On the minimal covering of infinite sets

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Abstract


Minimal covering of infinite sets is studied. It is shown that if \( \mathcal{A} \subseteq 2^X \) is closed, then \( \mathcal{A} \) contains a minimal covering of \( X \). If \( \mathcal{A} \) is a covering of \( X \), then \( \mathcal{A} \) is closed on \( X \) if and only if there is an admissible function \( f: X \to \mathcal{A} \) which has no chain.

For star type set systems Theorem 15 gives a sufficient condition to contain finite covering.

Notations. Let \( X \) be a set. We denote by \( 2^X \) the set of all subsets of \( X \). If \( B \subseteq A \) and \( B \neq A \), we write \( B \subset A \). For \( \mathcal{A} \subseteq 2^X \), \( Y \subseteq X \) let \( \mathcal{A} \cap Y = \{ M \cap Y : M \in \mathcal{A} \} \).

Definition 1. The set \( \mathcal{A} \) is a covering of the set \( X \) if

1. \( \mathcal{A} \subseteq 2^X \) and
2. \( \bigcup_{M \in \mathcal{A}} M = X \).

Definition 2. \( \mathcal{A} \) is a minimal covering of \( X \) if

1. \( \mathcal{A} \) is a covering of \( X \) and
2. \( \mathcal{A}' \subset \mathcal{A} \) implies that \( \mathcal{A}' \) is not a covering of \( X \).

Definition 3. Let \( \mathcal{A} \) be a covering of \( X \). \( \mathcal{A} \) is closed on \( X \) if there is a well-ordering \( \mu \) of \( \mathcal{A} \) with the following property: for all \( x \in X \), the set \( \{ M \in \mathcal{A} : x \in M \} \) has a maximal element according to the well-ordering \( \mu \).

Remarks. (1) If \( \mathcal{A} \) is a point finite covering of \( X \) (i.e., \( \forall x \in X \ | \{ M \in \mathcal{A} : x \in M \} \) is finite), then \( \mathcal{A} \) is closed on \( X \).

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$M_j < \infty$, then $\mathcal{M}$ is closed on $X$. In fact, any well-ordering of $\mathcal{M}$ satisfies the property given by Definition 3.

(2) $\mathcal{M}$ is a minimal covering of $X$ if and only if
   (1) $\mathcal{M}$ is a covering of $X$ and
   (2) $\forall M \in \mathcal{M}, \exists m \in M$ such that $M' \in \mathcal{M}$, $m \in M' \Rightarrow M' = M$.

(3) Clearly any finite covering of $X$ contains a minimal subcover, but this is not true in general. For example, let $\alpha$ be a limit ordinal, and let $M_{\beta} (\beta < \alpha)$ be sets such that $M_{\gamma} \subset M_{\beta}$ if $\gamma < \delta < \alpha$. Let $X = \bigcup_{\beta < \alpha} M_{\beta}$, $\mathcal{M} = \{M_{\beta}: \beta < \alpha\}$. Clearly $\mathcal{M}$ is a covering of $X$ which does not contain a minimal covering of $X$.

**Theorem 4.** Let $\mathcal{M} \subset 2^X$. If $\mathcal{M}$ is closed on $X$, then $\mathcal{M}$ contains a minimal covering of $X$.

**Proof.** Let $\mathcal{M} = \{M_{\xi}: \xi < \alpha\}$ in a well-ordering $\mu$ which attests that $\mathcal{M}$ is closed on $X$ according to Definition 3 (i.e., $M_{\xi} \mu M_\eta \Rightarrow \xi < \eta < \alpha$). We define $\mathcal{M}^* \subset \mathcal{M}$ by transfinite induction. Suppose that $\beta < \alpha$ and for all $\gamma < \beta$ we already decided, whether $M_{\gamma} \in \mathcal{M}^*$, or not. Let $M_{\beta} \in \mathcal{M}^*$ if and only if $\exists x \in M_{\beta}$ such that
   (1) $\beta = \max\{\eta: x \in M_\eta \in \mathcal{M}\}$ and
   (2) $x \in M_\eta$ if $\eta < \beta$, $M_\eta \notin \mathcal{M}^*$.

