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Note

A Short Proof of an Identity of Touchard's Concerning Catalan Numbers

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In this paper a very short proof is given of an identity concerning Catalan numbers due originally to Touchard.

The purpose of this note is to provide a short and simple proof of Touchard's identity. See Touchard [5], Riordan [3], Izbicki [1], and Shapiro [4] for other proofs.

Put n points around a circle with the condition that each point can either be painted red, painted green, or connected to another point. The connecting lines may not intersect. How many such configurations are possible?

The generating function, $S(x) = \sum_{n=0}^{\infty} S_n x^n$, satisfies the relation $S(x) = 1 + 2xS(x) + x^2S(x)^2$, where $2xS(x)$ represents removal of an unconnected point, the term $x^2S(x)^2$ comes from the removal of a connected point and the point connected to it. The removal of the connecting line leaves two halves of the circle and the generating function for each half is $S(x)$. Motzkin [2] contains several identities derived from similar considerations. However the generating function for the Catalan numbers, $C(x)$, satisfies $C(x) = 1 + xC(x)^2$ so that $C(x) = 1 + xS(x)$. We next observe that $2k$ points can be connected in pairs by nonintersecting lines in C_k ways. This is known [6] and is easily proved by generating functions.

We can now compute S_n be selecting $2k$ points and connecting them in pairs by nonintersecting lines. The remaining $n - 2k$ points can be colored either red or green. Thus,

$$\sum_{k \geqslant 0} \binom{n}{2k} 2^{n-2k} C_k = S_n = C_{n+1}$$

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and we have a proof of Touchard's identity as well as a new characterization of the Catalan numbers and an interpretation of the numbers $\binom{n}{2k} 2^{n-2k} C_k$.

References

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