{C}(u, v) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \overline{P}_0$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}.

Case 2.2.2. $s = 2$.
Using the same $\overline{P}_0$ and $\overline{P}_1$ as in Case 2.1.2., then $\overline{C}(u, v) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \overline{P}_0 \cup \overline{P}_1$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}.

Case 2.2.3. $s \geq 3$.
Using the same $\{\overline{P}_i \mid 0 \leq i \leq s-1\}$ as in Case 2.1.3., then $\overline{C}(u, v) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \{\overline{P}_i \mid 0 \leq i \leq s-1\}$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}. Please see Fig. 6 for an illustration.

Case 3. $v_{t-1} \leq k-2$ and $((k-2, k-1), (k-1, k-1)) \notin E(C(u, v))$ in Q_2^k.

Case 3.1. $t = 1$.
Let $P_0 = \langle (\overline{v}_0, k-2), (\overline{v}_0, k-1), I_{k-1}(\overline{v}_0, k-1), (k-1, k-1), H_{k-1}^{-1}(k-1, \overline{h}_0 + 1), (k-1, \overline{h}_0 + 1), (k, \overline{h}_0 + 1), H_k(\overline{h}_0 + 1, k-1), (k, k-1), (k+1, k-1), (0, k-1), I_{k-1}(0, \overline{v}_0 - 1), (\overline{v}_0 - 1, k-1), (\overline{v}_0 - 1, k), I_k^{-1}(\overline{v}_0 - 1, 0), (0, k), (k+1, k), (k, k), (k, k+1), (k-1, k+1), (k-1, k), I_k^{-1}(k-1, \overline{v}_0), (\overline{v}_0, k), (\overline{v}_0, k+1) \rangle$.

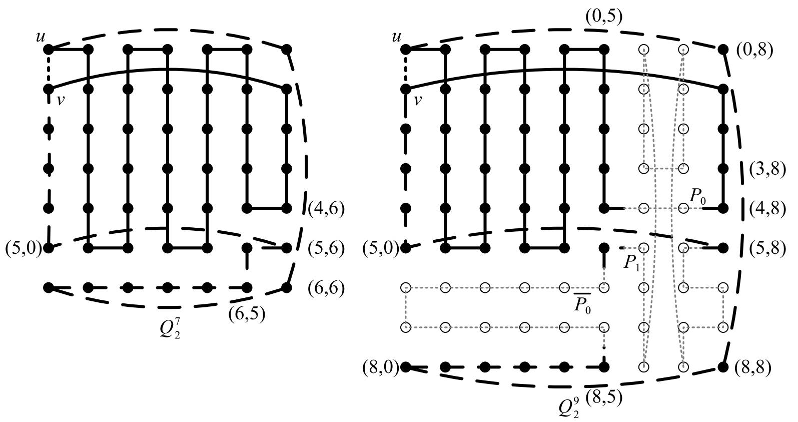


Fig. 7. An illustration for Case 3.2.1 of Lemma 7. Use the 3-DPC of Q_2^7 to construct the 3-DPC of Q_2^9, where $s = 1, t = 2, h_0 = 5, v_0 = 4, v_1 = 5$.

Case 3.1.1. $s = 1$.
Let $\overline{P}_0 = \langle (k-2, \overline{h}_0), (k-1, \overline{h}_0), H_{k-1}^{-1}(\overline{h}_0, 0), (k-1, 0), (k, 0), H_k(0, \overline{h}_0), (k, \overline{h}_0), (k+1, \overline{h}_0)\rangle$. Then $\overline{C}(u, v) \cup P_0 \cup \overline{P}_0$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}.

Case 3.1.2. $s \geq 2$.
Let $\overline{P}_i = \langle (k-2, \overline{h}_i), (k-1, \overline{h}_i), H_{k-1}^{-1}(\overline{h}_i, \overline{h}_{i+1}+1), (k-1, \overline{h}_{i+1}+1), (k, \overline{h}_{i+1}+1), H_k(\overline{h}_{i+1}+1, \overline{h}_i), (k, \overline{h}_i), (k+1, \overline{h}_i)\rangle$ for $0 \leq i \leq s-2$, and $\overline{P}_{s-1} = \langle (k-2, \overline{h}_{s-1}), (k-1, \overline{h}_{s-1}), H_{k-1}^{-1}(\overline{h}_{s-1}, 0), (k-1, 0), (k, 0), H_k(0, \overline{h}_{s-1}), (k, \overline{h}_{s-1}), (k+1, \overline{h}_{s-1})\rangle$. Then $\overline{C}(u, v) \cup P_0 \cup \{\overline{P}_i \mid 0 \leq i \leq s-1\}$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}.

Case 3.2. $t \geq 2$.
Let $P_i = \langle (\overline{v}_i, k-2), (\overline{v}_i, k-1), I_{k-1}(\overline{v}_i, \overline{v}_{i+1}-1), (\overline{v}_{i+1}-1, k-1), (\overline{v}_{i+1}-1, k), I_k^{-1}(\overline{v}_{i+1}-1, \overline{v}_i), (\overline{v}_i, k), (\overline{v}_i, k+1)\rangle$ for $0 \leq i \leq t-2$, and $P_{t-1} = \langle (\overline{v}_{t-1}, k-2), (\overline{v}_{t-1}, k-1), I_{k-1}(\overline{v}_{t-1}, k-1), (k-1, k-1), H_{k-1}^{-1}(k-1, \overline{h}_0+1), (k-1, \overline{h}_0+1), (k, \overline{h}_0+1), H_k(\overline{h}_0+1, k-1), (k, k-1), (k+1, k-1), (0, k-1), I_{k-1}(0, \overline{v}_0-1), (\overline{v}_0-1, k-1), (\overline{v}_0-1, k), I_k^{-1}(\overline{v}_0-1, 0), (0, k), (k+1, k), (k, k), (k, k+1), (k-1, k+1), (k-1, k), I_k^{-1}(k-1, \overline{v}_{t-1}), (\overline{v}_{t-1}, k), (\overline{v}_{t-1}, k+1)\rangle$.

Case 3.2.1. $s = 1$.
Using the same $\overline{P}_0$ as in Case 3.1.1, then $\overline{C}(u, v) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \overline{P}_0$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}. Please see Fig. 7 for an illustration.

Case 3.2.2. $s \geq 2$.
Using the same $\{\overline{P}_i \mid 0 \leq i \leq s-1\}$ as in Case 3.1.2., then $\overline{C}(u, v) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \{\overline{P}_i \mid 0 \leq i \leq s-1\}$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}.

Case 4. $v_{t-1} = k-1$ for some $t \geq 2$ and $v_0 = 0$.

Case 4.1. $t = 2$.
Let $P_0 = \langle (\overline{v}_0, k-2), (\overline{v}_0, k-1), I_{k-1}(\overline{v}_0, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \overline{v}_0), (\overline{v}_0, k), (\overline{v}_0, k+1)\rangle$, and $P_1 = \langle (k+1, k-2), (k+1, k-1), (k+1, k), (k+1, k+1)\rangle$.

Case 4.1.1. $s = 1$.
Using the same $\overline{P}_0$ as in Case 1.1., then $\overline{C}(u, v) \cup P_0 \cup P_1 \cup \overline{P}_0$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}.

