ADVANCES IN APPLIED MATHEMATICS 7, 435–454

Global Bifurcation and Attractivity of Stationary Solutions of a Degenerate Diffusion Equation

L. A. PELETIER

Department of Mathematics, University of Leiden, The Netherlands

AND

A. TESEI

Dipartimento di Matematica, II University of Rome, Italy

1. INTRODUCTION

We want to study the following problem:

$$\begin{aligned} u_t &= (u^m)_{xx} - a(x)u^p && \text{in } (0,\infty)\times\mathbb{R} \\ u &= u_0 && \text{in } \{0\}\times\mathbb{R}; \end{aligned} \tag{1.1}$$

here $m > 1$, $p \geq 1$, and $m > p$. We shall always consider $u_0 \geq 0$ with compact support; as for a, the following properties will be assumed:

(A_0) $a \in C^1(\mathbb{R})$ is even:

(A_1) a is increasing in $\mathbb{R}_+$; there exists $\alpha > 0$ such that $(x-\alpha)a(x) \geq 0$ for any $x \in \mathbb{R}_+$;

(A_2) there exists $A > \alpha$ such that $\sup_{x \geq A}(a^{-1/2})'(x) < \infty$.

The problem (1.1) was proposed in [6] as a model of mathematical population dynamics (see also [4]). The existence of an equilibrium solution, which exhibits a free boundary, was proved in [10]. The uniqueness of such solutions, whose support is connected, was proved in [11]. Both the existence and the uniqueness proofs hold in any space dimension.

In this paper we give a complete description of the set of stationary solutions—as well as of their attractivity properties—both for the Cauchy problem (1.1) and for the Dirichlet or Neumann initial-boundary value

0196-8858/86 $7.50

problems (see Sect. 4).[1] For this purpose we investigate the problem

$$y'' - a(x)y^{\beta} = 0 \qquad \text{in } \mathbb{R}_+$$
$$y(0) = \gamma, y'(0) = 0, \tag{1.2}$$

where $\gamma > 0$, $\beta \in (0,1)$ and $a \in C^1(\mathbb{R}_+)$ satisfies (A_1)–(A_2). Only classical nonnegative solutions of (1.2) will be considered. Problem (1.2) was investigated in [8] in the case $a(x) \geq 0$ $(x \geq 0)$.

Concerning the problem (1.2), the overall situation can be depicted as follows. Solutions of (1.2) are denoted by $y = y(x) = y(x, \gamma)$ $(x > 0; \gamma > 0)$.

THEOREM 1.1. *Let $a \in C^1(\mathbb{R}_+)$ satisfy (A_1)–(A_2). Then there exists a unique $\gamma^* > 0$ with the following properties:*

(i) *for any $\gamma \in (0, \gamma^*)$ there exists $\xi = \xi(\gamma)$ such that $y(\xi) = 0$, $y'(\xi) < 0$;*

(ii) *for $\gamma = \gamma^*$ there exists $\xi^* = \xi(\gamma^*)$ such that $y(\xi^*) = y'(\xi^*) = 0$;*

(iii) *for any $\gamma > \gamma^*$ there exists $\xi = \xi(\gamma)$ such that $y(\xi) > 0$, $y'(\xi) = 0$.*

In all cases $y(x) > 0$, $y'(x) < 0$ for any $x \in (0, \xi)$.

THEOREM 1.2. *Let $a \in C^1(\mathbb{R}_+)$ satisfy (A_1)–(A_2). Then $0 < \gamma_1 < \gamma_2$ implies $y(x, \gamma_1) < y(x, \gamma_2)$ for any $x \in [0, \min\{\xi(\gamma_1), \xi(\gamma_2)\}]$.*

We shall think of the graph of $\xi = \xi(\gamma)$ $(\gamma > 0)$ as a bifurcation diagram of solutions of (1.2). Define $\alpha^* > 0$ as follows:

$$\int_0^{\alpha^*} a(x)\,dx = 0; \tag{1.3}$$

then the following holds.

THEOREM 1.3. *Let $a \in C^1(\mathbb{R}_+)$ satisfy (A_1)–(A_2). Then*

(i) *$\xi \in C((0, \gamma^*) \cup (\gamma^*, \infty))$;*

(ii) *ξ is strictly increasing in $(0, (\gamma^*)$ and*

$$\xi(\gamma) = O(\gamma^{(1-\beta)/2}) \qquad \text{as } \gamma \to 0;$$

(iii) *ξ is strictly decreasing in (γ^*, ∞) and*

$$\xi(\gamma) - \alpha^* = O(\gamma^{\beta-1}) \qquad \text{as } \gamma \to \infty. \tag{1.4}$$

The above results are depicted in Fig. 1. It is interesting to remark that ξ has a very singular behavior near γ^*; as a detailed analysis shows, its graph exhibits a cusp at $\gamma = \gamma^*$, whose "width" depends on the ratio p/m (see Sect. 3). The results of Theorem 1.1 make the following definition sensible.

[1] Concerning the set of equilibrium solutions, a first result was recently given by Aronson [0], who considered piecewise constant functions a.

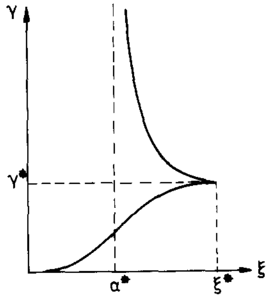

FIGURE 1

DEFINITION 1.1. The solution $y(\cdot, \gamma^*)$ of problem (1.2) is called a free boundary solution. By a Dirichlet (resp. Neumann) solution of (1.2) we mean any solution $y(\cdot, \gamma)$ such that $\gamma \in (0, \gamma^*)$ ($\gamma > \gamma^*$, respectively).

It is now clear how solutions of (1.2) can be put in one-to-one correspondence with (even) stationary solutions of the Cauchy problem (1.1)—or, respectively, of the Dirichlet and Neumann initial-boundary value problems in $(-L, L)$ (see Sect. 4). In particular, only (even) stationary solutions of the Dirichlet problem exist, whose support has a measure less than $2\xi^*$. Similarly, the measure of the support for (even) stationary solutions of the Neumann problem lies in the interval $(2\alpha^*, 2\xi^*)$. Concerning the attractivity properties of stationary solutions, optimal results are proved in Section 4.

The above results enable us to deal with the case where $\operatorname{supp} a_-$ is disconnected (here $a_- := \min\{a, 0\}$); the positivity of stationary solutions, versus the appearance of a "dead core," is discussed.

