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Metric nonlinear connections

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Abstract

For a system of second order differential equations we determine a nonlinear connection that is compatible with a given generalized Lagrange metric. Using this nonlinear connection, we can find the whole family of metric nonlinear connections that can be associated with a system of SODE and a generalized Lagrange metric. For the particular case when the system of SODE and the metric structure are Lagrangian, we prove that the canonical nonlinear connection of the Lagrange space is the only nonlinear connection which is metric and compatible with the symplectic structure of the Lagrange space. For this particular case, the metric tensor determines the symmetric part of the canonical nonlinear connection, while the symplectic structure determines the skew-symmetric part of the nonlinear connection.

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Introduction

Nonlinear connections and metric structures are important tools for the differential geometry of the tangent bundle. Using the dynamical derivative one can associate to a nonlinear connection, we introduce a compatibility condition between a nonlinear connection and a generalized Lagrange metric. This compatibility condition is a natural generalization of the well known metric compatibility of a linear connection in a Riemannian space, [5].

For the differential geometry of a system of SODE one can associate to it a nonlinear connection and the corresponding dynamical covariant derivative. Such nonlinear connections were introduced by M. Crampin [6] and J. Grifone [9]. A metric geometry of a system of SODE requires a nonlinear connection which is compatible with a given generalized Lagrange metric. If S is an SODE and g is a generalized Lagrange metric, we determine a metric nonlinear connection that corresponds to the pair (S, g). Using this nonlinear connection, we determine the whole family of metric nonlinear connections that correspond to the pair (S, g).

For the particular case of a Lagrange space, the canonical nonlinear connection, which can be associated with the Euler–Lagrange equations, is the unique nonlinear connection that is compatible with respect to the metric and symplectic structures of the space. This result does not contradict the fact that for the theory of Lagrange struc-

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tures the canonical nonlinear connection is uniquely determined just by the symplectic structure, [10], the required compatibility being different from ours. The compatibility of the nonlinear connection with respect to the Lagrange metric determines the symmetric part of the nonlinear connection. The compatibility of the nonlinear connection with respect to the symplectic structure of the Lagrange space is equivalent with the existence of an almost Kählerian structure on TM. The compatibility with the symplectic structure of the Lagrange space determines the skew-symmetric part of the nonlinear connection.

The metric compatibility of a semispray and the associated nonlinear connection with a generalized Lagrange metric has been studied by M. Crampin et al. [7], O. Krupkova [11], T. Mestdag et al. [13], W. Sarlet [17], J. Szilasi and Z. Muzsnay [18] and it is known as one of the Helmholtz condition for the inverse problem of Lagrangian Mechanics. In the above mentioned papers, the problem that is studied is as follows: for a given semispray and the associated nonlinear connection find if it exists a Lagrange metric with respect to which the nonlinear connection is compatible.

A generalized Lagrange metric does not posses a canonical nonlinear connection. R. Miron in [14] and [15] and recently T. Mestdag et al. in [13] and J. Szilasi in [19] studied classes of such metrics to which we can associate in a canonical way a nonlinear connection. However, in their work, the compatibility between the canonical nonlinear connection and the metric structure it is not studied.

In our work a system of SODE and a generalized Lagrange metric are given a priori and we associate to these structures a metric nonlinear connection. This nonlinear connection, in general, is different from the nonlinear connection one usually associates to a semispray that was introduced by M. Crampin [6] and J. Grifone [9]. These two nonlinear connections coincide if the metric structure is Lagrangian. A geometric theory of the pair (S, g) has been proposed also by B. Lackey in [12], using Cartan's method of equivalence. A different approach for studying metrizable nonlinear connection has been proposed by M. Anastasiei [2], but his approach coincides with ours only for the particular case of a Finsler space.

1. Geometric structures on tangent bundles

In this section we introduce the geometric structures we deal with in this paper: semisprays, nonlinear connections and metric structures. These structures are defined on the total space of a tangent bundle.