Case 4.1.2. $s \geq 2$.
Using the same $\{\overline{P}_i \mid 0 \leq i \leq s-1\}$ as in Case 1.2., then $\overline{C}(u, v) \cup P_0 \cup P_1 \cup \{\overline{P}_i \mid 0 \leq i \leq s-1\}$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}. Please see Fig. 8 for an illustration.

Case 4.2. $t \geq 3$.
Let $P_i = \langle (\overline{v}_i, k-2), (\overline{v}_i, k-1), I_{k-1}(\overline{v}_i, \overline{v}_{i+1}-1), (\overline{v}_{i+1}-1, k-1), (\overline{v}_{i+1}-1, k), I_k^{-1}(\overline{v}_{i+1}-1, \overline{v}_i), (\overline{v}_i, k), (\overline{v}_i, k+1)\rangle$ for $0 \leq i \leq t-3$, $P_{t-2} = \langle (\overline{v}_{t-2}, k-2), (\overline{v}_{t-2}, k-1), I_{k-1}(\overline{v}_{t-2}, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \overline{v}_{t-2}), (\overline{v}_{t-2}, k), (\overline{v}_{t-2}, k+1)\rangle$, and $P_{t-1} = \langle (k+1, k-2), (k+1, k-1), (k+1, k), (k+1, k+1)\rangle$.

Case 4.2.1. $s = 1$.
Using the same $\overline{P}_0$ as in Case 1.1., then $\overline{C}(u, v) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \overline{P}_0$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}.

Case 4.2.2. $s \geq 2$.
Using the same $\{\overline{P}_i \mid 0 \leq i \leq s-1\}$ as in Case 1.2., then $\overline{C}(u, v) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \{\overline{P}_i \mid 0 \leq i \leq s-1\}$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}.

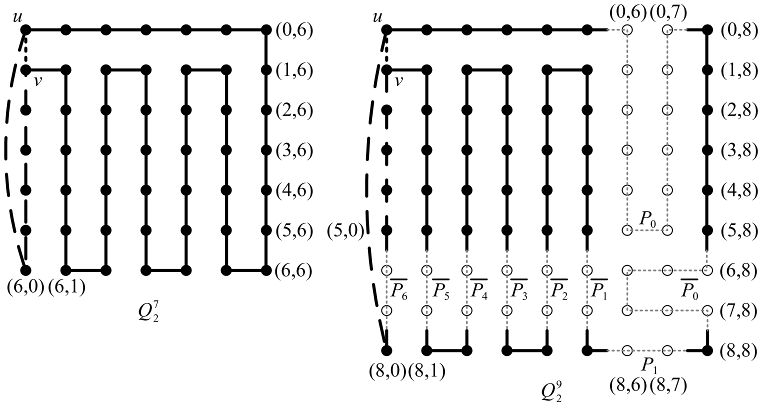


Fig. 8. An illustration for Case 4.1.2 of Lemma 7. Use the 3-DPC of Q_2^7 to construct the 3-DPC of Q_2^9, where $s = 7, t = 2, h_0 = 6, h_1 = 5, h_2 = 4, h_3 = 3, h_4 = 2, h_5 = 1, h_6 = 0, v_0 = 0, v_1 = 6$.

Case 5. $v_{t-1} = k - 1$ for some $t \geq 2$ and $v_0 \neq 0$.
Case 5.1. $t = 2$.
Let $P_0 = \langle(\overline{v}_0, k-2), (\overline{v}_0, k-1), I_{k-1}(\overline{v}_0, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \overline{v}_0), (\overline{v}_0, k), (\overline{v}_0, k+1)\rangle$, and $P_1 = \langle(k+1, k-2), (k+1, k-1), (k+1, k), (k+1, k+1)\rangle$, and $P_1 = \langle(k+1, k-2), (k+1, k-1), (0, k-1), I_{k-1}(0, \overline{v}_0 - 1), (\overline{v}_0 - 1, k-1), (\overline{v}_0 - 1, k), I_k^{-1}(\overline{v}_0 - 1, 0), (0, k), (k+1, k), (k+1, k+1)\rangle$.
Case 5.1.1. $s = 1$.
Using the same $\overline{P}_0$ as in Case 1.1., then $\overline{C}(u, v) \cup P_0 \cup P_1 \cup \overline{P}_0$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}.
Case 5.1.2. $s \geq 2$.
Using the same $\{\overline{P}_i \mid 0 \leq i \leq s-1\}$ as in Case 1.2., then $\overline{C}(u, v) \cup P_0 \cup P_1 \cup \{\overline{P}_i \mid 0 \leq i \leq s-1\}$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}.
Case 5.2. $t \geq 3$.
Let $P_i = \langle(\overline{v}_i, k-2), (\overline{v}_i, k-1), I_{k-1}(\overline{v}_i, \overline{v}_{i+1} - 1), (\overline{v}_{i+1} - 1, k-1), (\overline{v}_{i+1} - 1, k), I_k^{-1}(\overline{v}_{i+1} - 1, \overline{v}_i), (\overline{v}_i, k), (\overline{v}_i, k+1)\rangle$ for $0 \leq i \leq t-3$, $P_{t-2} = \langle(\overline{v}_{t-2}, k-2), (\overline{v}_{t-2}, k-1), I_{k-1}(\overline{v}_{t-2}, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \overline{v}_{t-2}), (\overline{v}_{t-2}, k), (\overline{v}_{t-2}, k+1)\rangle$, and $P_{t-1} = \langle(k+1, k-2), (k+1, k-1), (0, k-1), I_{k-1}(0, \overline{v}_0 - 1), (\overline{v}_0 - 1, k-1), (\overline{v}_0 - 1, k), I_k^{-1}(\overline{v}_0 - 1, 0), (0, k), (k+1, k), (k+1, k+1)\rangle$.
Case 5.2.1. $s = 1$.
Using the same $\overline{P}_0$ as in Case 1.1., then $\overline{C}(u, v) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \overline{P}_0$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}.
Case 5.2.2. $s \geq 2$.
Using the same $\{\overline{P}_i \mid 0 \leq i \leq s-1\}$ as in Case 1.2., then $\overline{C}(u, v) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \{\overline{P}_i \mid 0 \leq i \leq s-1\}$ is the 3-DPC (or 4-DPC) of Q_2^{k+2}. □

The following lemma for Q_2^k for any even integer $k \geq 6$ can be derived similarly.