The proofs make use of shooting techniques and similarity properties of the problem. We mention that analogous results may be worked out for a's without symmetry properties, as well as for higher space dimension.

2. STATIONARY SOLUTIONS

In order to prove Theorem 1.1, some preliminary lemmas are needed.

LEMMA 2.1. *Let $a \in C^1(\mathbb{R}_+)$ satisfy* (A_1). *Then, for any $x > 0$*

$$y(x) > \gamma\left(1 - \frac{kx}{\gamma^{(1-\beta)/2}}\right); \tag{2.1}$$

here $k := \sqrt{2|a(0)|/(\beta + 1)}$.

Proof. Due to (A_1), the quantity

$$E(x) := (y')^2(x) - \frac{2}{\beta+1}a(x)y^{\beta+1}(x)$$

is decreasing in $(0, \infty)$. For any $x \in (0, \alpha)$, the inequality

$$E(x) < E(0)$$

implies

$$(y')^2(x) < k^2\gamma^{\beta+1},$$

whence

$$y'(x) \geq -k\gamma^{(\beta+1)/2} \qquad (x \in (0, \alpha)). \tag{2.2}$$

Since $y''(x) \geq 0$ for any $x \in [\alpha, \infty)$, the inequality (2.2) holds for any $x > 0$; then the result follows.

Denote by $\mathscr{I}(\gamma)$ the maximal interval of existence of solutions to (1.2); moreover, set

$$\mathscr{I}_+(\gamma) := \{x \in \mathscr{I}(\gamma) | y(x,\gamma) > 0\} \qquad (\gamma > 0).$$

LEMMA 2.2. *Let $a \in C^1(\mathbb{R}_+)$ satisfy (A_1). Then the set*

$$S_- := \{\gamma > 0 | \exists \xi > 0 \text{ such that } y(x,\gamma) > 0\ \forall x \in [0, \xi), \\ y(\xi, \gamma) = 0,\ y'(\xi,\gamma) < 0\}$$

is nonempty.

Proof. It is clear from (2.1) that the intersection $\mathscr{I}_+(\gamma) \cap [0, \alpha/2)$ $(\gamma > 0)$ is nonempty. For any x in this intersection and any $\gamma > 0$, we get from (1.2),

$$(y')^2(x) = -2\int_0^x |a(t)| y^\beta(t) y'(t)\, dt \geq -2|a(\alpha/2)| \int_\gamma^{y(x)} u^\beta\, du,$$

having used (A_1). It follows that

$$y'(x) \leq -\sqrt{2|a(\alpha/2)|/(\beta+1)}\, \{\gamma^{\beta+1} - y^{\beta+1}(x)\}^{1/2} \\ (x \in \mathscr{I}_+(\gamma) \cap (0, \alpha/2)),$$

which entails

$$x \leq \gamma^{(1-\beta)/2}\sqrt{(\beta+1)/2|a(\alpha/2)|} \int_0^1 du/\sqrt{1-u^{\beta+1}} \\ (x \in \mathscr{I}_+(\gamma) \cap (0, \alpha/2)). \tag{2.3}$$

Now define

$$\gamma_0 := \left\{ \frac{2}{\alpha} \sqrt{(\beta+1)/2|a(\alpha/2)|} \int_0^1 du \sqrt{1-u^{\beta+1}} \right\}^{-2/(1-\beta)}$$

For any $\gamma \in (0, \gamma_0)$, it follows from (2.3) that $\mathscr{I}_+(\gamma) \subset (0, \alpha/2)$. Due to the above estimates, $\xi = \xi(\gamma) \in (0, \alpha/2)$ $(\gamma \in (0, \gamma_0))$ with the required properties is easily seen to exist. This completes the proof.

LEMMA 2.3. *Let $a \in C^1(\mathbb{R}_+)$ satisfy* (A_1). *Then the set*

$$S_+ := \{\gamma > 0 | \exists \xi > 0 \text{ such that } y(x,\gamma) > 0 \ \forall x \in [0, \xi], \\ y'(x,\gamma) < 0 \ \forall x \in (0, \xi),\ y'(\xi, \gamma) = 0\}$$

is nonempty.

Proof. Set $x_0 := \alpha^* + 1$, $\gamma_1 := (kx_0)^{2/(1-\beta)}$ (k being defined in Lemma 2.1). Were S_+ empty, we would have $y'(x, \gamma) < 0$ for any $x > 0$, $\gamma > \gamma_1$; in particular, this implies

$$\int_0^{x_0} a(x) y^\beta(x)\, dx < 0. \tag{2.4}$$

From (2.4) we get easily

$$\gamma^\beta \int_0^\alpha |a(x)| dx > \int_\alpha^{x_0} |a(x)| y^\beta(x)\, dx. \tag{2.5}$$

On the other hand, Lemma 2.1 implies the inequality

$$\int_\alpha^{x_0} |a(x)| y^\beta(x) dx > \gamma^\beta \int_\alpha^{x_0} |a(x)| \left(1 - \frac{kx}{\gamma^{(1-\beta)/2}}\right)^\beta dx; \tag{2.6}$$

observe that the integrand in the right-hand side is positive for any $x \in (0, x_0)$ and $\gamma > \gamma_1$.

It is easy to see that for any $\varepsilon > 0$ there exists $\tilde{\gamma} = \tilde{\gamma}(\varepsilon) > 0$ such that $\gamma > \tilde{\gamma}$ implies

$$\int_\alpha^{x_0} |a(x)| \left(1 - \frac{kx}{\gamma^{(1-\beta)/2}}\right)^\beta dx > \int_\alpha^{x_0} |a(x)| dx - \varepsilon. \tag{2.7}$$

Now (2.5)–(2.7) imply

$$\int_{\alpha^*}^{x_0} |a(x)| dx < \varepsilon;$$

from this contradiction the result follows.

The subsequent result follows from [11]; we give the proof in a form suitable for the present purposes.

LEMMA 2.4. *Let $a \in C^1(\mathbb{R}_+)$ satisfy* (A_1)–(A_2). *Then there is at most one $\gamma^* > 0$ with the following property: there exists $\xi^* > 0$ such that $y(\xi^*) = y'(\xi^*) = 0$.*

Proof. By contradiction, suppose there exist $0 < \gamma_1^* < \gamma_2^*$ with the above property; set $u := y(\cdot, \gamma_1^*)$, $v := y(\cdot, \gamma_2^*)$, $\xi_i^* := \xi(\gamma_i^*)$ $(i = 1, 2)$.