First we prove that $\mathcal{M}^*$ is a covering of $X$. Suppose for contradiction that $\exists y \in X$ such that $M_y \in \mathcal{M}^* \Rightarrow y \notin M_{\beta}$. Let $v = \max\{\xi: y \in M_\xi, \xi < \alpha\}$. By the definition of $\mathcal{M}^*$ we obtain $M_{\gamma} \in \mathcal{M}^*$, a contradiction.

We use Remark (2) to see that $\mathcal{M}^*$ is a minimal covering of $X$. Let $M \in \mathcal{M}^*$ ($M = M_{\beta}$), then $\exists x \in M$ which satisfies (1) and (2). It is obvious that $x \notin M_\eta$ if $M_\eta \in \mathcal{M}^*$, $\gamma \neq \beta$. Hence $x \in M', M' \in \mathcal{M}^* \Rightarrow M' = M$. □

**Definition 5.** Let $\mathcal{M} \subset 2^X$. The function $f: X \rightarrow \mathcal{M}$ is said to be admissible if $x \notin f(x)$ for all $x \in X$.

**Definition 6.** Let $\mathcal{M} \subset 2^X$ and $f: X \rightarrow \mathcal{M}$ be an admissible function. The sequence $(x_1, x_2, ...)$ of elements of $X$ is a chain of $f$ if

$$x_i \notin f(x_{i+1}) \neq f(x_i) \quad (i = 1, 2, ...),$$

and the sequence $f(x_1), f(x_2), ...$ contains infinitely many distinct elements.

Let $n \geq 2$. The finite sequence $(y_1, ..., y_n)$ is a cycle of $f$ if

$$y_i \notin f(y_{i+1}) \neq f(y_i) \quad (i = 1, 2, ..., n - 1)$$

and

$$y_n \notin f(y_1) \neq f(y_n).$$

**Note.** It is convenient to consider the indices modulo $n$, then the definition of the cycle reduces to $y_i \notin f(y_{i+1}) \neq f(y_i) \quad (i = 1, 2, ..., n)$. 
Theorem 7. Let $\mathcal{M}$ be a covering of $X$. $\mathcal{M}$ is closed on $X$ if and only if there is an admissible function $f : X \to \mathcal{M}$ which has no chain.

To prove Theorem 7 we need the following lemma.

Lemma 8. Let $\mathcal{M}$ be a covering of $X$. Suppose that there exists an admissible function $f$ which has no chain. Then there is an admissible function $f_1$ which has neither a chain nor a cycle.

Proof. Fix a well-ordering, $<$, of the range of $f (f(X) \subseteq \mathcal{M})$. Let $d \in X$. The cycle $(d, y_1, \ldots, y_j)$ of $f$ is called a unit cycle belonging to $d$ if $f(y_1) < f(d)$ and $f(y_j)$ is minimal. Define $f_1(d) = f(y_i)$ if $(d, y_1, \ldots, y_j)$ is a unit cycle belonging to $d$ and $f_1(d) = f(d)$ if there is no unit cycle belonging to $d$. Clearly, $f_1 : X \to \mathcal{M}$ is an admissible function.