Lemma 8. *For any even integer $k \geq 6$, Q_2^k is bi-3-DPC-able and bi-4-DPC-able.*

3.3. The disjoint path covers of Q_n^k with $n \geq 2$

Theorem 4. *Let $n \geq 2$ be an integer and $k \geq 3$ be an odd integer. Then Q_n^k is m-DPC-able, where $1 \leq m \leq 2n$.*

Proof. By Theorems 1 and 3, Q_n^k is 1-DPC-able and 2-DPC-able. Thus, it suffices to prove that Q_n^k is m-DPC-able for $3 \leq m \leq 2n$. With Lemmas 3, 5 and 7, Q_2^k is m-DPC-able for $3 \leq m \leq 4$. Thus the theorem holds for $n = 2$. We shall prove the theorem by mathematical induction on n. Using the induction hypothesis, we assume that $Q_{n-1}^{k,i}$ is m-DPC-able for $1 \leq m \leq 2n-2$, where $0 \leq i \leq k-1$. Given two distinct vertices $u, v \in V(Q_n^k)$, with $u \in Q_{n-1}^{k,j}$ and $v \in Q_{n-1}^{k,j'}$, we want to show that we can use the m-DPC in $Q_{n-1}^{k,i}$ to construct an $(m+2)$-DPC between u and v in Q_n^k.
Case 1. $j = j'$. W.L.O.G., let $j = j' = 0$.
Now, $u = u^0$ and $v = v^0$ are in $Q_{n-1}^{k,0}$. By the induction hypothesis, $Q_{n-1}^{k,0}$ is m-DPC-able, so there are m vertex disjoint paths between u and v, denoted by $\{P_i\}_{i=0}^{m-1}$, whose union covers all the vertices of $Q_{n-1}^{k,0}$ for all $1 \leq m \leq 2n-2$. According to Theorem 1, there is a path R between u^{k-1} and v^{k-1} covering all the vertices of $Q_{n-1}^{k,k-1}$. Let $P_m = \langle u, u^{k-1}, R, v^{k-1}, v\rangle$. By Lemma 2, there is a path S between u^1 and v^1 covering all the vertices of $Q_{n-1}^{k,i}$ for $1 \leq i \leq k-2$. Let $P_{m+1} = \langle u, u^1, S, v^1, v\rangle$. Hence, there exist $m+2$ vertex disjoint paths $\{P_i\}_{i=0}^{m+1}$ between u and v, whose union covers all the vertices of Q_n^k. Please see Fig. 9 for an illustration.

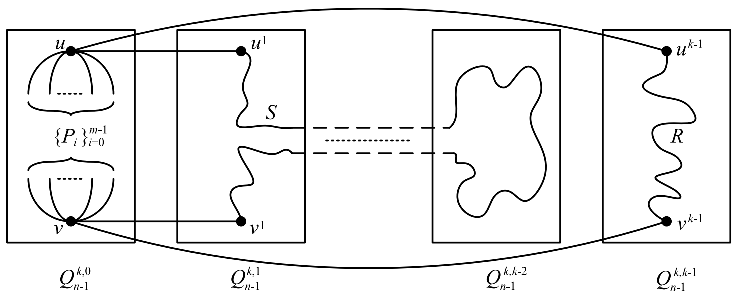


Fig. 9. An illustration for Case 1 of Theorem 4.

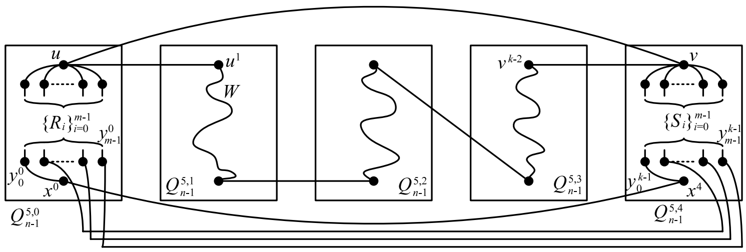


Fig. 10. An illustration for Case 2.1.2 of Theorem 4 when $k = 5$.

Case 2. $|j - j'| = 1$. W.L.O.G., let $j = 0$ and $j' = k - 1$.
Let $u = u^0$ be in $Q_{n-1}^{k,0}$ and $v = v^{k-1}$ in $Q_{n-1}^{k,k-1}$. We have the following three subcases.

Case 2.1. If $d_{Q_n^k}(u, v) = 1$.

Case 2.1.1. $m = 1$.
We let $P_0 = \langle u = u^0, v^{k-1} = v\rangle$. Given any vertex x^0 in $Q_{n-1}^{k,0} - \{u^0\}$. By Theorem 1, there is a path S between u^0 and x^0 covering all the vertices of $Q_{n-1}^{k,0}$, and a path T between x^{k-1} and v^{k-1} covering all the vertices of $Q_{n-1}^{k,k-1}$. Then, we set $P_1 = \langle u = u^0, S, x^0, x^{k-1}, T, v^{k-1} = v\rangle$. According to Lemma 1, there is a path U between u^1 and v^{k-2} covering all the vertices of $Q_{n-1}^{k,i}$ for $1 \le i \le k - 2$. Let $P_2 = \langle u = u^0, u^1, U, v^{k-2}, v^{k-1} = v\rangle$. Hence, there are three vertex disjoint paths $\{P_0, P_1, P_2\}$ between u and v, whose union covers all the vertices of Q_n^k.

Case 2.1.2. $m \ge 2$.
By the induction hypothesis, $Q_{n-1}^{k,0}$ is m-DPC-able, so there are m vertex disjoint paths between u^0 and x^0, denoted by $\{R_i\}_{i=0}^{m-1}$, whose union covers all the vertices of $Q_{n-1}^{k,0}$. Besides, there are m vertex disjoint paths between x^{k-1} and v^{k-1}, denoted by $\{S_i\}_{i=0}^{m-1}$, whose union covers all the vertices of $Q_{n-1}^{k,k-1}$. Set $R_i = \langle u^0, T_i, y_i^0, x^0\rangle$, and $S_i = \langle x^{k-1}, y_i^{k-1}, U_i, v^{k-1}\rangle$. We let $P_0 = \langle u = u^0, R_0, x^0, x^{k-1}, S_0, v^{k-1} = v\rangle$ and $P_i = \langle u = u^0, T_i, y_i^0, y_i^{k-1}, U_i, v^{k-1} = v\rangle$ for $1 \le i \le m - 1$. By Lemma 1, there is a path W between u^1 and v^{k-2} covering all the vertices of $Q_{n-1}^{k,i}$ for $1 \le i \le k-2$. Set $P_m = \langle u = u^0, u^1, W, v^{k-2}, v^{k-1} = v\rangle$. Finally, let $P_{m+1} = \langle u = u^0, v^{k-1} = v\rangle$. Therefore, we construct $m + 2$ vertex disjoint paths $\{P_i\}_{i=0}^{m+1}$ between u and v, whose union covers all the vertices of Q_n^k. Please see Fig. 10 for an illustration.

Case 2.2. If $d_{Q_n^k}(u, v) = 2$.

Case 2.2.1. $m = 1$.
By Theorem 1, there is a path R between u^0 and v^0 covering all the vertices of $Q_{n-1}^{k,0}$, and a path S between u^{k-1} and v^{k-1} covering all the vertices of $Q_{n-1}^{k,k-1}$. W.L.O.G., we let $R = \langle u^0, T, x^0, v^0\rangle$ and $S = \langle u^{k-1}, y^{k-1}, U, v^{k-1}\rangle$. Let $P_0 = \langle u = u^0, u^{k-1}, v^{k-1} = v\rangle$ and $P_1 = \langle u = u^0, v^0, v^{k-1} = v\rangle$. According to Lemma 1, there exists a path W between x^1 and y^{k-2} covering all the vertices of $Q_{n-1}^{k,i}$ for $1 \le i \le k - 2$. So, we set $P_2 = \langle u = u^0, T, x^0, x^1, W, y^{k-2}, y^{k-1}, U, v^{k-1} = v\rangle$. Therefore, there exist three vertex disjoint paths $\{P_0, P_1, P_2\}$ between u and v, whose union covers all the vertices of Q_n^k.