We start by considering M a real, n-dimensional manifold of C^∞-class and denote by (TM, π, M) its tangent bundle. If $(U, \phi = (x^i))$ is a local chart at $p \in M$ from a fixed atlas of C^∞-class, then we denote by $(\pi^{-1}(U), \Phi = (x^i, y^i))$ the induced local chart at $u \in \pi^{-1}(p) \subset TM$. The linear map $\pi_{*,u} : T_uTM \to T_{\pi(u)}M$ induced by the canonical submersion π is an epimorphism of linear spaces for each $u \in TM$. Therefore, its kernel determines a regular, n-dimensional, integrable distribution $V : u \in TM \mapsto V_uTM := \operatorname{Ker}(\pi_{*,u}) \subset T_uTM$, which is called the *vertical distribution*. For every $u \in TM$, $\{\partial/\partial y^i|_u\}$ is a basis of V_uTM, where $\{\partial/\partial x^i|_u, \partial/\partial y^i|_u\}$ is the natural basis of T_uTM induced by a local chart. Denote by $\mathcal{F}(TM)$ the ring of real-valued functions over TM and by $\mathcal{X}(TM)$ the $\mathcal{F}(TM)$-module of vector fields on TM. We shall consider also $\mathcal{X}^v(TM)$ the $\mathcal{F}(TM)$-module of vertical vector fields on TM. An important vertical vector field is $\mathbb{C} = y^i(\partial/\partial y^i)$, which is called the *Liouville vector field*.

The mapping $J : \mathcal{X}(TM) \to \mathcal{X}(TM)$ given by $J = (\partial/\partial y^i) \otimes dx^i$ is called the *tangent structure* and it has the following properties: $\operatorname{Ker} J = \operatorname{Im} J = \mathcal{X}^v(TM)$; $\operatorname{rank} J = n$ and $J^2 = 0$.

A vector field $S \in \chi(TM)$ is called a *semispray*, or a second order vector field, if $JS = \mathbb{C}$. In local coordinates a semispray can be represented as follows:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}. \tag{1}$$

We refer to the functions $G^i(x, y)$ as to the local coefficients of the semispray S. Integral curves of a semispray S are solutions of the following system of SODE:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0. \tag{2}$$

A *nonlinear connection* N on TM is an n-dimensional distribution $N : u \in TM \mapsto N_uTM \subset T_uTM$ that is supplementary to the vertical distribution. This means that for every $u \in TM$ we have the direct decomposition:

$$T_uTM = N_uTM \oplus V_uTM. \tag{3}$$

The distribution induced by a nonlinear connection is called the *horizontal distribution*. We denote by h and v the horizontal and the vertical projectors that correspond to the above decomposition and by $\mathcal{X}^h(TM)$ the $\mathcal{F}(TM)$-module of horizontal vector fields on TM. For every $u=(x,y)\in TM$ we denote by $\delta/\delta x^i|_u = h(\partial/\partial x^i|_u)$. Then $\{\delta/\delta x^i|_u, \partial/\partial y^i|_u\}$ is a basis of T_uTM adapted to the decomposition (3). We call it the *Berwald basis* of the nonlinear connection. With respect to the natural basis $\{\partial/\partial x^i|_u, \partial/\partial y^i|_u\}$ of T_uTM, the horizontal components of the Berwald basis have the expression:

$$\left.\frac{\delta}{\delta x^i}\right|_u = \left.\frac{\partial}{\partial x^i}\right|_u - N_i^j(u)\left.\frac{\partial}{\partial y^j}\right|_u, \quad u\in TM. \tag{4}$$

The functions $N_j^i(x,y)$, defined on domains of induced local charts, are called the *local coefficients* of the nonlinear connection. The dual basis of the Berwald basis is $\{dx^i, \delta y^i = dy^i + N_j^i dx^j\}$.

It has been shown by M. Crampin [6] and J. Grifone [9] that every semispray determines a nonlinear connection. Local coefficients of the induced nonlinear connection are $N_j^i = \partial G^i/\partial y^j$.

A *generalized Lagrange metric*, or a GL-metric for short, is a $(2,0)$-type symmetric d-tensor field $g = g_{ij}(x,y)dx^i\otimes dx^j$ of rank n on TM. Throughout this paper by a d-tensor field we mean a tensor field on TM, whose components, under a change of coordinates on TM, behave like the components of a tensor field on the base manifold M. One can use the generalized Lagrange metric to define a metric structure on the vertical subbundle VTM, that is we can consider $g^v = g_{ij}\delta y^i\otimes \delta y^j$. Then, $G = g + g^v$ is a metric structure on TM with respect to which the horizontal and the vertical distributions are orthogonal to each other.