Suppose for contradiction that $(x_1, x_2, \ldots)$ is a chain of $f_1$. Insert a unit cycle belonging to $x_{i+1}$ between $x_i$ and $x_{i+1}$ whenever it is possible. More exactly, consider the sequence

$$(x_1, y_1^{(1)}, y_2^{(1)}, \ldots, y_i^{(j)}, x_2, y_3^{(2)}, \ldots, y_i^{(j)}, x_3, \ldots)$$

where $(x_1, y_1^{(1)}, \ldots, y_i^{(j)})$ is a unit cycle belonging to $x_i$ if such a unit cycle exists at all and $r_i = 0$ otherwise. Delete $x_i$ if $f(x_i) = f_1(x_{i+1})$, i.e., if $f(x_i) = f(y_i^{(j+1)})$ or if $r_{i+1} = 0$ and $f(x_i) = f(x_{i+1})$. Note that, if $x_i$ is deleted, then $f(x_i) = f_1(x_{i+1}) \neq f(x_i)$, and so $x_i$ has a unit cycle. It follows that after deleting the $x_i$ we are still left with an infinite sequence $(z_1, z_2, \ldots)$.

We will prove that $(z_1, z_2, \ldots)$ is a chain of $f$. It is clear that $|\{ f(z_1), f(z_2), \ldots \}| = \infty$.

We distinguish five cases.

Case 1. If $z_i = y_i^{(j)}$, $z_{i+1} = y_i^{(k)}$ or if $z_i = y_i^{(k)}$, $z_{i+1} = x_k$, then $z_i \in f(z_{i+1}) \neq f(z_i)$ follows from the definition of the unit cycle.

Case 2. If $z_i = x_i$, $z_{i+1} = y_i^{(j+1)}$, then $f(y_i^{(j+1)}) = f_1(x_{i+1})$ and we can use the fact that $(x_1, x_2, \ldots)$ is a chain of $f_1$.

Case 3. If $z_i = x_i$, $z_{i+1} = x_{i+1}$, then we can use the same fact.

Case 4. If $z_{i+1} = y_i^{(j+1)}$ and $z_i \neq x_j$, then $f_1(x_j) \neq f_1(x_{j+1}) = f(y_i^{(j+1)}) = f(x_j)$ proves that $x_j$ has a unit cycle and $z_i = y_i^{(j)}$. Now $z_i \in f(z_{i+1}) \neq f(z_i)$ follows from the definition of the unit cycle again.

Case 5. If $z_{i+1} = x_{j+1}$, $r_{j+1} = 0$ and $z_i \neq x_j$, then $f_1(x_j) \neq f_1(x_{j+1}) = f(x_{j+1}) = f(x_j)$, hence $z_i = y_i^{(j)}$ and the proof is similar to that of Case 4. This shows that $(z_1, z_2, \ldots)$ is a chain of $f$, which is a contradiction.

To finish the proof we will verify that $f_1$ also has no cycle. Suppose that $(x_1, \ldots, x_j)$ is a cycle of $f_1$. We will prove $f_1(x_1) < f_1(x_2) < \cdots < f_1(x_j) < f_1(x_1)$, a contradiction. Apply the same construction to obtain a finite sequence $(z_1, \ldots, z_j)$. One can check that it is a cycle of $f$ with a similar approach to the one used in the case of the chain.

Suppose first that $x_j$ is in the new cycle, i.e., $x_j = z_j$ for some $j$. If $z_{j+1} = y_j^{(j+1)}$,
then \( f_1(x_i) \leq f(y_i(i+1)) \) follows from the definition of \( f_1 \) and the fact that \( (z_1, \ldots, z_r) \) is a cycle of \( f \). Combining this with \( f(y_i(i+1)) = f_1(x_{i+1}) \neq f_1(x_i) \) we obtain \( f_1(x_i) < f_1(x_{i+1}) \). If \( z_j+1 \neq y_j(i+1) \), then \( t_{j+1} = 0 \) and \( z_{j+1} = x_{j+1} \), hence \( f_1(x_i) < f_1(x_{i+1}) = f_1(x_{i+1}) \neq f_1(x_i) \) implies \( f_1(x_i) < f_1(x_{i+1}) \).

Suppose finally that \( x_i \) was deleted during the construction of \( (z_1, \ldots, z_r) \). Then \( f_1(x_i) = f_1(x_{i+1}) \neq f_1(x_i) \), which again implies \( f_1(x_i) < f_1(x_{i+1}) \). We obtain a contradiction since \( f_1(x_3) < f_1(x_{3+1}) = f_1(x_3) \).