Case 2.2.2. $m \ge 2$.
By the induction hypothesis, $Q_{n-1}^{k,r}$ is m-DPC-able, so there are m vertex disjoint paths between u^r and v^r, denoted by $\{R_i^r\}_{i=0}^{m-1}$, whose union covers all the vertices of $Q_{n-1}^{k,r}$ where $0 \le r \le k-1$. W.L.O.G., we let $R_0^r = \langle u^r, v^r\rangle$ and $R_i^r = \langle u^r, x_i^r, S_i^r, y_i^r, v^r\rangle$ for $1 \le i \le m - 1$. Let $P_0 = \langle u = u^0, v^0, v^{k-1}\rangle$. We set $P_i = \langle u = u^0, x_i^0, S_i^0, y_i^0, y_i^1, (S_i^1)^{-1}, x_i^1, \ldots, x_i^{k-1}, S_i^{k-1}, y_i^{k-1}, v^{k-1} = v\rangle$ for $1 \le i \le m - 1$. We let $P_m = \langle u = u^0, u^1, v^1, v^2, u^2, \ldots, u^{k-2}, v^{k-2}, v^{k-1} = v\rangle$, and $P_{m+1} = \langle u = u^0, u^{k-1}, v^{k-1}\rangle$.

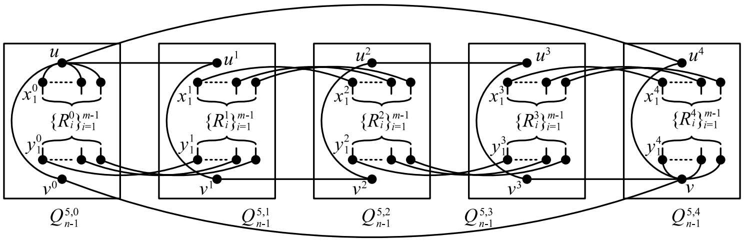


Fig. 11. An illustration for Case 2.2.2 of Theorem 4 when $k = 5$.

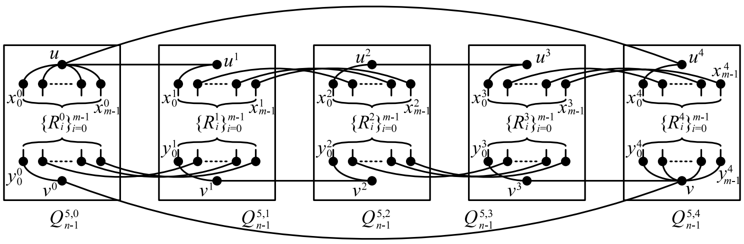


Fig. 12. An illustration for Case 2.3.2 of Theorem 4 when $k = 5$.

Therefore, we construct $m + 2$ vertex disjoint paths $\{P_i\}_{i=0}^{m+1}$ between u and v, whose union covers all the vertices of Q_n^k. Please see Fig. 11 for an illustration.

Case 2.3. If $d_{Q_n^k}(u, v) \geq 3$.

Case 2.3.1. $m = 1$.
By Theorem 1, there exists a path R between u^0 and v^0 covering all the vertices of $Q_{n-1}^{k,0}$, and a path S between u^{k-1} and v^{k-1} covering all the vertices of $Q_{n-1}^{k,k-1}$. According to Lemma 1, there is a path W between u^1 and v^{k-2} covering all the vertices of $Q_{n-1}^{k,i}$ for $1 \leq i \leq k-2$. We let $P_0 = \langle u = u^0, R, v^0, v^{k-1} = v\rangle$, $P_1 = \langle u = u^0, u^{k-1}, S, v^{k-1} = v\rangle$, and $P_2 = \langle u = u^0, u^1, W, v^{k-2}, v^{k-1} = v\rangle$. There are three vertex disjoint paths $\{P_0, P_1, P_2\}$ between u and v, whose union covers all the vertices of Q_n^k.

Case 2.3.2. $m \geq 2$.
By the induction hypothesis, $Q_{n-1}^{k,r}$ is m-DPC-able, so there are m vertex disjoint paths between u^r and v^r, denoted by $\{R_i^r\}_{i=0}^{m-1}$, whose union covers all the vertices of $Q_{n-1}^{k,r}$ where $0 \leq r \leq k-1$. W.L.O.G., we let $R_i^r = \langle u^r, x_i^r, S_i^r, y_i^r, v^r\rangle$ for $0 \leq i \leq m-1$. Let $P_0 = \langle u = u^0, R_0^0, v^0, v^{k-1} = v\rangle$, and $P_i = \langle u = u^0, x_i^0, S_i^0, y_i^0, y_i^1, (S_i^1)^{-1}, x_i^1, \ldots, x_i^{k-1}, S_i^{k-1}, y_i^{k-1}, v^{k-1} = v\rangle$ for $1 \leq i \leq m-1$. Then, we set $P_m = \langle u = u^0, u^1, R_0^1, v^1, v^2, (R_0^2)^{-1}, u^2, \ldots, u^{k-2}, R_0^{k-2}, v^{k-2}, v^{k-1} = v\rangle$, and $P_{m+1} = \langle u = u^0, u^{k-1}, R_0^{k-1}, v^{k-1} = v\rangle$. Hence, we construct $m + 2$ vertex disjoint paths $\{P_i\}_{i=0}^{m+1}$ between u and v, whose union covers all the vertices of Q_n^k. Please see Fig. 12 for an illustration.

Case 3. $|j - j'| \geq 2$. W.L.O.G., let $j = 0$ and j' be even.
Now, $u = u^0 \in Q_{n-1}^{k,0}$ and $v = v^{j'} \in Q_{n-1}^{k,j'}$. Assume that $0 \leq h \leq j'$. By the induction hypothesis, $Q_{n-1}^{k,h}$ is m-DPC-able, so there are m vertex disjoint paths between u^h and v^h, denoted by $\{R_i^h\}_{i=0}^{m-1}$, whose union covers all the vertices of $Q_{n-1}^{k,h}$. We set $R_i^h = \langle u^h, x_i^h, S_i^h, y_i^h, v^h\rangle$. Let $P_i = \langle u = u^0, x_i^0, S_i^0, y_i^0, y_i^1, (S_i^1)^{-1}, x_i^1, \ldots, x_i^{j'}, S_i^{j'}, y_i^{j'}, v^{j'} = v\rangle$ for $0 \leq i \leq m-1$. By Lemma 2, there is a path T between $u^{j'+1}$ and $v^{j'+1}$ covering all the vertices of $Q_{n-1}^{k,i}$, for $j' + 1 \leq i \leq k-2$. Set $P_m = \langle u = u^0, u^1, \ldots, u^{j'}, u^{j'+1}, T, v^{j'+1}, v^{j'} = v\rangle$. Finally, according to Theorem 1, there is a path U between u^{k-1} and v^{k-1} covering all the vertices of $Q_{k-1}^{k,k-1}$. We let $P_{m+1} = \langle u = u^0, u^{k-1}, U, v^{k-1}, v^0, v^1, \ldots, v^{j'-1}, v^{j'} = v\rangle$. Therefore, we construct the $m + 2$ vertex disjoint paths $\{P_i\}_{i=0}^{m+1}$ between u and v, whose union covers all the vertices of Q_n^k. Please see Fig. 13 for an illustration. □

With Theorem 4, we have shown that Q_n^k is m-DPC-able for $1 \leq m \leq 2n$, where $k \geq 3$ is an odd integer and $n \geq 2$ is an integer. The result is optimal since each vertex of Q_n^k has exactly $2n$ neighbors. The construction scheme in Theorem 4 cannot be applied to Q_n^k for $k \geq 4$ being an even integer. In fact, it is much more difficult to prove that Q_n^k is bi-m-DPC-able for $1 \leq m \leq 2n$ when $k \geq 2$ is even. Thus the detailed derivation is given below.