(a) Let us first prove that $0 < \xi_1^* < \xi_2^*$ and $u(x) < v(x)$ for any $x \in [0, \xi_1^*]$. Suppose to the contrary that $x_0 \in (0, \min\{\xi_1^*, \xi_2^*\}]$ exists, such that

$$u(x) < v(x) \qquad \text{for any } x \in [0, x_0),\ u(x_0) = v(x_0).$$

For any $\lambda > 0$ define

$$u_\lambda(x) := \lambda^{-2/(1-\beta)} u(\lambda x) \qquad (x \in [0, \xi_1^*/\lambda]); \tag{2.8}$$

it is easily seen that u_λ satisfies the equation

$$u_\lambda'' - a(\lambda x) u_\lambda^\beta = 0. \tag{2.9}$$

Moreover, according to (2.8), we can choose $\lambda \in (0,1)$ so small that $u_\lambda(x) > v(x)$ for any $x \in [0, x_0]$. Define

$$\bar{\lambda} := \sup\{\lambda > 0 \,|\, u_\lambda(x) > v(x)\ \forall x \in [0, x_0]\};$$

then there exists $\bar{x} \in [0, x_0)$ such that

$$u_{\bar{\lambda}}(\bar{x}) = v(\bar{x}). \tag{2.10}$$

Actually, for any $\lambda \in (0,1)$, the fact that u is decreasing and definition (2.8) imply

$$u_\lambda(x) > u(\lambda x) > u(x) \qquad (x \in [0, \xi_1^*]). \tag{2.11}$$

Due to (2.11), the equality

$$u_{\bar{\lambda}}(x_0) = v(x_0) = u(x_0)$$

would imply $\bar{\lambda} = 1$, which contradicts the definition of $\bar{\lambda}$.

Since $\bar{\lambda} \in (0,1)$ and a is increasing, from (2.9)–(2.10) we get

$$(u_\lambda - v)''(\bar{x}) = [a(\bar{\lambda}\bar{x}) - a(\bar{x})] v^\beta(\bar{x}) < 0; \tag{2.12}$$

the contradiction proves the claim.

(b) According to (a), there exists $\lambda \in (0,1)$ such that supp $u_\lambda \supset [0, \xi_2^*]$ and $u_\lambda(x) > v(x)$ for any $x \in [0, \xi_2^*]$. Define

$$\tilde{\lambda} := \sup\{\lambda > 0 | u_\lambda(x) > v(x)\ \forall x \in [0, \xi_2^*]\};$$

then either

(i) $u_{\tilde{\lambda}}(\tilde{x}) = v(\tilde{x})$ for some $\tilde{x} \in [0, \xi_2^*)$, or

(ii) $u_{\tilde{\lambda}}(\xi_2^*) = \mathrm{v}(\xi_2^*)$.

In case (i) a contradiction is reached as in (2.12). In case (ii) we get

$$\begin{aligned}\left(u_{\tilde{\lambda}}^{(1-\beta)/2}\right)'(\xi_2^*) &= \left(u^{(1-\beta)/2}\right)'(\tilde{\lambda}\xi_2^*)\\ &= -(1-\beta)/\sqrt{2(1+\beta)}\sqrt{a(\tilde{\lambda}\xi_2^*)}\\ &> -(1-\beta)/\sqrt{2(1+\beta)}\sqrt{a(\xi_2^*)}\\ &= \left(v^{(1-\beta)/2}\right)'(\xi_2^*),\end{aligned} \tag{2.13}$$

since $\tilde{\lambda} > 1$ and a is increasing (here use of Proposition 2.4 in [8] has been made). The inequality (2.13) contradicts again the definition of $\tilde{\lambda}$; this completes the proof.

Let us now prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. According to Lemmas 2.2 and 2.3, the sets S_-, S_+ are nonempty and relatively open. It is easily seen that the intersection $S_- \cap S_+$ is empty. Then there exists $\gamma^* > 0$ such that either claim (ii) is true, or the corresponding solution $y(\cdot, \gamma^*)$ of (1.2) satisfies

$$y(x,\gamma^*) > 0 \quad \text{for any } x \in [0,\infty)$$

$$y'(x,\gamma^*) < 0 \quad \text{for any } x \in (0,\infty).$$

Moreover, the second possibility is ruled out by assumption (A_2) (see Theorem 2.5 in [8]).

This proves the existence of $\gamma^* > 0$ such that (ii) holds; its uniqueness follows by Lemma 2.4. The other claims are an immediate consequence of Lemmas 2.2, 2.3. This completes the proof.

Proof of Theorem 1.2. It follows from Theorem 1.1 that any solution of (1.2) is decreasing in $(0, \xi(\gamma))$ $(\gamma > 0)$. Then the conclusion follows by the argument given in the proof of Lemma 2.4, part (a).

3. The Function ξ

Let $y = y(\cdot, \gamma)$ denote the solution of problem (1.2) $(\gamma > 0)$. As it has been shown in the previous section, for one value $\gamma^* > 0$ there exists

$\xi^* \in \mathbb{R}_+$ such that

(i) $y > 0$, $y' < 0$ in $(0, \xi^*)$,

(ii) $y(x, \gamma^*) \to 0$, $y'(x, \gamma^*) \to 0$ as $x \to \xi^*$.

In addition, a function $\xi:[0, \infty) \to [0, \xi^*]$ was defined such that

(i′) for $\gamma \in (0, \gamma^*)$,

$$\xi(\gamma) = \sup\{x \in \mathbb{R}_+ | y(x, \gamma) > 0\};$$

(ii′) for $\gamma > \gamma^*$,

$$\xi(\gamma) = \sup\{x \in \mathbb{R}_+ | y'(x, \gamma) < 0\};$$

moreover, $\xi(\gamma^*) = \xi^*$.

In this section we shall establish a number of qualitative properties of this function ξ. To start with, let us prove Theorem 1.3.

Proof of Theorem 1.3. (i) Let $\gamma \in (0, \gamma^*)$. Integrating twice the differential equation in (1.2) we obtain

$$\varphi(\gamma, \xi) := \gamma + \int_0^\xi (\xi - x) a(x) y^\beta(x, \gamma)\, dx = 0.$$

Observe that

$$\varphi_\xi(\gamma, \xi) = \int_0^\xi a(x) y^\beta(x, \gamma)\, dx = y'(\xi, \gamma) < 0,$$

due to Theorem 1.1. Since $y(\cdot, \gamma)$ depends continuously on γ, it follows from the implicit function theorem that ξ depends continuously on γ near every $\gamma \in (0, \gamma^*)$.