The geometry of the pair $(M, g_{ij}(x,y))$ is called the geometry of a generalized Lagrange space. This geometry has been studied by R. Miron in [15,16]. However, in this work no compatibility condition between the generalized Lagrange metric and a nonlinear connection is required.

2. Metric nonlinear connections

Nonlinear connections, semisprays and metric structures are important tools in the geometry of tangent bundles. There are situations, as in the geometry of generalized Lagrange spaces, [13–15], where these structures are considered, but no condition of compatibility is required for them. Using the dynamical derivative one can associate to a semispray S and a nonlinear connection N, we introduce a compatibility condition between the pair (S,N) and a generalized Lagrange metric g. This compatibility is a natural generalization of the well known metric compatibility of a linear connection in a Riemannian space, [5]. As the metric compatibility is not enough to determine the Levi-Civita connection of a Riemannian space, similarly the metric compatibility does not uniquely determine a nonlinear connection. A whole family of metric nonlinear connections is determined when a generalized Lagrange metric and a semispray are fixed. The problem of compatibility between a system of second order differential equations and a metric structure has been studied by numerous authors, [7,12] and it is known as one of the Helmholtz conditions from the inverse problem of Lagrangian Mechanics, [7,11,17,18]. In this section we approach the Helmholtz condition in a different way: for a given semispray and a generalized Lagrange metric we determine the whole family of nonlinear connections that are compatible with the metric tensor.

Let N be a nonlinear connection with local coefficients $N_j^i(x,y)$ and let S be a semispray. We determine the whole family of nonlinear connections one can associate to the semispray S and that are compatible with a generalized Lagrange metric g. The dynamical derivative that corresponds to the pair (S,N) is defined by $\nabla : \chi^v(TM)\longrightarrow \chi^v(TM)$ through:

$$\nabla\left(X^i\frac{\partial}{\partial y^i}\right) = \left(S(X^i) + X^jN_j^i\right)\frac{\partial}{\partial y^i}. \tag{5}$$

In terms of the natural basis of the vertical distribution we have

$$\nabla\left(\frac{\partial}{\partial y^i}\right) = N_i^j\frac{\partial}{\partial y^j}. \tag{6}$$

Hence, N_j^i are also local coefficients of the dynamical derivative. Dynamical derivative ∇ is the same with the covariant derivative D in [7] or $\mathcal{D}_\Gamma$ in [11], where it is called the Γ-derivative. Dynamical derivative ∇ has the following properties:

(1) $\nabla(X+Y)=\nabla X+\nabla Y,\ \forall X,Y\in\chi^v(TM)$,
(2) $\nabla(fX)=S(f)X+f\nabla X,\ \forall X\in\chi^v(TM),\ \forall f\in\mathcal{F}(TM)$.

It is easy to extend the action of ∇ to the algebra of d-tensor fields by requiring for ∇ to preserve the tensor product. For a GL-metric g, which is a $(2,0)$-type d-tensor field, its dynamical derivative is given by

$$(\nabla g)(X,Y)=S\big(g(X,Y)\big)-g(\nabla X,Y)-g(X,\nabla Y),\quad \forall X,Y. \tag{7}$$

In local coordinates, we have:

$$g_{ij|}:=(\nabla g)\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)=S(g_{ij})-g_{im}N^m_j-g_{mj}N^m_i. \tag{8}$$

Definition 2.1. Let S be a semispray, N a nonlinear connection and ∇ the associated covariant derivative. The nonlinear connection N is metric or compatible with the metric tensor g if $\nabla g=0$, which is equivalent to:

$$S\big(g(X,Y)\big)=g(\nabla X,Y)+g(X,\nabla Y),\forall X,Y.$$

For a semispray S with local coefficients G^i one can associate to it a nonlinear connection with local coefficients $N^i_j=\partial G^i/\partial y^j$. In general this nonlinear connection is not metric with respect to g. For the fixed semispray S, we determine first a nonlinear connection that is metric with respect to g and then we determine the whole family of nonlinear connections with this property.

Let us consider the following Obata operators one can associate to a GL-metric g_{ij}:

$$O^{ij}_{kl}=\frac{1}{2}\big(\delta^i_k\delta^j_l-g^{ij}g_{kl}\big)\quad\text{and}\quad O^{*ij}_{kl}=\frac{1}{2}\big(\delta^i_k\delta^j_l+g^{ij}g_{kl}\big). \tag{9}$$

The coordinate form (9) of Obata operators for a generalized metric structure was considered in [16], a coordinate-free expression of this operators appears in [13]. In both works, these operators were used to determine the family of linear connections on TM that preserve the horizontal and vertical distributions and are compatible with respect to the GL-metric.