Theorem 5. *Let $n \geq 2$ be an integer and $k \geq 4$ be an even integer. Then Q_n^k is bi-m-DPC-able, where $1 \leq m \leq 2n$.*

Proof. According to Theorems 2, 3 and Lemmas 4, 6 and 8, the theorem holds for any even integer $k \geq 4$ when $n = 2$. We will give the proof of the theorem by mathematical induction on n. By the induction hypothesis, assume that $Q_{n-1}^{k,i}$ is

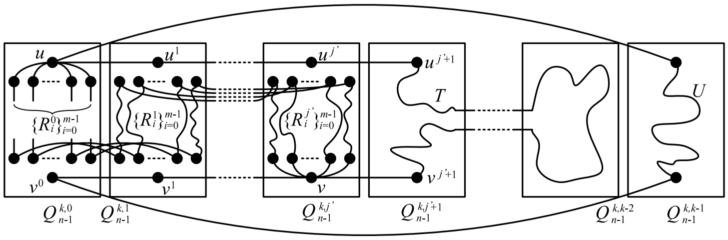


Fig. 13. An illustration for Case 3 of Theorem 4.

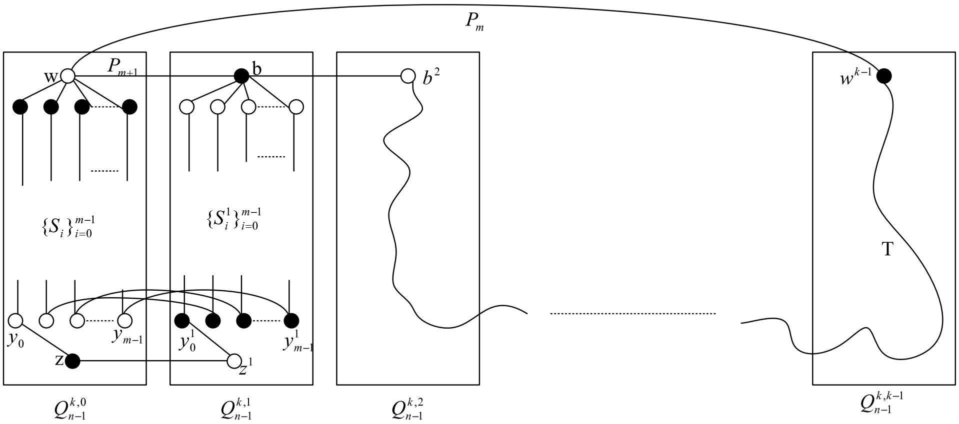


Fig. 14. The illustration for Case 2.1.2 of Theorem 5.

bi-m-DPC-able for $1 \le m \le 2n-2$, where $0 \le i \le k-1$. Given a white vertex $w \in V(Q_{n-1}^{k,j})$ and a black vertex $b \in V(Q_{n-1}^{k,j'})$. We will show that we can use the m-DPC of $Q_{n-1}^{k,j}$ to construct an $(m+2)$-DPC of Q_n^k between w and b.

Case 1. For $j = j'$. W.L.O.G., we let $j = j' = 0$.

In this case, we have $\{w, b\} \in Q_{n-1}^{k,0}$. By the induction hypothesis, there are m vertex disjoint paths $\{P_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q_{n-1}^{k,0}$ between w and b for $1 \le m \le 2n-2$. By Lemma 2, the exists a path S covering all vertices of $Q_{n-1}^{k,i}$ for $1 \le i \le k-2$ between w^1 and b^1. We can let $P_m = \langle w, w^1, S, b^1, b\rangle$. In $Q_{n-1}^{k,k-1}$, there exist a hamiltonian path R joining from w^{k-1} to b^{k-1} by Theorem 2. Also, we can let $P_{m+1} = \langle w, w^{k-1}, R, b^{k-1}, b\rangle$. Therefore, there are $m+2$ vertex disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of Q_n^k between w and b.

Case 2. For $|j - j'| = 1$. W.L.O.G., we let $j = 0$ and $j' = 1$.

We have the following two cases.

Case 2.1. Suppose that $d_{Q_n^k}(w, b) = 1$. It is easy to see that we can let $P_{m+1} = \langle w, b\rangle$.

Case 2.1.1. If $m = 1$.

Let z be any black vertex of $Q_{n-1}^{k,0}$. By Theorem 2, there exist a hamiltonian path S of $Q_{n-1}^{k,0}$ from w to z, and a hamiltonian path T of $Q_{n-1}^{k,1}$ from z^1 to b. So we set $P_0 = \langle w, S, z, z^1, T, b\rangle$. According to Lemma 1, a hamiltonian path R between $w^{k-1} \in Q_{n-1}^{k,k-1}$ and $b^2 \in Q_{n-1}^{k,2}$ covers all vertices of $Q_{n-1}^{k,i}$ for $2 \le i \le k-1$. We can write P_1 as $\langle w, w^{k-1}, R, b^2, b\rangle$. Hence, there are three vertex disjoint paths $\{P_0, P_1, P_2\}$ whose union covers all vertices of Q_n^k between w and b.

Case 2.1.2. If $m \ge 2$.

According to the induction hypothesis, given any black vertex $z \in V(Q_{n-1}^{k,0} - N(w))$, there exist m vertex disjoint paths $\{R_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q_{n-1}^{k,0}$ between w and z for $2 \le m \le 2n-2$. Let $R_i = \langle w, S_i, y_i, z\rangle$ for $0 \le i \le m-1$. We set $P_0 = \langle w, S_0, y_0, z, z^1, y_0^1, (S_0^1)^{-1}, b\rangle$ and $P_i = \langle w, S_i, y_i, y_i^1, (S_i^1)^{-1}, b\rangle$ for $1 \le i \le m-1$. By Lemma 1, there is a hamiltonian path T between $w^{k-1} \in Q_{n-1}^{k,k-1}$ and $b^2 \in Q_{n-1}^{k,2}$ covering all vertices of $Q_{n-1}^{k,i}$ for $2 \le i \le k-1$. Set $P_m = \langle w, w^{k-1}, T, b^2, b\rangle$. Consequently, there are $m+2$ vertex disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of Q_n^k between w and b. Please see Fig. 14 for an illustration.

Case 2.2. Suppose that $d_{Q_n^k}(w, b) \ge 3$.

Case 2.2.1. If $m = 1$.