Proof of Theorem 7. Suppose first that \( \mathcal{M} \) is closed on \( X \), with an appropriate well-ordering. If \( x \in X \), let \( f(x) \) be the maximal element of \( \mathcal{M} \) which contains \( x \). Clearly \( f: X \to \mathcal{M} \) is an admissible function. If \( (x_1, x_2, \ldots) \) was a chain of \( f \), then \( f(x_1) > f(x_2) > \cdots \) would be an infinite descending chain in \( \mathcal{M} \), a contradiction.

To prove the converse suppose that \( f: X \to \mathcal{M} \) is an admissible function which has no chain. By the lemma, there is an admissible function \( f_1: X \to \mathcal{M} \) which has neither a chain nor a cycle. Define a partial ordering on \( \mathcal{M}_0 = f(X) \) in which \( M < N \) holds if there are \( x_1, \ldots, x_r \in X \) such that \( x_i \in f_1(x_{i+1}) \neq f_1(x_i) \) (\( i = 1, 2, \ldots, r - 1 \)) and \( f_1(x_1) = N, f_1(x_r) = M \). It is transitive: if \( x_1, \ldots, x_r, y_1, \ldots, y_s \) attest that \( N > M \) and \( M > L \) respectively, then \( x_1, \ldots, x_r, y_1, \ldots, y_s \) attest that \( N > L \). It is irreflexive since \( f_1 \) has no cycle. \( \mathcal{M}_0 \) has no \( \omega \)-type sequence (a sequence \( M_1 > M_2 > \cdots \)) because \( f_1 \) has no chain. Thus we can extend this partial ordering to a well-ordering of \( \mathcal{M}_0 \).

Choosing a well-ordering of \( \mathcal{M} \setminus \mathcal{M}_0 \) and defining \( M < N \) if \( M \in \mathcal{M} \setminus \mathcal{M}_0, N \in \mathcal{M}_0 \) we get a well-ordering of \( \mathcal{M} \). This well-ordering attests that \( \mathcal{M} \) is closed on \( X \). To see this, let \( x \in X \). If \( x \in f_1(y) \in \mathcal{M}_0 \), then \( f_1(y) \leq f_1(x) \) from the definition of the partial ordering of \( \mathcal{M}_0 \). Thus one can see from the definition of the well-ordering of \( \mathcal{M} \) that \( f_1(x) \) is the maximal element of \( \mathcal{M} \) which contains \( x \).

Theorem 9. Let \( \mathcal{M} \) be a covering of \( X \). Suppose that \( M \in \mathcal{M} \Rightarrow |M| \leq k \) for some \( k \in \mathbb{N} \). Then \( \mathcal{M} \) is closed on \( X \) and \( \mathcal{M} \) contains a minimal covering of \( X \).

Proof. We use induction on \( k \). The case \( k = 1 \) is trivial. Suppose that the theorem is proved for \( k \leq n \). We prove it for \( k = n + 1 \).

Consider the pairs \( (x, M) \) for which \( x \in M \in \mathcal{M} \). Using Zorn’s lemma there is a maximal set \( T = \{ (x_\alpha, M_\alpha): \alpha \in I \} \) such that \( \alpha \neq \beta \Rightarrow x_\alpha \neq x_\beta, x_\alpha \notin M_\beta \). Let \( X_0 = \{ x_\alpha: \alpha \in I \} \), \( \mathcal{M}_0 = \{ M_\alpha: \alpha \in I \} \). Let \( \mathcal{M}_1 = \{ M \in \mathcal{M}: M \cap X_0 \neq \emptyset \} \). Obviously, \( \mathcal{M}_1 \supseteq \mathcal{M}_0 \). It follows from the maximality of \( T \) that \( \mathcal{M}_1 \) is a covering of \( X \). Let \( A = \bigcup_{M \in \mathcal{M}_0} M, B = X \setminus A \). Of course, \( A \supseteq X_0 \).