Theorem 2.2. *Let S be a semispray with local coefficients G^i. There is a metric nonlinear connection N^c, whose coefficients N^{ci}_j are given by*:

$$N^{ci}_j=\frac{1}{2}g^{ik}S(g_{kj})+O^{ik}_{sj}\frac{\partial G^s}{\partial y^k}. \tag{10}$$

Proof. One can write coefficients N^{ci}_j from expression (10) into the following equivalent form

$$N^{ci}_j=\frac{1}{2}g^{ik}g_{kj|}+\frac{\partial G^i}{\partial y^j}. \tag{11}$$

In expression (11) the dynamical derivative $g_{kj|}$ is taken with respect to the pair $(G^i,\partial G^i/\partial y^j)$. Since $\partial G^i/\partial y^j$ are local coefficients of a nonlinear connection and $g^{ik}g_{kj|}$ are components of a d-tensor field of $(1,1)$-type we have that N^{ci}_j are also the local coefficients of a nonlinear connection. Consider now the dynamical derivative ∇ one can associate to the pair (G^i,N^{ci}_j). It is a straightforward calculation to check that

$$S(g_{ij})-g_{im}N^{cm}_j-g_{mj}N^{cm}_i=0,$$

which means that $\nabla g=0$ and hence, the nonlinear connection N^c is metric. □

Local coefficients of the metric nonlinear connection given by expression (10) can be written as follows:

$$N^{ci}_j=\frac{1}{2}g^{ik}S(g_{kj})+\frac{1}{2}\left(\frac{\partial G^i}{\partial y^j}-g^{ik}g_{mj}\frac{\partial G^m}{\partial y^k}\right), \tag{12}$$

which coincides with the nonlinear connection determined by B. Lackey in [12].

Proposition 2.3. *Let S be a semispray with local coefficients G^i, N the associated nonlinear connection with local coefficients $N^i_j = \partial G^i/\partial y^j$, and N^c the metric nonlinear connection given by expression* (10). *The nonlinear connection N is metric if and only if $N = N^c$.*

The metric compatibility of the nonlinear connection $N^i_j = \partial G^i/\partial y^j$ reads as follows:

$$S(g_{ij}) - g_{im}\frac{\partial G^m}{\partial y^j} - g_{mj}\frac{\partial G^m}{\partial y^i} = 0, \tag{13}$$

which is one of the Helmholtz conditions for the inverse problem in Lagrangian Mechanics, [17].

Now, we can determine the whole family of metric nonlinear connections one can associate to a semispray.

Theorem 2.4. *Consider S a semispray with local coefficients G^i and N^c the metric nonlinear connection with local coefficients given by expression* (10). *The family of all nonlinear connections that are metric with respect to the GL-metric g_{ij} is given by*:

$$N^i_j = N^{ci}_j + O^{ki}_{jm} X^m_k, \tag{14}$$

where X^m_k are the components of an arbitrary $(1,1)$-type d-tensor field.

Proof. The condition that both nonlinear connections N^{ci}_j and N^i_j are metric with respect to the metric tensor g can be written as $S(g_{ij}) = g_{mj}N^{cm}_i + g_{im}N^{cm}_j$ and $S(g_{ij}) = g_{mj}N^m_i + g_{im}N^m_j$. If we subtract these two equations we obtain $O^{*is}_{jm}(N^m_i - N^{cm}_i) = 0$. Using the fact that Obata operators (9) are projectors, which implies $O^{ij}_{kl}O^{*km}_{pj} = 0$, we obtain that the solution of this tensorial equation is given by expression (14). □

It is possible to define a dynamical derivative ∇ given by expression (5) by considering a nonlinear connection N, only, without considering an arbitrary semispray S. In such a case the semispray S is the horizontal vector field $S = y^i(\delta/\delta x^i)$ with local coefficients $2G^i(x,y) = N^i_j(x,y)y^j$. All results obtained in this section can be reformulated within the new particular framework. However, this does not allow us to determine a canonical metric nonlinear connection for a generalized Lagrange space.