Given any black vertex z in $Q_{n-1}^{k,0}$, by Theorem 2, there is a hamiltonian path R of $Q_{n-1}^{k,0}$ joining from w to z. So there is also a hamiltonian path S of $Q_{n-1}^{k,1}$ between w^1 to z^1. We can set $S = \langle w^1, S_1', b, S_2', z^1\rangle$. By Lemma 1, there exists a hamiltonian path

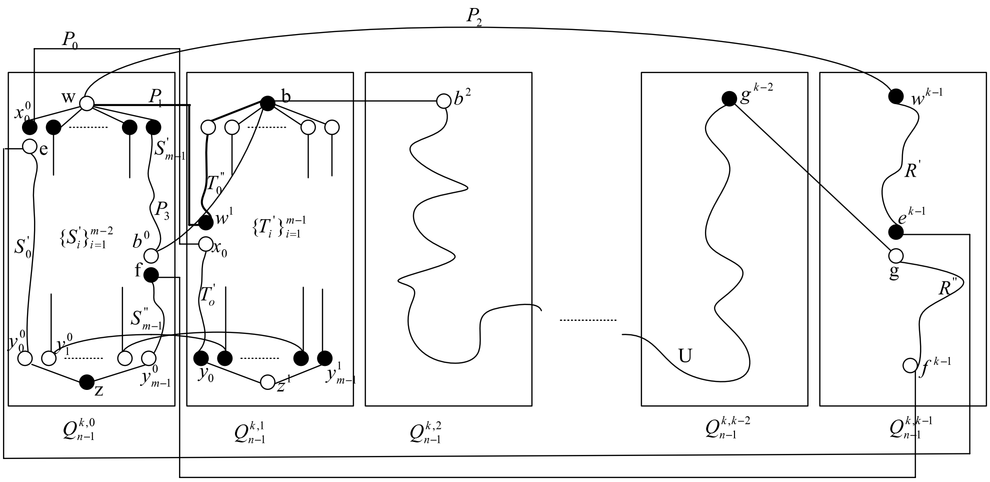


Fig. 15. The illustration for Case 2.2.2 of Theorem 5 when $b^0 \notin V(S_0)$.

T between $w^{k-1} \in Q_{n-1}^{k,k-1}$ and $b^2 \in Q_{n-1}^{k,2}$ covering all vertices of $Q_{n-1}^{k,i}$ for $2 \le i \le k-1$. We let $P_0 = \langle w, R, z, z^1, (S_2')^{-1}, b\rangle$, $P_1 = \langle w, w^1, S_1', b\rangle$, and $P_2 = \langle w, w^{k-1}, T, b^2, b\rangle$. Therefore, there are three vertex disjoint paths $\{P_0, P_1, P_2\}$ whose union covers all vertices of Q_n^k between w and b.

Case 2.2.2. If $m \ge 2$.
Let z be a black vertex of $V(Q_{n-1}^{k,0} - N(w))$. In $Q_{n-1}^{k,0}$, according to the induction hypothesis, there exist m vertex disjoint paths $\{S_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q_{n-1}^{k,0}$ between w and z for $2 \le m \le 2n-2$. So as in $Q_{n-1}^{k,1}$, there exist m vertex disjoint paths $\{T_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q_{n-1}^{k,1}$ between z^1 and b for $2 \le m \le 2n-2$. Let $T_0 = \langle z^1, y_0, T_0', x_0, w^1, T_0'', b\rangle$ and $T_i = \langle z^1, y_i, T_i', b\rangle$ for $1 \le i \le m-1$ in $Q_{n-1}^{k,1}$.

If $b^0 \notin V(S_0)$, W.L.O.G., let $b^0 \in V(S_{m-1})$. In $Q_{n-1}^{k,0}$, we also let $S_0 = \langle w, x_0^0, e, S_0', y_0^0, z\rangle$, $S_i = \langle w, S_i', y_i^0, z\rangle$ for $1 \le i \le m-2$, and $S_{m-1} = \langle w, S_{m-1}', b^0, f, S_{m-1}'', y_{m-1}^0, z\rangle$. A hamiltonian path R is embedded in $Q_{n-1}^{k,k-1}$ between w^{k-1} and f^{k-1} by Theorem 2. Write R as $\langle w^{k-1}, R', e^{k-1}, g, R'', f^{k-1}\rangle$. Notice that g^{k-2} is a black vertex and b^2 is a white vertex. According to Lemma 1, there is a hamiltonian path U between g^{k-2} and b^2 covering all vertices of $Q_{n-1}^{k,i}$ for $2 \le i \le k-2$. We can set $P_0 = \langle w, x_0^0, x_0, (T_0')^{-1}, y_0, z^1, y_{m-1}, T_{m-1}, b\rangle$, $P_1 = \langle w, w^1, T_0'', b\rangle$, $P_2 = \langle w, w^{k-1}, R', e^{k-1}, e, S_0', y_0^0, z, y_{m-1}^0, (S_{m-1}'')^{-1}, f, f^{k-1}, (R'')^{-1}, g, g^{k-2}, U, b^2, b\rangle$, $P_3 = \langle w, S_{m-1}', b^0, b\rangle$, and $P_i = \langle w, S_{i-3}', y_{i-3}^0, y_{i-3}, T_{i-3}', b\rangle$ for $4 \le i \le m+1$. So, there are $m+2$ vertex disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of Q_n^k between w and b. Please see Fig. 15 for an illustration.

If $b^0 \in V(S_0)$, let $S_0 = \langle w, x_0^0, e, S_0', b^0, f, S_0'', y_0^0, z\rangle$, and $S_i = \langle w, S_i', y_i^0, z\rangle$ for $1 \le i \le m-1$. A hamiltonian path R is embedded in $Q_{n-1}^{k,k-1}$ between w^{k-1} and f^{k-1} by Theorem 2. R is written as $\langle w^{k-1}, R', e^{k-1}, g, R'', f^{k-1}\rangle$. Notice that g^{k-2} is a black vertex and b^2 is a white vertex. According to Lemma 1, there is a hamiltonian path U between g^{k-2} and b^2 covering all vertices of $Q_{n-1}^{k,i}$ for $2 \le i \le k-2$. We let $P_0 = \langle w, x_0^0, x_0, (T_0')^{-1}, y_0, z^1, y_{m-1}, T_{m-1}', b\rangle$, $P_1 = \langle w, w^1, T_0'', b\rangle$, $P_2 = \langle w, w^{k-1}, R', e^{k-1}, e, S_0', b^0, b\rangle$, $P_3 = \langle w, S_{m-1}', y_{m-1}^0, z, y_0^0, (S_0'')^{-1}, f, f^{k-1}, (R'')^{-1}, g, g^{k-2}, U, b^2, b\rangle$, and $P_i = \langle w, S_{i-3}', y_{i-3}^0, y_{i-3}, T_{i-3}', b\rangle$ for $4 \le i \le m+1$. Hence, there are $m+2$ vertex disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of Q_n^k between w and b. Please see Fig. 16 for an illustration.