If \( M \in \mathcal{M}_0 \), then \( (A \setminus X_0) \cap \mathcal{M}_0 \leq n \) and therefore by the induction hypothesis \( \mathcal{M}_0 \cap (A \setminus X_0) \) is closed on \( A \setminus X_0 \), with an appropriate well-ordering \( \mu_1 \). For \( x \in X_0 \), there is one and only one \( M \in \mathcal{M}_0 \) with \( x \in M \), hence \( \mathcal{M}_0 \cap A \) is closed on \( A \). It is also obvious that \( \mathcal{M}_1 \setminus \mathcal{M}_0 \) is closed on \( B \), with an appropriate well-ordering \( \mu_2 \). Let \( \mu \) be a well-ordering of \( \mathcal{M} \setminus \mathcal{M}_1 \). Connecting these well-orderings with \( M_1 < M_2 < M_3 \) if \( M_1 \in \mathcal{M} \setminus \mathcal{M}_1, M_2 \in \mathcal{M} \setminus \mathcal{M}_0, M_3 \in \mathcal{M}_0 \), we get a well-ordering of \( \mathcal{M} \) which attests that \( \mathcal{M} \) is closed on \( X \). Therefore, \( \mathcal{M} \) contains a minimal covering by Theorem 4.  \( \square \)
Theorem 10. If $\kappa$ is an infinite regular cardinal, $|X| = \kappa$, $k < \omega$, and $\mathcal{M}$ is a covering of $X$ such that

1. $A \subset X$, $|A| = k \Rightarrow |\{ M : A \subset M \in \mathcal{M} \}| = \kappa$ and
2. $B \subset X$, $|B| = k + 1 \Rightarrow |\{ M : B \subset M \in \mathcal{M} \}| < \kappa$,

then there is a covering $\mathcal{M}' \subset \mathcal{M}$ such that $|\{ M \in \mathcal{M}' : x \in M \}| < \kappa$ for all $x \in X$.

Proof. It is trivial if $k = 0$. Suppose for induction that the theorem holds for $k \leq n$. We will prove it for $k = n + 1$.

Let $X = \{ x_0, x_1, \ldots \}$. Let $\mathcal{A} = \{ A_0, A_1, \ldots \}$ be the set of all $(n + 1)$-element subsets of $X$ ($|\mathcal{A}| = \kappa$). For $m < \kappa$, there exists $M_m \in \mathcal{M}$ such that $A_m \subset M_m$ and $A_m \cup A_r \not\subset M_m$ if $r < m$, as it is guaranteed by (1) and (2) with $k = n + 1$. Of course, $m \neq n$ implies $M_m \neq M_n$. Let $\mathcal{M}_0 = \{ M_0, M_1, \ldots \}$. For a fixed $r$, $|\{ M \in \mathcal{M}_0 : A_r \subset M \}| < \kappa$.

However, if $C \subset X$, $|C| = n$, then $|\{ m < \kappa : C \subset A_m \subset M_m \}| = \kappa$. Hence $\mathcal{M}_0 \subset \mathcal{M}$ satisfies the conditions of the theorem for $k = n$, and we can find $\mathcal{M}' \subset \mathcal{M}_0 \subset \mathcal{M}$ with the desired property. $\square$

There is a natural duality between the notion of a minimal covering of a set by a family of sets and that of a minimal point covering for a family of sets. This duality is best expressed in the language of bipartite graphs. We will denote by $(X, Y, \mathcal{E})$ the bipartite graph with vertex set $X \cup Y$ and edge set $\mathcal{E}$, in which the edges join points of $X$ to points of $Y$.