Not let $\gamma > \gamma^*$. Integrating over $(0, \xi)$ yields

$$\psi(\gamma, \xi) := \int_0^\xi a(x) y^\beta(x, \gamma)\, dx = 0. \tag{3.1}$$

Therefore $\xi > \alpha^* > \alpha$. In fact, because y is decreasing in $(0, \xi)$, from (3.1) we obtain

$$\begin{aligned} y^\beta(\alpha) \int_0^\alpha |a(x)| dx &< \int_0^\alpha |a(x)| y^\beta(x)\, dx \\ &= \int_\alpha^\xi a(x) y^\beta(x)\, dx < y^\beta(\alpha) \int_\alpha^\xi a(x)\, dx. \end{aligned}$$

Thus

$$\int_0^\xi a(x)\, dx > 0,$$

which proves the claim. It follows that

$$\psi_\xi(\gamma, \xi) = a(\xi) y^\beta(\xi, \gamma) > 0; \tag{3.2}$$

the implicit function theorem now implies $\xi \in C(\gamma^*, \infty)$.

(ii) is an immediate consequence of the monotonicity of $y(\cdot, \gamma)$ with respect to γ (see Theorem 1.2) and of inequality (2.3).

(iii) Recall that $\xi > \alpha^*$ for any $\gamma > \gamma^*$ (see (i) above). To show that $\xi \to \alpha^*$ as $\gamma \to \infty$, it is convenient to write $z := y/\gamma$. Then

$$\begin{aligned} 0 &= -\int_0^\alpha |a(x)| z^\beta(x)\, dx + \int_\alpha^\xi |a(x)| z^\beta(x)\, dx \\ &> -\int_0^\alpha |a(x)| dx + \int_\alpha^\xi |a(x)| z^\beta(x)\, dx. \end{aligned}$$

Due to inequality (2.1), for any $\varepsilon > 0$ and γ sufficiently large we have

$$\int_\alpha^\xi |a(x)| z^\beta(x)\, dx > \int_\alpha^\xi |a(x)| dx - \varepsilon.$$

This in turn implies

$$\int_0^\xi a(x)\, dx = \int_{\alpha^*}^\xi a(x)\, dx < \varepsilon,$$

which proves the claim.

Now fix any $\gamma_1 > \gamma^*$. Suppose that $\xi(\gamma) > \xi(\gamma_1)$ for some $\gamma > \gamma_1$. Since ξ is continuous on (γ^*, ∞) and $\xi \to \alpha^*$ as $\gamma \to \infty$, there would exist $\gamma_2 > \gamma_1$ such that $\xi(\gamma_2) = \xi(\gamma_1)$. The corresponding solutions $y(\cdot, \gamma_1)$ and $y(\cdot, \gamma_2)$ of (1.2) would satisfy the problem

$$\begin{gathered} y'' - a(x) y^\beta = 0 \qquad \text{in } (0, \xi) \\ y'(0) = y'(\xi) = 0, \end{gathered}$$

there $\xi = \xi(\gamma_1) = \xi(\gamma_2)$; moreover, $y(x, \gamma_1) < y(x, \gamma_2)$ for any $x \in [0, \xi]$ by Theorem 1.2. Due to the maximum principle proved in [11, Theorem 1.1], this situation cannot arise; hence ξ is strictly decreasing in (γ^*, ∞).

Let us now prove the estimate (1.4). Integrating the equation in z twice we obtain

$$z(x) = 1 + \gamma^{\beta-1} \int_0^x (x - t) a(t) z^\beta(t)\, dt \geq 1 - \gamma^{\beta-1} \xi^* \int_0^{\xi^*} |a(t)|\, dt,$$

where we have used the fact that $\xi < \xi^*$. Thus

$$\left|\int_0^{\alpha^*} a(x) z^\beta(x)\, dx\right| = \left|\int_0^{\alpha^*} a(x)\{z^\beta(x) - 1\}\, dx\right| \le C\gamma^{\beta-1}$$

for some constant $C > 0$. On the other hand, as $\gamma \to \infty$

$$\int_{\alpha^*}^{\xi} a(x) z^\beta(x)\, dx \to \int_{\alpha^*}^{\xi} a(x)\, dx \ge a(\alpha^*)(\xi - \alpha^*),$$

where $a(\alpha^*) > a(\alpha) = 0$. Thus the equality

$$\int_0^{\alpha^*} a(x) z^\beta(x)\, dx + \int_{\alpha^*}^{\xi} a(x) z^\beta(x)\, dx = 0$$

implies

$$\xi(\gamma) - \alpha^* < C^*\gamma^{\beta-1}$$

for some constant $C^* > 0$. This completes the proof.

Theorem 1.3 gives a description of the function $\xi(\gamma)$ away from the point $\gamma = \gamma^*$. Numerical evidence suggests that the graph of ξ has a cusp there. In the remainder of this section we shall show that this is indeed so, and we shall give estimates for $\xi(\gamma)$ as $\gamma \nearrow \gamma^*$ and $\gamma \searrow \gamma^*$. For convenience we shall assume that for some $\delta > 0$,

$$a(x) \equiv \bar{a} > 0 \qquad \text{when } \xi^* - \delta \le x \le \xi^*.$$

We shall proceed in two steps. In the next lemma we shall translate changes in y at $x = 0$ to changes in y and y' at $x = \xi^* - \delta$. Thereafter we translate these changes into changes in ξ.

LEMMA 3.1. *There exist positive constants k_1 and k_2 such that*

$$|y(\xi^* - \delta, \gamma^* + \varepsilon) - y(\xi^* - \delta, \gamma^*)| \le k_1|\varepsilon|.$$
$$|y'(\xi^* - \delta, \gamma^* + \varepsilon) - y'(\xi^* - \delta, \gamma^*)| \le k_2|\varepsilon|,$$

Proof. Write $y_\varepsilon = y(\cdot, \gamma^* + \varepsilon)$ and $y_0 = y(\cdot, \gamma^*)$. Then, since $y_0(x)$ is bounded away from zero on $[0, \xi^* - \delta]$, $h = \partial y/\partial\gamma$ exists at $\gamma = \gamma^*$:

$$y_\varepsilon(x) = y_0(x) + \varepsilon h(x) + w(x, \varepsilon),$$

where $\|h\|$ is bounded and $\|w(\cdot, \varepsilon)\| = o(\varepsilon)$ as $\varepsilon \to 0$ ($\|\cdot\|$ denotes the supremum norm on $(0, \xi^* - \delta)$). This yields the first estimate.