Case 3. For $|j - j'| \ge 2$. W.L.O.G., we let $j = 0$ and $2 \le j' \le \frac{k}{2}$ be even.
Because $b \in Q_{n-1}^{k,j'}$ where j' is even, b^i is a white (resp. black) vertex in $Q_{n-1}^{k,i}$ for $0 \le i \le k-1$ when i is odd (resp. even). It is easy to see that w^i is a black (resp. white) vertex in $Q_{n-1}^{k,i}$ for $0 \le i \le k-1$ when i is odd (resp. even). By the induction hypothesis, there exist m vertex disjoint paths $\{R_p^i\}_{p=0}^{m-1}$ of $Q_{n-1}^{k,i}$ between w^i and b^i for $0 \le i \le j'$. Let $R_p^i = \langle w^i, x_p^i, U_p^i, y_p^i, b^i\rangle$ for $0 \le p \le m-1$ and $0 \le i \le j'$. According to Lemma 2, a hamiltonian path S covers all vertices of $Q_{n-1}^{k,i}$ for $j'+1 \le i \le k-2$ joining from $w^{j'+1}$ to $b^{j'+1}$. There is a hamiltonian path T of $Q_{n-1}^{k,k-1}$ from w^{k-1} to b^{k-1} by Theorem 2. Hence, we can write $P_p = \langle w = w^0, x_p^0, U_p^0, y_p^0, y_p^1, (U_p^1)^{-1}, x_p^1, x_p^2, U_p^2, \ldots, (U_p^{j'-1})^{-1}, x_p^{j'-1}, x_p^{j'}, U_p^{j'}, y_p^{j'}, b^{j'} = b\rangle$ for $0 \le p \le m-1$, $P_m = \langle w = w^0, w^1, w^2, \ldots, w^{j'}, w^{j'+1}, S, b^{j'+1}, b^{j'} = b\rangle$, and $P_{m+1} = \langle w = w^0, w^{k-1}, T, b^{k-1}, b^0, b^1, \ldots, b^{j'-1}, b^{j'} = b\rangle$. Therefore, there are $m+2$ vertex disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of Q_n^k between w and b. Please see Fig. 17 for an illustration.

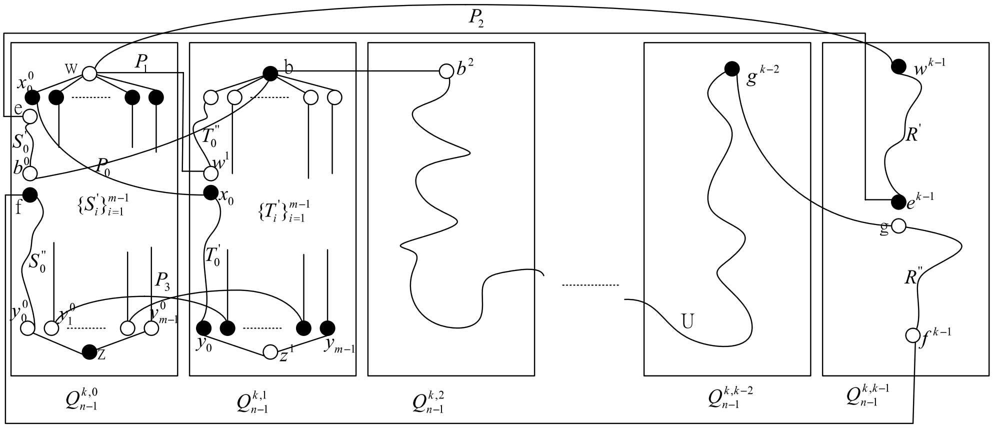


Fig. 16. The illustration for Case 2.2.2 of Theorem 5 when $b^0 \in V(S_0)$.

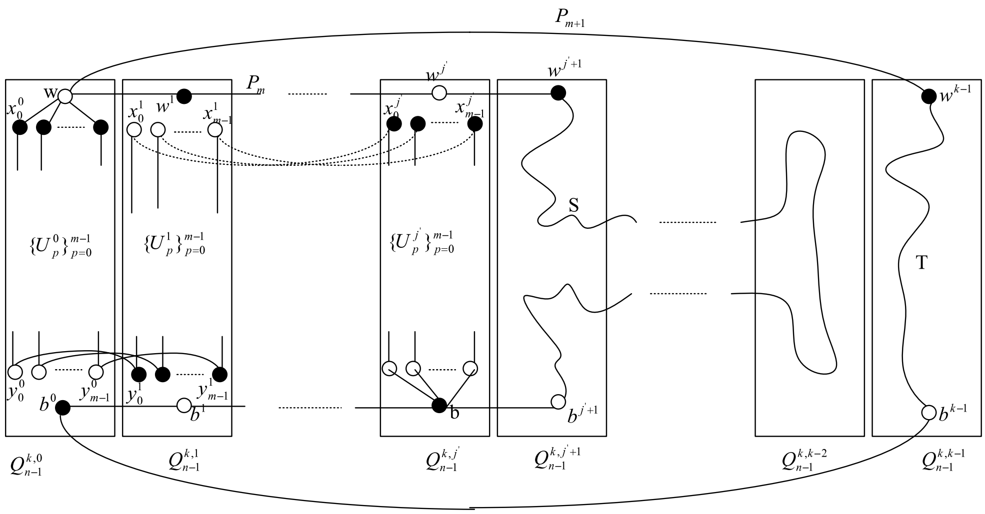


Fig. 17. The illustration for Case 3 of Theorem 5.

Case 4. For $|j-j'| \geq 2$. W.L.O.G., we let $j=0$ and $3 \leq j' \leq \frac{k}{2}+1$ be odd.

Case 4.1. If $m=1$.

Choosing a black vertex z of $Q_{n-1}^{k,0}$, by Theorem 2, there is a hamiltonian path R of $Q_{n-1}^{k,0}$ joining from w to z. In $Q_{n-1}^{k,k-1}$, there exists a hamiltonian path S of $Q_{n-1}^{k,k-1}$ between w^{k-1} and z^{k-1}. We can let $S = \langle w^{k-1}, S', e, b^{k-1}, S'', z^{k-1}\rangle$, where b^{k-1} is a black vertex of $Q_{n-1}^{k,k-1}$, so e is a white vertex of $Q_{n-1}^{k,k-1}$. By Theorem 2, there is a hamiltonian path T of $Q_{n-1}^{k,k-2}$ joining from e^{k-2} to b^{k-2}. Let $T = \langle e^{k-2}, W, f^{k-2}, b^{k-2}\rangle$. In $Q_{n-1}^{k,i}$, we also have a hamiltonian path T^i between e^i and b^i for $j' \leq i \leq k-3$, so we let $T^i = \langle e^i, W^i, f^i, b^i\rangle$. According to Lemma 1, there is a hamiltonian path U between a black vertex $w^1 \in Q_{n-1}^{k,1}$ and a white vertex $b^{j'-1} \in Q_{n-1}^{k,j'-1}$ covering all vertices of $Q_{n-1}^{k,i}$ for $2 \leq i \leq j'-1$. We set $P_0 = \langle w, w^1, U, b^{j'-1}, b\rangle$, $P_1 = \langle w, R, z, z^{k-1}, (S'')^{-1}, b^{k-1}, b^{k-2}, \ldots, b^{j'+1}, b^{j'} = b\rangle$, and $P_2 = \langle w, w^{k-1}, S', e, e^{k-2}, W, f^{k-2}, f^{k-3}, (W^{k-3})^{-1}, e^{k-3}, e^{k-4}, W^{k-4}, f^{k-4}, \ldots, e^{j'+1}, W^{j'+1}, f^{j'+1}, f^{j'}, W^{j'}, b^{j'} = b\rangle$. Hence, there are three vertex disjoint paths $\{P_0, P_1, P_2\}$ whose union covers all vertices of Q_n^k between w and b. Please see Fig. 18 for an illustration.

Case 4.2. If $m \geq 2$.