Definition 11. Let $(X, Y, \mathcal{E})$ be a bipartite graph. $X_0 \subset X$ is a covering of $Y$ if $Y = \{ y : (x, y) \in \mathcal{E}, x \in X_0 \}$. $X_0$ is a minimal covering of $Y$ if it is a covering of $Y$ and $X_0 \subset X_0'$ implies that $X_0'$ is not a covering of $Y$.

Examples. (1) Let $X \subset 2^Y$ and $(x, y) \in \mathcal{E}$ iff $y \in x$. Then $X_0 \subset X$ is a covering of $Y$ iff $\bigcup_{x \in X_0} x = Y$, and this coincides with the original notion.

(2) Let $Y \subset 2^X$ and $(x, y) \in \mathcal{E}$ iff $x \in y$. Here the notion of minimal covering corresponds to that of the minimal point cover of a set system.

(3) Let $X \subset 2^A$, $Y \subset 2^A$ and $(x, y) \in \mathcal{E}$ iff $x \subset y$. This leads to another well-known minimum problem.

The notion of minimal point cover mentioned in the second example is closely related to the Teichmüller–Tukey lemma, and our paper is based on the analysis of a well-known proof of the lemma. $\mathcal{F} \subset 2^X$ is a property of finite type on $X$ if $A \in \mathcal{F}$ iff $A_0 \subset A$, $|A_0| < \kappa_0$ implies $A_0 \in \mathcal{F}$. If $\mathcal{F}$ is a property of finite type on $X$, then there is a maximal subset $M$ of $X$ for which $A \subset M$, $|A| < \kappa_0$, and $A \in \mathcal{F}$, as the Teichmüller–Tukey lemma claims. Then $X \setminus M$ is a minimal point cover of the set system consisting of all finite subsets $A$ of $X$, $A \in \mathcal{F}$. Our second observation is this: if $\mathcal{M} \subset 2^X$ and $X$ has a well-ordering such that every element of $\mathcal{M}$ has a maximal element, then there is a minimal point cover of $\mathcal{M}$. This is the dual of Theorem 4.
The notion of minimal point cover can be related to that of property-B.

**Definition 12.** A set system \( \mathcal{M} \subseteq 2^X \) has property-B if there exist disjoint subsets \( Y \) and \( Z \) of \( X \) such that \( X = Y \cup Z \) and \( H \cap Y \neq \emptyset, H \cap Z \neq \emptyset \) for every \( H \in \mathcal{M} \).

If \( \mathcal{M} \subseteq 2^X \) has a minimal point cover and \( M_1, M_2 \in \mathcal{M} \) implies \( |M_1 \cap M_2| \neq 1 \), then \( \mathcal{M} \) has property-B, as it follows directly from the definition. Now we modify Theorems 9 and 10 to get results concerning set systems having property-B.

1. Let \( \mathcal{M}, \mathcal{N} \subseteq 2^X \) satisfy
   1. \( M \in \mathcal{M} \Rightarrow |M| < \infty \).
   2. \( \exists k \in \mathbb{N} \forall x \in X \{|N_1 : x \in N \in \mathcal{N}_1\}| \leq k \).
   3. \( M, N \in \mathcal{M} \cup \mathcal{N} \Rightarrow |N \cap M| \neq 1 \).

   Then \( \mathcal{M} \cup \mathcal{N} \) has property-B.

   This is a consequence of Theorem 4, because the dual of Theorem 9 implies that \( X \) has a well-ordering \( \nu \) such that every element of \( \mathcal{N} \) has a maximal element, and the elements of \( \mathcal{M} \) (being finite sets) have maximal elements, too.

2. Let \( \mathcal{M} \subseteq 2^X, |\mathcal{M}| \leq \aleph_0 \) and \( k \in \mathbb{N} \) such that the intersection of any \( k \) elements of \( \mathcal{M} \) is infinite, but the intersection of any \( k+1 \) elements of \( \mathcal{M} \) is finite (maybe empty). Then \( \mathcal{M} \) has property-B.