If we integrate the equation for y_ε and for y_0 over $(0, \xi^* - \delta)$ and subtract, we obtain

$$|y_\varepsilon'(\xi^* - \delta) - y_0'(\xi^* - \delta)| = |\int_0^{\xi^* - \delta} a(x)\{y_\varepsilon^\beta(x) - y_0^\beta(x)\}\,dx|$$
$$\leq 2\varepsilon\beta y_0^{\beta-1}(\xi^* - \delta)\int_0^{\xi^*}|a(x)|\,|h(x)|dx,$$

which yields the second estimate. This completes the proof.

To study $y(x, \gamma)$ for $x > \xi^* - \delta$ we transform to the variable $u = y^{2/(1-\beta)}$. This yields the equation

$$uu'' + \frac{1+\beta}{1-\beta}u'^2 = \frac{1-\beta}{2}a(x) \qquad (x > \xi^* - \delta).$$

We write this as a first-order system

$$u' = v$$
$$uv' = -\frac{1+\beta}{1-\beta}v^2 + \frac{1-\beta}{2}a(x),$$

or, when we introduce the new independent variable t by means of the transformation $dt/dx = 1/u$, as

$$u' = uv$$
$$v' = -\frac{1+\beta}{1-\beta}v^2 + \frac{1-\beta}{2}a(x)$$
$$x' = u, \tag{3.3}$$

where the primes now denote differentiation with respect to t (see also [7]).

The set of critical points of (3.3) near $x = \xi^*$ consists of the line segments

$$l_\pm = \{(0, \pm v^*, x)|\xi^* - \delta \leq x \leq \xi^*\},$$

where

$$v^* = (1-\beta)/\sqrt{2(1+\beta)}\sqrt{a}\,.$$

The orbit entering the point

$$P^* = (0, -v^*, \xi^*)$$

is given by the line

$$l^* = \{(v^*(\xi^* - x), -v^*, x)|\xi^* - \delta \leq x \leq \xi^*\}.$$

It passes through the plane $\{x = \xi^* - \delta\}$ at the point

$$P_0 = (v^*\delta, -v^*, \xi^* - \delta).$$

Let $(u(t), v(t), x(t))$ be an orbit which starts near the point $((\gamma^*)^{2/(1-\beta)}, 0, 0)$. By Lemma 3.1 it passes through the plane $\{x = \xi^* - \delta\}$ at a point P_ε near P_0 which we write as

$$P_\varepsilon = (v^*\delta(1 + p\varepsilon), -v^*(1 - q\varepsilon), \xi^* - \delta);$$

here $\varepsilon = \gamma - \gamma^*$ and p and q are constants. Taking P_ε as initial value, the equation for v in (3.3) yields

$$v(t) = -v^* \frac{1 - ke^{\lambda t}}{1 + ke^{\lambda t}},$$

where

$$k := \frac{q\varepsilon}{2 - q\varepsilon}, \qquad \lambda := \mu v^*, \qquad \mu := 2\frac{1 + \beta}{1 - \beta}.$$

It follows that

(i) if $q\varepsilon > 0$, then $v(t) \to 0$ as $t \to T$, where

$$1 - ke^{\lambda T} = 0 \qquad \text{or} \qquad T = \frac{1}{\lambda}\log\left(\frac{2 - q\varepsilon}{q\varepsilon}\right);$$

(ii) if $q\varepsilon < 0$, then $v(t) \to -\infty$ as $t \to T$, where

$$1 - |k|e^{\lambda T} = 0 \qquad \text{or} \qquad T = \frac{1}{\lambda}\log\left(\frac{2 + |q\varepsilon|}{|q\varepsilon|}\right).$$

Recall that if $\varepsilon > 0$, $v(t)$ vanishes after finite time, whence q must be a *positive* constant.

Next we use the first equation of (3.3) to deduce $u(t)$. We obtain

$$\log\frac{u(t)}{u(0)} = \int_0^t v(s)\,ds = -v^*t + \frac{2}{\mu}\log\left(\frac{1 + ke^{\mu t}}{1 + k}\right),$$

whence

$$u(t) = u(0)\left(1 - \tfrac{1}{2}q\varepsilon\right)^{2/\mu} e^{-v^*t}\left(1 + ke^{\lambda t}\right)^{2/\mu}.$$

Finally, we use the equation $x' = u$ to obtain $x(T)$:

$$x(T) = x(0) + \int_0^T u(t)\,dt = x(0) + \frac{u(0)}{v^*}\left(1 - \frac{1}{2}q\varepsilon\right)^{2/\mu} \cdot I(k),$$

where

$$I(k) := \nu(k)\int_0^{1/\nu(k)} (1 + (k/|k|), t^{-\mu})^{2/\mu}\,dt, \qquad \nu(k) := |k|^{1/\mu}.$$

Observe that

$$\begin{aligned} I_{\pm} &= \nu(1/\nu - 1) + \nu\int_1^{1/\nu}\left\{(1 \pm t^{-\mu})^{2/\mu} - 1\right\}dt \\ &= 1 - \nu + \nu\int_1^{\infty}\left\{(1 \pm t^{-\mu})^{2/\mu} - 1\right\}dt + O(\nu^{\mu}) \end{aligned}$$

as $\nu \to 0$ because $\mu > 2$. Writing

$$I_{\pm}(\mu) := \int_1^{\infty}\left\{(1 \pm t^{-\mu})^{2/\mu} - 1\right\}dt, \tag{3.4}$$

we then obtain

$$\begin{aligned} x(T) &= x(0) + (u(0)/v^*)\left(1 - \tfrac{1}{2}q\varepsilon\right)^{2/\mu} \\ &\quad \times\left\{1 - [1 - I_{\pm}(\mu)]|k|^{1/\mu} + O(|k|)\right\}. \end{aligned}$$

Substituting $x(0)$ and $u(0)$ gives

$$\xi^* - x(T) = \delta q^{1/\mu}[1 - I_{\pm}(\mu)]|\varepsilon|^{1/\mu} + O(\varepsilon) \qquad \text{as } \varepsilon \to 0,$$

where the + sign applies if $\varepsilon > 0$ and the − sign if $\varepsilon < 0$.

Finally, in the following proposition we list a number of properties of the integrals I_+ and I_- defined in (3.4).

