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A Conjecture of Glazman

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1. INTRODUCTION

The purpose of this note is to show that by combining some previous work of the author [5, 6] with a recent theorem of Allegretto [2] a problem posed by Glazman may be solved.

In [3, pp. 158–159], Glazman outlined a connection between the oscillation of solutions of the differential equation

$$-\Delta u + q(x)\, u = 0 \tag{1.1}$$

and the finiteness of the negative spectrum of the associated selfadjoint operator $-\Delta + q$ in $L^2(\mathbb{R}^m)$. He showed that if the negative spectrum of $-\Delta + q$ is finite, then (1.1) is nonoscillatory.

In [5] the author strengthened the idea of nonoscillation for (1.1). The main result was that if (1.1) is *strongly* nonoscillatory it follows that the negative spectrum of $-\Delta + q$ is finite. This idea was used in [6] to sharpen specific criteria due to F. Brownell and M. Birman for the finiteness of the negative spectrum of $-\Delta + q$.

The theorem of Allegretto [2] shows that the two definitions for nonoscillation are equivalent provided $q(x)$ is sufficiently smooth near infinity. The question of their equivalence had been posed but not answered by the author in [5].

With certain restrictions on the degree of local singularity of q it follows that the finiteness of the negative spectrum of $-\Delta + q$ to the left of the first point of its essential spectrum depends solely on the behavior of q near infinity.

This in turn solves a problem posed by Glazman [3, pp. 69–70]. Let Γ be a smooth closed hypersurface in $\mathbb{R}^m$ and let Ω denote its exterior. Let $L = L(B, \Gamma)$ be a selfadjoint realization of $-\Delta + q$ in $L^2(\Omega)$ with the boundary condition

$$Bu = \frac{\partial u}{\partial n} + \rho(x)\, u = 0 \tag{1.2}$$

on Γ. Here $(\partial u)/(\partial n)$ is the outward (from Ω) directed normal derivative of u and $0 \leqslant \rho(x) \leqslant +\infty$. The problem was to show that the finiteness of the spectrum

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of L to the left of the essential spectrum is invariant under certain distortions of Γ, perturbations of ρ, and perturbations of q over compact subsets of $\Omega \cup \Gamma$. But if the finiteness of the lower discrete spectrum only depends on the behavior, fq for large $|x|$, the conjectured invariance is obvious.

The results of [5, 6] are easily extended to the case where a boundary condition (1.2) is present. Allegretto's theorem need be applied only in a neighborhood of infinity. To simplify matters, and without fear of a loss of generality, we will only discuss the case where $\Omega = \mathbb{R}^m$ and $m \geqslant 3$.

We will consider the more general symmetric differential operator

$$Lu = -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + q(x)\,u,$$

where $q(x)$ is real and $\{a_{ij}(x)\}$ is real symmetric with positive eigenvalues. In Theorem 2.1 local degeneracy of ellipticity is permitted for the sake of completeness. But for the final result of the paper we assume that $\{a_{ij}(x)\}$ is uniformly positive definite as x varies over any compact subset of $\mathbb{R}^m$.

2. The Selfadjoint Realization of L and Criterion for Finite Negative Spectrum

In this section we will give a precise definition of the selfadjoint realization of the differential operator L. No proofs will be given since they are straightforward extensions of the proofs of the analogous results in [6] for $L = -\Delta + q(x)$.

First let $\lambda(x)$ be a nonnegative function so that

$$\lambda(x) \sum_{j=1}^{m} |\xi_j|^2 \leqslant \sum_{j,k=1}^{m} a_{jk}(x)\,\xi_j\bar{\xi}_k \tag{2.1}$$

for each x and each vector $(\xi_1,\ldots,\xi_m)$ in C^m. Also $q_1(x)$ will be a nonnegative function in $L^2_{\mathrm{loc}}(\mathbb{R}^m)$ and $q_2(x)$ is defined by setting

$$q(x) = q_1(x) + q_2(x).$$

We denote by $L^2(\mathbb{R}^m)$ the usual Hilbert space of square-integrable functions. W will be the closure of $C_0^\infty(\mathbb{R}^m)$, the space of infinitely differentiable functions with compact supports, relative to the norm $\|\phi\|_W$,

$$\|\phi\|_W^2 = \int\left[\sum_{i,j=1}^{m} a_{ij}\phi_i\bar{\phi}_j + (q_1+1)|\phi|^2\right]dx$$

where $\phi_l = (\partial\phi)/(\partial_{x_l})$, $l = 1,\ldots, m$. We finally define the quadratic form b by setting

$$b(\phi,\psi) = \int\left(\sum_{i,j} a_{ij}\phi_i\bar{\psi}_j + q\phi\bar{\psi}\right)dx.$$

Now suppose that inequalities of the following form hold:

$$| b(\phi, \psi)| \leqslant c_1 \| \phi \|_W \| \psi \|_W , \tag{2.2}$$

$$b(\phi, \phi) \geqslant c_2 \| \phi \|_W{}^2 - c_3 \| \phi \|^2, \tag{2.3}$$

where c_j are positive constants, and ϕ and ψ are arbitrary in W. Then it is well known, see, e.g., [7], that the form b defines a selfadjoint operator A which is bounded from below. A is characterized by specifying that u be in the domain of A and $v = Au$ if and only if

$$b(w, u) = (w, v) \qquad \text{for each} \quad w \text{ in } W.$$

This A will be used throughout the rest of the paper.