There are classes of generalized Lagrange spaces that posses canonical nonlinear connections. However, these nonlinear connections are not compatible with the generalized Lagrange metric. Such spaces are called regular generalized Lagrange spaces and they were introduced by R. Miron in [14,15] and studied recently by Mestdag et al. [13] and J. Szilasi in [19].

3. Lagrange spaces

The variational problem of a Lagrange space determines a canonical semispray. In this section we prove that for the canonical semispray of a Lagrange space, there is a unique nonlinear connection that is metric and it is compatible with the symplectic structure of the space.

Consider $L^n = (M, L)$ a Lagrange space. This means that $L : TM \longrightarrow \mathbb{R}$ is a regular Lagrangian. In other words, the $(2,0)$-type, symmetric, d-tensor field with components

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j} \tag{15}$$

has rank n on TM. The regularity of the Lagrangian L is also equivalent with the fact that the Cartan two-form

$$\omega = \frac{1}{2} d\left(\frac{\partial L}{\partial y^i} dx^i\right) \tag{16}$$

is a symplectic structure on TM. The variational problem for the Lagrangian L determines the Euler–Lagrange equations:

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) = 0. \tag{17}$$

Under the assumption of regularity for the Lagrangian L, the system of equations (17) is equivalent with the following system of SODE:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0. \tag{18}$$

The functions G^i are local coefficients of a semispray S on TM, and they are given by:

$$G^i = \frac{1}{4} g^{ik}\left(\frac{\partial^2 L}{\partial y^k \partial x^h} y^h - \frac{\partial L}{\partial x^k}\right). \tag{19}$$

We refer to this semispray as to the canonical semispray of the Lagrange space. The canonical semispray S of the Lagrange space L^n is the unique vector field that satisfies the equation $i_S\omega = -(1/2)dE_L$, where $E_L = y^i(\partial L/\partial y^i) - L$ is the energy of the Lagrange space L^n. The semispray S determines a canonical nonlinear connections, which depends only on the fundamental function L. The local coefficients of this nonlinear connection are given by [9]

$$N^i_j = \frac{\partial G^i}{\partial y^j}. \tag{20}$$

For the canonical semispray S and the canonical nonlinear connection N consider ∇ the induced dynamical derivative (5).

Theorem 3.1. *For a Lagrange space L^n, the canonical nonlinear connection (20) is the unique nonlinear connection N that satisfies*:

(1) *The horizontal subbundle NTM is a Lagrangian subbundle of TTM, which means that*:

$$\omega(hX, hY) = 0, \quad \forall X, Y \in \chi(TM). \tag{21}$$

(2) *The metric tensor g_{ij} of the Lagrange space is covariant constant with respect to the dynamical derivative induced by (S, N), which is equivalent to*:

$$\nabla g = 0. \tag{22}$$

Proof. First we prove that conditions (21) and (22) uniquely determine a nonlinear connection. Then, we show that this nonlinear connection is the canonical nonlinear connection of the Lagrange space.

Consider N a nonlinear connection with local coefficients N^i_j. We want to express the symplectic form ω using the adapted cobasis $\{dx^i, \delta y^i\}$. If we use expression (16) and replace $dy^i = \delta y^i - N^i_j\, dx^j$ we obtain:

$$\begin{aligned}\omega &= g_{ij}\left(\delta y^j - N^j_k\, dx^k\right) \wedge dx^i + \frac{1}{4}\left(\frac{\partial^2 L}{\partial y^i \partial x^j} - \frac{\partial^2 L}{\partial x^i \partial y^j}\right) dx^j \wedge dx^i \\ &= g_{ij}\delta y^j \wedge dx^i + \frac{1}{2}\left[-N_{ij} + N_{ji} + \frac{1}{2}\left(\frac{\partial^2 L}{\partial y^i \partial x^j} - \frac{\partial^2 L}{\partial x^i \partial y^j}\right)\right] dx^j \wedge dx^i \\ &= g_{ij}\delta y^j \wedge dx^i + \left[-N_{[ij]} + \frac{1}{4}\left(\frac{\partial^2 L}{\partial y^i \partial x^j} - \frac{\partial^2 L}{\partial x^i \partial y^j}\right)\right] dx^j \wedge dx^i,\end{aligned} \tag{23}$$

where $N_{[ij]}$ denotes the skew symmetric part of $N_{ij} := g_{ik}N^k_j$. We have that condition (21) is true if and only if the second term of the right-hand side of expression (23) vanishes. Consequently, we have that condition (21) is true if and only if:

$$N_{[ij]} = \frac{1}{2}(N_{ij} - N_{ji}) = \frac{1}{4}\left(\frac{\partial^2 L}{\partial y^i \partial x^j} - \frac{\partial^2 L}{\partial x^i \partial y^j}\right). \tag{24}$$