PROPOSITION 3.1. (i) $I_{\pm} \in C^1([2, \infty))$;

(ii) $I'_+ < 0$ and $I'_- > 0$ on $[2, \infty)$;

(iii) $I_{\pm}(2) = \pm 1$; $I_{\pm}(\infty) = 0$.

The proofs of (i)–(iii) are all elementary, and we shall omit them. Thus, in summary we have proved the behavior of $\xi(\gamma)$ near γ^* with the following result:

THEOREM 3.1. *Let* (A_1)–(A_2) *be satisfied. Then there exists a constant k such that*

$$\xi^* - \xi(\gamma) = KA_{\pm}(\beta)|\gamma - \gamma^*|^{(1-\beta)/2(1+\beta)} + O(|\gamma - \gamma^*|)$$

as $\gamma \to \gamma^*$. Here the + sign applies when $\gamma > \gamma^*$ and the − sign when $\gamma < \gamma^*$; the constants $A_\pm(\beta)$ have the properties

$$0 < A_+(\beta) < 1 < A_-(\beta) < 2;$$
$$A_\pm(\beta) \to 1 \qquad \text{if } \beta \to 1;$$

and

$$\left.\begin{array}{l} A_+(\beta) \to 0 \\ A_-(\beta) \to 2 \end{array}\right\} \text{ if } \beta \to 0.$$

Proof. The expression for $\xi^* - \xi(\gamma)$ follows from the derivation given above. The properties of the constants $A_\pm(\beta)$ follow from the expression

$$A_\pm(\beta) = 1 - I_\pm(\mu(\beta)),$$

due to Proposition 3.1.

The expression for $\xi^* - \xi(\gamma)$ given in Theorem 3.1 indeed implies that the graph of $\xi(\gamma)$ has a cusp at $\gamma = \gamma^*$.

Notice that, because $1/A_+$ varies from 1 to ∞ and $1/A_-$ from 1 to $\frac{1}{2}$ as β goes from 0 to 1, the top of the cusp is much more sensitive to changes of β than the bottom (see Fig. 2).

4. Attractivity Results

Let us investigate the set of stationary solutions, both for (1.1) and for the Dirichlet or Neumann initial-boundary value problems. By a solution of (1.1) we mean any function $u:(0, \infty) \times \mathbb{R} \to \overline{\mathbb{R}}_+$ such that:

(i) u is bounded and continuous in $[0, T] \times \mathbb{R}$ for any $T > 0$;

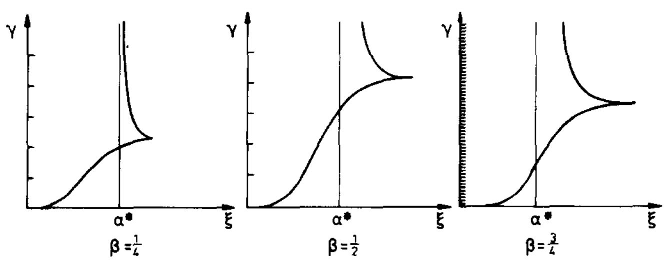

FIG. 2. Bifurcation diagram: $a(x) = 25[-1 + 2H(x - \frac{1}{4})]$.

(ii) for any $t \in (0, T]$ and any interval $(-L, L) \subset \mathbb{R}$ the following holds:

$$\int_{-L}^{L} u(t)\eta(t) - \int_0^t \int_{-L}^{L} [u\eta_t + u^m \eta_{xx}]$$
$$+ \int_0^t [u^m(\cdot, L)\eta_x(\cdot, L) - u^m(\cdot, -L)\eta_x(\cdot, -L)]$$
$$= \int_{-L}^{L} u_0 \eta(0) + \int_0^t \int_{-L}^{L} a(x) u^p \eta \tag{4.1}$$

for any $\eta \in C^{1,2}(\overline{Q}_T)$ such that $\eta \geq 0, \eta = 0$ on Σ_T (here $Q_T := (0, T] \times (-L, L)$, $\Sigma_T := (0, T] \times \{-L, L\}$, $L > 0$). Supersolutions and subsolutions of (1.1) are defined replacing in (4.1) equality by $\leq$ (resp. $\geq$).

A similar definition holds for stationary solutions of (1.1), namely for solutions of the problem

$$(u^m)'' - a(x)u^p = 0 \qquad (x \in \mathbb{R}). \tag{4.2}$$

We shall prove the following result.

THEOREM 4.1. *Let* (A_0)–(A_2) *be satisfied. Then there exists a unique nontrivial stationary solution* $u^* \geq 0$ *of problem* (1.1).

Proof. According to Theorem 1.1, there exists a unique free boundary solution $y(\cdot, \gamma^*)$ of problem (1.2) with $\beta = p/m$. Define

$$u^*(x) := \begin{cases} y^{1/m}(|x|, \gamma^*) & |x| \leq \xi^* \\ 0 & |x| > \xi^* \end{cases}. \tag{4.3}$$

Then, since a is even, u^* is a stationary solution and it is unique among the even ones. But any stationary solution must necessarily be even in the present case; therefore u^* is the only stationary solution of (1.1).

Concerning the attractivity of u^*, the following result holds.

THEOREM 4.2. *Let* (A_0)–(A_2) *be satisfied. Then the stationary solution* u^* *attracts in the supremum norm any solution of* (1.1) *such that* $\operatorname{supp} u_0 \cap (-\alpha, \alpha)$ *is nonempty.*

Proof. For any $\lambda \in (0,1)$ define

$$u_\lambda^*(x) := \lambda^{-2/(m-p)} u^*(\lambda x) \qquad (x \in \mathbb{R});$$

observe that

$$\operatorname{supp} u_\lambda^* = (1/\lambda) \operatorname{supp} u^*.$$

Due to the properties of u^* and a, u_λ^* is easily seen to be a supersolution of (1.1) ($\lambda \in (0,1)$). Now choose λ so small that

$$u_0 \leq u_\lambda^*;$$

it follows from the comparison principle that

$$u(t, u_0) \leq u_\lambda^* \qquad \text{for any } t \geq 0.$$

Consider the Cauchy–Dirichlet problem

$$\begin{aligned} &u_t = (u^m)_{xx} - a(x)u^p && \text{in } (0,\infty) \times \Omega \\ &u = 0 && \text{in } (0,\infty) \times \partial\Omega \\ &u = u_0 && \text{in } \{0\} \times \Omega, \end{aligned} \tag{4.4}$$

where $\Omega = \Omega(\lambda) = (-\xi^*/\lambda, \xi^*/\lambda)$. Due to the above remarks, the solution $u(t, u_0)$ of the Cauchy problem (1.1) is also a solution of (4.4). Clearly, u_λ^* is a supersolution of (4.4) for any $\lambda \in (0,1)$; since $\operatorname{supp} u_0 \cap (-\alpha, \alpha)$ is nonempty, there exists a subsolution $\underline{u}$ of the same problem such that

$$\operatorname{supp} \underline{u} \subseteq \operatorname{supp} u_0 \cap (-\alpha, \alpha), \qquad \underline{u} \leq u_0.$$