Given a white vertex z in $Q_{n-1}^{k,j'}$ such that z is adjacent to b. So z^i is a black (resp. white) vertex and w^i is a white (reps. black) vertex of $Q_{n-1}^{k,i}$ if $0 \leq i \leq j'-1$ when i is even (resp. odd). By the induction hypothesis, there exist m vertex disjoint paths $\{R_i\}_{i=0}^{m-1}$ of $Q_{n-1}^{k,0}$ between w and z^0. We write $R_0 = \langle w, x_0(1), x_0(2), \ldots, x_0(\alpha), z^0\rangle$, and $R_p = \langle w, x_p, S_p, y_p, z^0\rangle$ for $1 \leq p \leq m-1$. Again, by the induction hypothesis, there exist m vertex disjoint paths $\{T_p^i\}_{p=0}^{m-1}$ of $Q_{n-1}^{k,i}$ between w^i and z^i for $2 \leq i \leq j'-1$. We let $T_p^i = \langle w^i, x_p^i, U_p^i, t_p^i, z^i\rangle$ for $0 \leq p \leq m-1$ and $2 \leq i \leq j'-1$. Notice that $b^{j'-1}$ is adjacent to $z^{j'-1}$, W.L.O.G., we let $t_{m-1}^{j'-1} = b^{j'-1}$. In $Q_{n-1}^{k,j'}$, there are m vertex disjoint paths $\{W_i\}_{i=0}^{m-1}$ from b to z by the induction

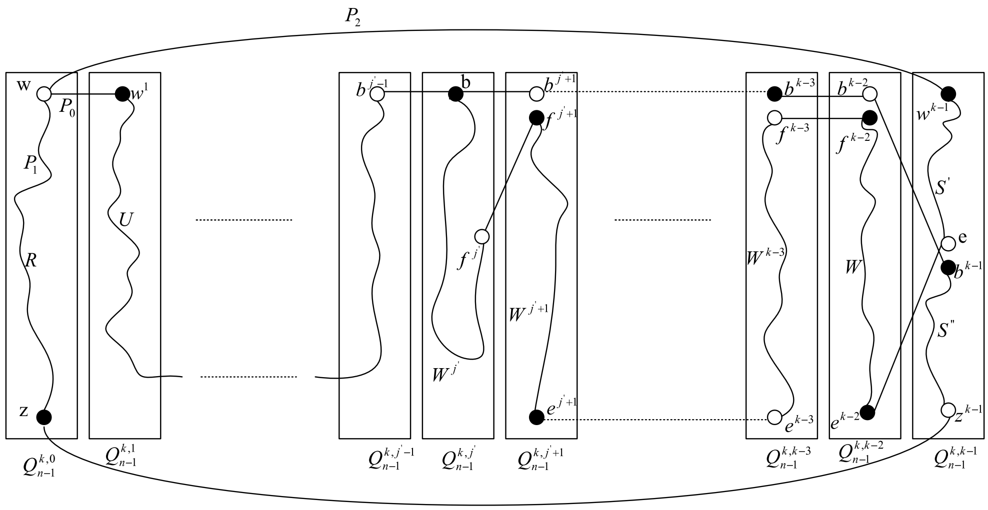


Fig. 18. The illustration for Case 4.1 of Theorem 5.

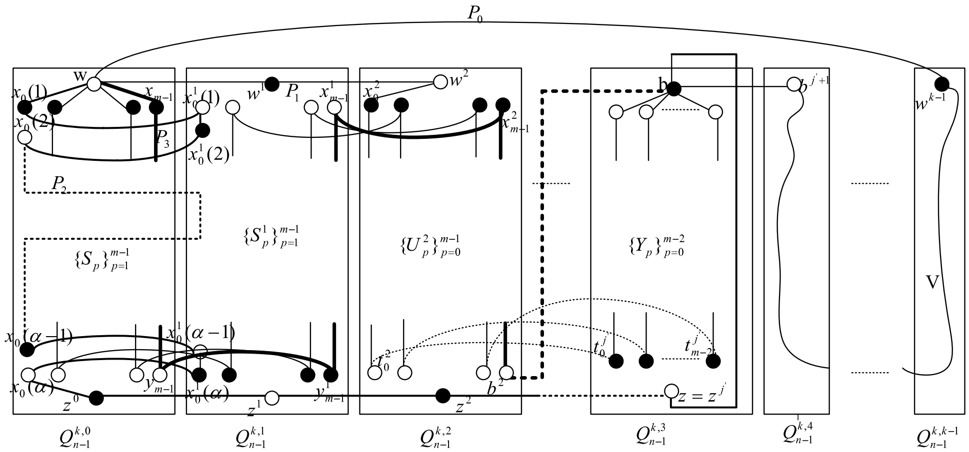


Fig. 19. The illustration for Case 4.2 of Theorem 5.

hypothesis. We can write $W_p = \langle z, t_p^{j'}, Y_p, b\rangle$ for $0 \le p \le m-2$ and $W_{m-1} = \langle z, b\rangle$. According to Lemma 1, there is a hamiltonian path V between $w^{k-1} \in Q_{n-1}^{k,k-1}$ and $b^{j'+1} \in Q_{n-1}^{k,j'+1}$ covering all vertices of $Q_{n-1}^{k,i}$ for $j'+1 \le i \le k-1$. Set $P_0 = \langle w, w^{k-1}, V, b^{j'+1}, b\rangle$, $P_1 = \langle w, w^1, w^2, x_0^2, U_0^2, t_0^2, t_0^3, (U_0^3)^{-1}, x_0^3, w^3, w^4, \ldots, w^{j'-1}, x_0^{j'-1}, U_0^{j'-1}, t_0^{j'-1}, t_0^{j'}, Y_0, b\rangle$, $P_2 = \langle w, x_0(1), x_0^1(1), x_0^1(2), x_0(2), \ldots, x_0(\alpha-1), x_0^1(\alpha-1), x_0^1(\alpha), x_0(\alpha), z^0, z^1, \ldots, z^{j'}, b\rangle$, $P_3 = \langle w, x_{m-1}, S_{m-1}, y_{m-1}, y_{m-1}^1, (S_{m-1}^1)^{-1}, x_{m-1}^1, x_{m-1}^2, U_{m-1}^2, t_{m-1}^2, t_{m-1}^3, (U_{m-1}^3)^{-1}, x_{m-1}^3, \ldots, x_{m-1}^{j'-1}, U_{m-1}^{j'-1}, t_{m-1}^{j'-1} = b^{j'-1}, b\rangle$, and $P_i = \langle w, x_{i-3}, S_{i-3}, y_{i-3}, y_{i-3}^1, (S_{i-3}^1)^{-1}, x_{i-3}^1, x_{i-3}^2, U_{i-3}^2, t_{i-3}^2, t_{i-3}^3, (U_{i-3}^3)^{-1}, x_{i-3}^3, \ldots, x_{i-3}^{j'-1}, U_{i-3}^{j'-1}, t_{i-3}^{j'-1}, t_{i-3}^{j'}, Y_{i-3}, b\rangle$ for $4 \le i \le m+1$. So, there are $m+2$ vertex disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of Q_n^k between w and b. Please see Fig. 19 for an illustration. □

With Theorem 5, we have shown that Q_n^k is bi-m-DPC-able for $1 \le m \le 2n$, where $k \ge 4$ is an even integer and $n \ge 2$ is an integer. The result is optimal since each vertex of Q_n^k has exactly $2n$ neighbors.

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