Expression (24) tells us that the skew symmetric part of N_{ij} is uniquely determined by condition (21) and hence $N_{[ij]}$ is uniquely determined by the symplectic structure ω. Next, we prove that the symmetric part of N_{ij} is perfectly determined by metric condition (22). In local coordinates, condition (22) is equivalent to:

$$S(g_{ij}) = g_{mj}N^m_i + g_{im}N^m_j = N_{ij} + N_{ji} = 2N_{(ij)}. \tag{25}$$

Expressions (24) and (25) uniquely determine the local coefficients N^i_j of the nonlinear connection N that satisfies conditions (21) and (22). These coefficients are given by:

$$\begin{aligned} N^i_j &= g^{ik}N_{kj} = g^{ik}(N_{(kj)} + N_{[kj]}) \\ &= \frac{1}{2}g^{ik}\left[S(g_{kj}) + \frac{1}{2}\left(\frac{\partial^2 L}{\partial y^k \partial x^j} - \frac{\partial^2 L}{\partial x^k \partial y^j}\right)\right]. \end{aligned} \tag{26}$$

We prove now that the nonlinear connection (20) of a Lagrange space is the unique one that satisfies conditions (21) and (22). For this we have to show that the nonlinear connection with local coefficients given by expression (20) satisfies (26). The coefficients N^i_j of the canonical nonlinear connection (20) of a Lagrange space can be written as:

$$N^i_j = \frac{\partial G^i}{\partial y^j} = \frac{1}{4}\frac{\partial g^{ip}}{\partial y^j}\left(\frac{\partial^2 L}{\partial y^p \partial x^m}y^m - \frac{\partial L^2}{\partial x^p}\right) + \frac{1}{4}g^{ip}\left(2\frac{\partial g_{jp}}{\partial x^m}y^m - \frac{\partial^2 L}{\partial y^j \partial x^p}\right) + \frac{1}{4}g^{ip}\frac{\partial^2 L}{\partial y^p \partial x^j}.$$

If we multiply the above formula by g_{is} we obtain:

$$N_{sj} := g_{si}N^i_j = -\frac{\partial g_{is}}{\partial y^j}G^i + \frac{1}{2}\frac{\partial g_{sj}}{\partial x^i}y^i + \frac{1}{4}\left(\frac{\partial^2 L}{\partial y^s \partial x^j} - \frac{\partial^2 L}{\partial x^s \partial y^j}\right),$$

which is equivalent to:

$$N_{ij} = \frac{1}{2}S(g_{ij}) + \frac{1}{4}\left(\frac{\partial^2 L}{\partial y^i \partial x^j} - \frac{\partial^2 L}{\partial x^i \partial y^j}\right). \tag{27}$$

We can see that expressions (27) and (26) are equivalent to one another. From expression (27) it follows that the canonical nonlinear connection of a Lagrange space with local coefficients (20) satisfies the two axioms of the theorem. □

Theorem 3.1 shows that the canonical nonlinear connection of a Lagrange space has:

(1) the skew-symmetric part $N_{[ij]} = (1/2)a_{ij}$ uniquely determined by the symplectic form $\omega = g_{ij}\delta y^j \wedge dx^i + (1/2)a_{ij}\,dx^j \wedge dx^i$.
(2) the symmetric part $N_{(ij)} = S(g_{ij})$ uniquely determined by the semispray S and the metric tensor g.