It follows that

$$u(t, \underline{u}) \leq u(t, u_0) \leq u(t, u_\lambda^*) \qquad \text{for any } t \geq 0.$$

Since

$$u(t, \underline{u}) \to u^*, \qquad u(t, u_\lambda^*) \to u^*$$

in the supremum norm as $t \to \infty$ [2], the claim follows.

The above argument carries over to higher space dimensions. Essentially the same result may be proved by well-known techniques, which involve the concept of ω-limit set [1] (see also [5]). The result is optimal, as the following proposition proves.

PROPOSITION 4.1. *Let* (A_0)–(A_2) *be satisfied. Then there exist* u_0 *such that*: (i) $\operatorname{supp} u_0 \cap (-\alpha, \alpha)$ *is empty*, (ii) *the corresponding solution of* (1.1) *tends to zero uniformly as* $t \to \infty$.

Proof. Let $\bar{x}$ be any number greater than α. Then, according to [8], there exists a unique nontrivial solution $z \geq 0$ of the problem

$$\begin{aligned} z'' - a(x)z^{p/m} &= 0 \qquad \text{in } (\bar{x}, \infty) \\ z(\bar{x}) = z'(\bar{x}) &= 0, \end{aligned}$$

moreover, $z(x) \to \infty$ as $x \to \infty$. Plainly, for $\lambda \in (0,1)$ small enough, the function

$$v_\lambda(x) := \begin{cases} \min\{z^{1/m}(x), u_\lambda^*(x)\} & x \in [\bar{x}, \infty), \\ 0 & x \in (-\infty, \bar{x}), \end{cases}$$

is a (nontrivial) supersolution of (1.1). Hence $u_0 \le v_\lambda$ implies $u(t, u_0) \le v_\lambda$ for any $t \ge 0$.

Since v_λ is a supersolution of (4.4) in $\Omega = (\bar{x}, \xi^*/\lambda)$, the conclusion follows as in the proof of Theorem 4.2. The case of general (compact) supp u_0 can be dealt with similarly. This completes the proof.

Observe that if supp $u_0 \cap (-\alpha, \alpha)$ is empty, yet u_0 is "large" in a suitable neighborhood of $(-\alpha, \alpha)$, $u(t, u_0)$ may converge to u^* as $t \to \infty$.

Finally, let us consider the Cauchy-Neumann problem

$$\begin{cases} u_t = (u^m)_{xx} - a(x)u^p & \text{in } (0,\infty) \times \Omega \\ u_x = 0 & \text{in} (0,\infty) \times \partial\Omega \\ u = u_0 & \text{in} \{0\} \times \Omega \end{cases} \tag{4.5}$$

where $\Omega = (-\mathrm{L}, \mathrm{L})$ $(\mathrm{L} > 0)$. For stationary solutions of (4.4) or (4.5) the following holds.

THEOREM 4.3. *Let* (A_0)–(A_2) *be satisfied. Then*

(i) *for any* $L > 0$ *there exists a unique even (nontrivial) stationary solution of* (4.4) *in* $(-L, L)$;

(ii) *for any* $L > \alpha^*$ *there exists a unique even (nontrivial) stationary solution of* (4.5) *in* $(-L, L)$.

In either case the support of the above solutions is $[-L, L] \cap [-\xi^*, \xi^*]$.

Proof. (i) Due to Theorem 1.3, for any $L \in (0, \xi^*]$ there is a unique $\bar{\gamma} \in (0, \gamma^*]$ $(\bar{\gamma} = \bar{\gamma}(L))$ such that $\xi(\bar{\gamma}) = L$. Then define

$$\bar{u}(x, \bar{\gamma}) := y^{1/m}(|x|, \bar{\gamma}) \qquad (|x| \le L);$$

the claim follows in this case by Theorem 1.1, part (i), due to assumption (A_0). If $L > \xi^*$, the function $\bar{u}(x, \gamma^*)$ can be defined on the all of $[-L, L]$ by setting $\bar{u} \equiv 0$ for $|x| > \xi^*$. This proves (i); the proof of (iii) is similar.

Let us mention that the attractivity of the above stationary solutions of (4.4), (4.5) can be investigated as we did before for the Cauchy problem.

5. More General Functions a

In this section we briefly discuss stationary solutions of (1.1) under more general assumptions on a. For simplicity, we consider the case when a is a

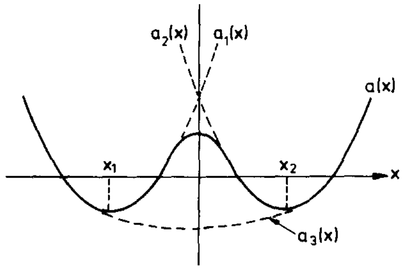

FIGURE 3

smooth approximation to $\min\{b_{x_1}, b_{x_2}\}$, where

$$b_{x_i}(x) := b(x - x_i) \qquad (x_i \in \mathbb{R} \text{ fixed; } x \in \mathbb{R},\ i = 1,2)$$

and b satisfies (A_0)–(A_2). Similar results hold in more general cases.

More precisely, let $a \in C^1(\mathbb{R})$ satisfy the following conditions (see Fig. 3):

(B_0) There exist $x_1 < x_2$ and $\eta_1, \eta_2 > 0$ such that $\eta_1 + \eta_2 \leq x_2 - x_1$,

$$a(x_i - x) = a(x_i + x) \qquad \text{for any } x \in [-\eta_i, \eta_i]\ (i = 1,2).$$

Moreover,

$$a(x) \leq \min_{i=1,2}\{a(2x_i - x)\} \qquad \text{for any } x \in [x_1 + \eta_1, x_2 - \eta_2].$$

(B_1) There exist $\alpha_i \in (0, \eta_i)$ such that $a(x) < 0$ for any $x \in [x_i - \alpha_i, x_i + \alpha_i]$ $(i = 1,2)$. In addition, a is decreasing in $(-\infty, x_1)$, increasing in (x_2, ∞).