   This follows immediately from the dual of Theorem 10 if \( |M| = \aleph_0 \). The case \( |M| < \infty \) is trivial.

**Definition 13.** Let \( M \subseteq X, \mathcal{H} \subseteq 2^X \). \( M \) is discrete over \( \mathcal{H} \) if \( |H \cap M| \leq 1 \) for all \( H \in \mathcal{H} \).

**Definition 14.** Let \( \mathcal{H} \subseteq 2^X \). The sequence \( \{(x_0, H_0), (x_1, H_1), \ldots \} \) is a strong chain in \( \mathcal{H} \) if the \( x_i \) are distinct elements of \( X \), the \( H_i \) are distinct elements of \( \mathcal{H} \) for \( i = 0, 1, \ldots \) and \( x_i \in H_i \) for \( i \leq n \). \( \mathcal{H} \) is of star type if there is no strong chain in \( \mathcal{H} \).

**Theorem 15.** Let \( \mathcal{H} \subseteq 2^X \) be of star type and suppose that whenever \( \mathcal{H}' \subseteq \mathcal{H} \) is a covering of \( X \), then every discrete set over \( \mathcal{H}' \) is finite. If \( \mathcal{H}_0 \subseteq \mathcal{H} \) is a covering of \( X \), then \( \mathcal{H}_0 \) contains a minimal covering of \( X \).

**Note.** This minimal covering must be finite, because every discrete set over it is finite.

**Proof.** First we note that if \( \mathcal{H}' \subseteq 2^X \) is a covering of \( X \) and \( \mathcal{H}' \cap (X \setminus M) \) is a point finite covering of \( X \setminus M \) for some \( M \subseteq X \) which is covered by a finite number of elements of \( \mathcal{H}' \), then \( \mathcal{H}' \) contains a minimal covering of \( X \). Indeed, this follows from Remark (1) and Theorem 4.

Suppose indirectly that there are no such \( \mathcal{H}' \) and \( M \). We define inductively sets \( M_n, N_n \subseteq X, \mathcal{H}_n, \mathcal{B}_n \subseteq 2^X \) for \( i = 0, 1, \ldots \) such that \( |M_n| = |\mathcal{B}_n| < \infty \), \( \mathcal{B}_n \) is a covering of \( X \) and \( \bigcup_{i=0}^{n} \mathcal{B}_i \) is a covering of \( \bigcup_{i=0}^{n} N_i \) for \( n \in \mathbb{N} \). Let \( \mathcal{B}_0 = M_0 = N_0 = \emptyset \) and
suppose that for \( i \leq n \) we have defined \( M_i, N_i, \mathcal{H}_i, \mathcal{G}_i \) with the desired properties. There exists \( p_n \in X \setminus \bigcup_{i=0}^{n} N_i \) which is contained in infinitely many elements of \( \mathcal{H}_n \). Let \( M_{n+1} \subseteq X \setminus \bigcup_{i=0}^{n} N_i \) be a maximal discrete set over \( \mathcal{H}_n \), which contains \( p_n \), then \( M_{n+1} \) is finite. Let \( \mathcal{H}'_{n+1} \) consist of those elements of \( \mathcal{H}_n \) which intersect \( M_{n+1} \). \( \mathcal{H}'_{n+1} = \mathcal{H}_{n+1} \cup \mathcal{G}_0 \cup \ldots \cup \mathcal{G}_n \) is a covering of \( X \). For \( p \in M_{n+1} \) let \( G_p \in \mathcal{H}_{n+1} \) such that \( p \in G_p \). Let \( \mathcal{G}_{n+1} = \{ G_p : p \in M_{n+1} \} \) and \( N_{n+1} = \bigcup_{p \in M_{n+1}} G_p \). This defines \( M_i, N_i, \mathcal{H}_i, \mathcal{G}_i \) for \( i \leq n + 1 \) with the desired properties.