Consequently, we can generalize Theorem 3.1 as follows:

Theorem 3.2. *Consider (M, g) a GL-space. Let S be a semispray and ω a symplectic structure on TM for which the vertical subbundle VTM is a Lagrangian subbundle. There exists a unique nonlinear connection N on TM such that*:

(1) *The horizontal subbundle NTM is a Lagrangian subbundle of TTM, which means that*

$$\omega(hX, hY) = 0, \quad \forall X, Y \in \chi(TM).$$

(2) *The metric tensor g_{ij} of the generalized Lagrange space is covariant constant, which means that*

$$\nabla g = 0.$$

Symplectic structures on a GL-space that satisfy condition (1) in the above theorem were studied by M. Anastasiei in [1]. With respect to the adapted cobasis $\{dx^i, \delta y^i = dy^i + N^i_j\,dx^j\}$ of the canonical nonlinear connection the symplectic form ω of a Lagrange space L^n has a simple form:

$$\omega = g_{ij}\delta y^j \wedge dx^i. \tag{28}$$

Expression (28) is equivalent to (21), which says that symplectic form ω vanishes if both of its arguments are horizontal vector fields. One can see also from expression (28) that $\omega(X,Y)=0$ if both vectors X and Y are vertical vector fields. Therefore both horizontal and vertical subbundles are Lagrangian subbundles for the tangent bundle TTM.

A nonlinear connection is perfectly determined by an almost complex structure $\mathbb{F}$ given by:

$$\mathbb{F}=\frac{\delta}{\delta x^i}\otimes \delta y^i-\frac{\partial}{\partial y^i}\otimes dx^i. \tag{29}$$

For a GL-metric g one can use a nonlinear connection N to define a nondegenerate metric tensor $\mathbb{G}$ on TM that preserves the horizontal and vertical distributions:

$$\mathbb{G}=g_{ij}\,dx^i\otimes dx^j+g_{ij}\delta y^i\otimes \delta y^j. \tag{30}$$

The compatibility condition (21) implies

$$\omega(X,Y)=\mathbb{G}(\mathbb{F}X,Y)\quad\text{and}\quad \mathbb{G}(X,Y)=\mathbb{G}(\mathbb{F}X,\mathbb{F}Y),\quad \forall X,Y\in\chi(TM).$$

Consequently, the pair $(\mathbb{G},\mathbb{F})$ is an almost Kählerian structure on TM.

Theorem 3.1 shows that the canonical nonlinear connection of a Lagrange space is metric. Next, as we did for a GL-metric, we can determine the family of all metric nonlinear connections for a Lagrange space. For this we do not require anymore the compatibility of the nonlinear connection with the symplectic structure.

Proposition 3.3. *The family of all nonlinear connections that are compatible with the metric tensor of a Lagrange space is given by*:

$$N^i_j=N^{ci}_j+O^{ki}_{jm}X^m_k. \tag{31}$$

Here X^m_k are the components of an arbitrary $(1,1)$-type d-tensor field, O^{ki}_{jm} is the Obata operator (9) and N^{ci}_j are the local coefficients of the canonical nonlinear connection of the Lagrange space.

The proof of this result is similar with that of Theorem 2.4. A result of similar type has been obtained by O. Krupkova in [11], where the difference between an arbitrary metric nonlinear connection and the canonical one is given by expression (7.22).

4. Discussions

In this work we start with a system of second order differential equations and a generalized Lagrange metric and we determine a metric nonlinear connection.

This is a different approach of the inverse problem of Lagrangian Mechanics, where, for a given system of SODE and an associated nonlinear connection, we seek for a metric tensor that makes the nonlinear connection metric.

The metric nonlinear connection we determine in Theorem 2.2 is not unique. However its symmetric part is uniquely determined by the metric compatibility. The metric nonlinear connection given by expression (10) depends on both structures: semispray and generalized Lagrange metric. Hence, this nonlinear connection is different from the nonlinear connection, given by expression (20), which one usually associate to a semispray. The two nonlinear connections coincide for the particular case when the metric tensor is Lagrangian.

A metric nonlinear connection is uniquely determined if we add a condition that determines its skew-symmetric part. This can be done if we require the compatibility of the nonlinear connection with a symplectic structure as we did in Theorem 3.2. For a Lagrange space we prove in Theorem 3.1 that the metric structure and the symplectic structure uniquely determine the nonlinear connection we associate with Euler–Lagrange equations.

In general for a system of SODE, the induced nonlinear connection is used to build another connection, which is linear and it is called the Berwald connection, that gives more information about the system, [3,4,6,8]. For the future I intend to use the metric nonlinear connection we can associate to a system of second order differential equations S and a GL-metric g to determine corresponding Berwald connection. The geometric invariants of the pair (S,g) will be obtained from the Berwald connection as it has been done in [3]. Also the integrability conditions of the pair (S,g) can be studied using the corresponding Berwald connection, using the techniques developed in [4].

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