A fairly simple sufficient condition for the above construction is provided by the following theorem. Here $\| f \|_{p,K}$ denotes the L^p-norm of f over the set K.

THEOREM 2.1. *Let t be fixed in the interval $m/2 < t \leqslant +\infty$. Let $B_r(y)$ denote the open ball of radius r and with center at y.*

Suppose the following conditions hold:

$$\sup_y \| q_2 \|_{mt/(2t-m), B_r(y)} < +\infty \tag{2.4}$$

$$\lim_{r \to 0} \sup_y \left\{ \left\| \frac{1}{\lambda} \right\|_{t, B_r(y)} \| q_2 \|_{mt/(2t-m), B_r(y)} \right\} = 0. \tag{2.5}$$

Then inequalities (2.2) *and* (2.3) *hold.*

The special case of

$$\begin{aligned} a_{ij}(x) &= 1 \qquad \text{if} \quad i = j \\ &= 0 \qquad \text{if} \quad i \neq j \end{aligned}$$

is proved in detail in [6]. The same method yields the above theorem when proper account is taken of the variable modulus of ellipticity $\lambda(x)$. The conclusion still holds when the limit in (2.5) is less than a specific positive constant depending on t and m.

For simplicity we will assume from here on that $a_{ij}(x) \in C^1(\mathbb{R}^m)$ and $\lambda(x) > 0$ for each x. Local degeneracy could still be permitted at the expense of more complicated definitions of strong nonoscillation.

DEFINITION. We say that the equation,

$$Lu = 0,$$

is *strongly nonoscillatory at infinity* if there is a function $v(x)$ defined for $|x| > R$ and satisfying the following conditions

(i) v is in the local Sobolev space $H^{2,\frac{2}{3}m}_{\text{loc}}(|x| > R)$;

(ii) $v(x) > 0$ and $1/v$ is in $L^{\infty}_{\text{loc}}(|x| > R)$;

(iii) $Lv(x) \geqslant 0$ for, a.e., x with $|x| > R$.

The relevant result is

THEOREM 2.2. *If the equation $Lu = 0$ is strongly nonoscillatory at infinity, it follows that the negative spectrum of Λ is finite.*

Some results in [4] are necessary for the proof of Theorem 2.2.

3. ALLEGRETTO'S THEOREM

In this section and Section 4 we will require considerably more smoothness on the coefficients near infinity. Let $[a]$ denote the greatest integer not exceeding a. We assume that

$$a_{ij}(x) \text{ is in } C^{2n+1}(|x| > R)$$

and

$$q(x) \text{ is in } C^{2n}(|x| > R), \tag{3.1}$$

where

$$n = \left[\frac{1}{2}\left[\frac{m+6}{2}\right]\right].$$

We should point out that Allegretto proves the theorem under the stronger conditions that $a_{ij} \in C^{3n+1}$ and $q \in C^{3n}$. A slight change in his proof and an application of local regularity results for solutions of elli₊tic equations contained in [1] yield the more general conditions (3.1).

First we need another definition.

DEFINITION. The equation $Lu = 0$ is *nonoscillatory at infinity* if for some $R > 0$ and any bounded smooth domain B in $\{x \mid |x| > R\}$, the Dirichlet problem

$$\begin{aligned} Lu &= 0 \quad \text{in } B \\ u &= 0 \quad \text{on } \partial B \end{aligned} \tag{3.2}$$

has no nontrivial solution.

Then in our terminology Allegretto's theorem is as follows.

THEOREM 3.1. *If the smoothness conditions* (3.1) *are satisfied, and L is uniformly elliptic over compact sets, then* $Lu = 0$ *is strongly nonoscillatory whenever it is oscillatory.*

4. THE MAIN RESULT

We assume in this concluding section that the operators $L^{(\alpha)}$,

$$L^{(\alpha)}u = -\sum \frac{\partial}{\partial x_j}\left(a_{jk}^{(\alpha)}(x)\frac{\partial u}{\partial x_k}\right) + q^{(\alpha)}(x)\,u, \qquad \alpha = 1, 2, \tag{4.1}$$

satisfy the conditions of both Sections 2 and 3. Let $\Lambda^{(\alpha)}$ denote their respective selfadjoint realizations.

THEOREM 4.1. *Suppose for the operators* $L^{(\alpha)}$ *in* (4.1) *that*

$$a_{jk}^{(1)}(x) = a_{jk}^{(2)}(x) \qquad \text{and} \qquad q^{(1)}(x) = q^{(2)}(x)$$

for all sufficiently large $|x|$. *Then it follows that* $\Lambda^{(2)}$ *has finite negative spectrum if* $\Lambda^{(1)}$ *has.*

The proof is clear since the finiteness of the negative spectrum of $\Lambda^{(1)}$ implies that $L^{(1)}u = 0$ is nonoscillatory at infinity, see [3, p. 159] or [5] Now Theorem 3.1 tells us that $L^{(1)}u = 0$, and hence $L^{(2)}u = 0$, is *strongly* nonoscillatory at infinity. Finally by Theorem 2.2 the negative spectrum of $\Lambda^{(2)}$ is finite.

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