Observe that \( \| \mathcal{H}_i \| = \infty \) \( (i \in \mathbb{N}) \) because of \( \mathcal{H}_1 \supseteq \mathcal{H}_2 \supseteq \ldots \).

**Claim 1.** For \( m = 1, 2, \ldots \) there exist \( z_1 \in M_1, \ldots, z_m \in M_m \) such that \( z_1, \ldots, z_m \in H \) for infinitely many \( H \in \mathcal{H}_0 \).

**Proof.** Suppose that for all \( z_i \in M_i \) \((i = 1, \ldots, m)\) there are only a finite number of elements of \( \mathcal{H}_m \) (and hence only a finite number of elements of \( \mathcal{H}_m \)) which contain \( z_1, \ldots, z_m \). Because each element of \( \mathcal{H}_m \) intersects \( M_i \) \((i \leq m)\) and \( |M_i| < \infty \), we obtain that \( |\mathcal{H}_m| < \infty \), a contradiction.

We call a sequence \( (x_1, \ldots, x_n) \) \( m \)-good if \( x_i \in M_i \) \((1 \leq i \leq n)\) and there are elements \( z_j \in M_{n+1} \) \((1 \leq j \leq m)\) such that \( \{ x_1, \ldots, x_n, z_1, \ldots, z_m \} \subseteq H \) for infinitely many \( H \in \mathcal{H}_0 \).

**Claim 2.** If \( (x_1, \ldots, x_n) \) is \( m \)-good for every \( m = 1, 2, \ldots \), then there is \( x_{n+1} \in M_{n+1} \) such that \( (x_1, \ldots, x_{n+1}) \) is also \( m \)-good for every \( m \).

**Proof.** Suppose indirectly that for every \( y_j \in M_{n+1} \) there is an \( m_j \in \mathbb{N} \) such that for all \( z_1 \in M_{n+2}, \ldots, z_m \in M_{n+m_j+1} \) there are only finitely many elements of \( \mathcal{H}_0 \) which contains \( x_1, \ldots, x_n, y_j, z_1, \ldots, z_m \). Then the condition of the claim does not hold for \( m = 1 + \max \{ m_j : y_j \in M_{n+1} \} \).

Using Claims 1 and 2 we can obtain \( x_i \in M_i \) \((i = 1, 2, \ldots)\) such that every finite subset of \( \{ x_1, x_2, \ldots \} \) is contained in infinitely many elements of \( \mathcal{H}_0 \), hence there is a strong chain in \( \mathcal{H}_0 \), and also in \( \mathcal{H} \). This contradiction completes the proof of the theorem.

To formulate the dual of Theorem 15 we use the following definition.

**Definition 14'.** Let \( \mathcal{H} \subseteq 2^X \). The sequence \( (x_0, H_0), (x_1, H_1), \ldots \) is an inverse strong chain in \( \mathcal{H} \) if the \( x_i \) are distinct elements of \( X \), the \( H_i \) are distinct elements of \( \mathcal{H} \) for \( i = 0, 1, \ldots \) and \( x_i \in H_i \) for \( i \geq n \).

**Theorem 15'.** Let \( \mathcal{H} \subseteq 2^X \). Suppose there is no inverse strong chain in \( \mathcal{H} \) and suppose also that, whenever \( X' \subseteq X \) is a point cover of \( \mathcal{H} \), then \( \mathcal{H} \cap X' \) does not contain infinitely many pairwise disjoint members. Then any point cover \( X_0 \subseteq X \) of \( \mathcal{H} \) contains a minimal (finite) point cover.

As an application of Theorem 15 we give the following topological result.
Definition 16. A topological space $X$ is of star type if every open covering of $X$ has an open covering refinement which is of star type in the sense of Definition 14.

The metacompact topological spaces are of star type.

Theorem 17. Suppose that the countably compact topological space $X$ is of star type. Then $X$ is compact.

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References