(B_2) $\int_{-\infty}^{x_1 - \alpha_1} a(x)\,dx = \int_{x_2+\alpha_2}^{\infty} a(x)\,dx = \infty$.

(B_3) There exist $A_1 < x_1 - \alpha_1$, $A_2 > x_2 + \alpha_2$ such that

$$\sup_{x \leq A_1}(a^{-1/2})'(x) < \infty, \qquad \sup_{x \geq A_2}(a^{-1/2})'(x) < \infty.$$

Define $\alpha_1^*, \alpha_2^* > 0$ as follows:

$$\int_{x_1}^{x_1+\alpha_1^*} a(x)\,dx = \int_{x_2-\alpha_2^*}^{x_2} a(x)\,dx = 0;$$

then the following holds.

THEOREM 5.1. *Let $a \in C^1(\mathbb{R})$ satisfy* (B_0)–(B_3); *moreover, assume*

$$\alpha_1^* + \alpha_2^* \geq x_2 - x_1. \tag{5.1}$$

Then there exists a stationary solution $u^* \geq 0$ *of* (1.1), *which is strictly positive in* $[x_1 + \alpha_1, x_2 - \alpha_2]$.

Proof. Define two functions a_1, a_2 as follows:

$$a_1(x) := \begin{cases} a(x) & x \in (-\infty, x_1], \\ a(2x_1 - x) & x \in (x_1, \infty); \end{cases}$$

$$a_2(x) := \begin{cases} a(x) & x \in [x_2, \infty), \\ a(2x_2 - x) & x \in (-\infty, x_2); \end{cases}$$

thus $a_i(x) \geq a(x)$ for any $x \in \mathbb{R}$ ($i = 1,2$). In addition, let a_3 denote any function such that (i) $a_3(x) \leq a(x)$ for any $x \in \mathbb{R}$, (ii) a_3 satisfies (A_0)–(A_2) with a suitable choice of the origin. Such an a_3 is easily seen to exist under the present assumptions.

Then the free boundary solution $y_i^* = y(\cdot, \gamma_i^*)$ of the problem

$$y_i'' - a_i(x) y_i^{\beta} = 0 \qquad \text{in } \mathbb{R}_+$$
$$y_i(0) = \gamma_i, \; y_i'(0) = 0$$

($i = 1,2,3$; $\beta = p/m$) is a subsolution of (1.2) for $i = 1,2$, respectively, a supersolution for $i = 3$. Define u_i^* ($i = 1,2,3$) on $\mathbb{R}$ as in (4.3); then $\underline{u} := \max\{u_1^*, u_2^*\}$ is a subsolution, u_3^* a supersolution of (1.1). In view of (5.1), it follows from Theorem 1.3 that $\underline{u}$ is strictly positive in $[x_1 + \alpha_1, x_2 - \alpha_2]$, from which the conclusion follows easily.

Observe that in particular (5.1) is satisfied if

$$\int_{x_1}^{x_2} a(x)\, dx < 0.$$

Let us discuss a specific example. Set $m = 2$, $p = 1$; let $\tilde{a}$ be any function on $\mathbb{R}$ such that

(i) $\tilde{a}(x) = \cos(3\pi x/2d)$ in $[-4d/3, 4d/3]$ ($d > 0$);

(ii) $\tilde{a}$ is decreasing in $(-\infty, -4d/3)$, increasing in $(4d/3, \infty)$ and satisfies (B_2)–(B_3). Define

$$a(x) := \begin{cases} \rho a(x) & x \in (-d/3, d/3), \\ a(x) & x \in (-\infty, -d/3] \cup [d/3, \infty), \end{cases}$$

where $\rho > 0$. It is easily seen that

$$\sin \frac{3\pi\alpha_1^*}{2d} = \sin \frac{3\pi\alpha_2^*}{2d} = \frac{\rho - 1}{\rho};$$

hence Theorem 5.1 applies for any $\rho \in (0,1)$.

It can be observed that Theorem 5.1 ensures that no dead core of u^* exists, using an estimate "from below" of the support for the free boundary solution of (1.2). Sufficient conditions for the existence of a dead core, involving estimates "from above," can be derived as in [3]. Results of this kind were proved in [9], to which we also refer for the attractivity problem.

ACKNOWLEDGMENT

It is a pleasure to thank Don Aronson for helpful discussions.

REFERENCES

0. D. G. ARONSON, Bifurcation phenomena associated with non-linear diffusion mechanisms, in "Partial Differential Equations and Dynamical Systems" (W. E. Fitzgibbon, III, Ed.), Pitman, New York, 1984.
1. D. G. ARONSON, M. CRANDALL, AND L. A. PELETIER, Stabilization of solutions of a degenerate nonlinear diffusion problem, *Nonlinear Anal.* **6** (1982), 1001–1022.
2. P. DE MOTTONI, A. SCHIAFFINO, AND A. TESEI, Attractivity properties of nonnegative solutions for a class of nonlinear degenerate parabolic problems, *Ann. Mat. Pura Appl.* (4) **136** (1984), 35–48.
3. J. I. DIAZ AND J. HERNÁNDEZ, On the existence of a free boundary for a class of reaction-diffusion systems, *SIAM J. Math. Anal.* **15** (1984), 670–685.
4. M. E. GURTIN AND R. C. MACCAMY, On the diffusion of biological populations, *Math. Biosci.* **33** (1977), 35–49.
5. M. LANGLAIS AND D. PHILLIPS, Stabilization of solutions of nonlinear and degenerate evolution equations, 1984, preprint.
6. T. NAMBA, Density-dependent dispersal and spatial distribution of a population, *J. Theoret. Biol.* **86** (1980), 351–363.
7. L. A. PELETIER AND D. TERMAN, A very singular solution of the porous media equation with absorption, 1985, preprint.
8. L. A. PELETIER AND A. TESEI, Diffusion in inhomogeneous media: Localization and positivity, 1984, preprint.
9. M. A. POZIO AND A. TESEI, Support properties of solutions for a class of degenerate parabolic problems, 1984, preprint.
10. M. SCHATZMAN, Stationary solutions and asymptotic behavior of a quasilinear degenerate parabolic equation, *Indiana Univ. Math. J.* **33** (1984), 1–30.
11. J. SPRUCK, Uniqueness in a diffusion model of population biology, *Comm. Partial Differential Equations* **8** (1983), 1605–1620.