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The mod 2 cohomology of fixed point sets of anti-symplectic involutions

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Abstract

Let M be a compact, connected symplectic manifold with a Hamiltonian action of a compact n-dimensional torus $G = T^n$. Suppose that σ is an anti-symplectic involution compatible with the G-action. The real locus of M is X, the fixed point set of σ. Duistermaat uses Morse theory to give a description of the ordinary cohomology of X in terms of the cohomology of M. There is a residual $G_{\mathbb{R}} = (\mathbb{Z}/2\mathbb{Z})^n$ action on X, and we can use Duistermaat's result, as well as some general facts about equivariant cohomology, to prove an equivariant analogue to Duistermaat's theorem. In some cases, we can also extend theorems of Goresky–Kottwitz–MacPherson and Goldin–Holm to the real locus.

Keywords: Symplectic manifolds; Equivariant cohomology; Real locus

1. Introduction

Atiyah observed in [A] that if M is a compact symplectic manifold and τ a Hamiltonian action of an n-dimensional torus G on M, then the cohomology groups of M can be computed from the cohomology groups of the fixed point set M^G of τ. Explicitly, if $\Phi : M \to \mathfrak{T}^*$ is the moment map for τ, then a generic component Φ^ξ of Φ

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is a perfect Bott–Morse function. Using Φ^{ξ}, we may compute

$$H^*(M;\mathbb{R}) = \sum_{i=1}^{N} H^{*-d_i}(F_i;\mathbb{R}), \tag{1.1}$$

where the F_i are the connected components of M^G and d_i is the Bott–Morse index of F_i. This result is also true in equivariant cohomology [AB,BV]:

$$H_G^*(M;\mathbb{R}) = \sum_{i=1}^{N} H_G^{*-d_i}(F_i;\mathbb{R}) = \sum_{i=1}^{N} H^{*-d_i}(F_i \times BG;\mathbb{R}). \tag{1.2}$$

This is a consequence of Atiyah's result and Kirwan's equivariant formality theorem for Hamiltonian G-manifolds, as shown in [Ki].

In [Du], Duistermaat proved a "real form" version of (1.1). Let $\sigma : M \to M$ be an anti-symplectic involution with the property that

$$\sigma \circ \tau_g = \tau_{g^{-1}} \circ \sigma \tag{1.3}$$

and let $X = M^\sigma$ be the fixed point set of σ. We call X the *real locus* of M. The motivating example of this setup is a complex manifold M with a complex conjugation σ. Duistermaat proved that

$$H^*(X;\mathbb{Z}/2\mathbb{Z}) = \sum_{i \in I} H^{*-\frac{d_i}{2}}(F_i^\sigma;\mathbb{Z}/2\mathbb{Z}), \tag{1.4}$$

where $I \subseteq \{1,\dots,N\}$ is the set for which σ preserves F_i. The $\mathbb{Z}/2\mathbb{Z}$ coefficients are essential here; the theorem does not hold with real coefficients. (See the comments at the end of this section.)

The first of the four main theorems of this paper is an equivariant analogue of (1.4) similar to the equivariant analogue (1.2) of Atiyah's result (1.1). By (1.3), the group

$$G_{\mathbb{R}} = \{g \in G \mid g^2 = \mathrm{id}\} \cong (\mathbb{Z}/2\mathbb{Z})^n \tag{1.5}$$

acts on X and we will prove the following theorem in Section 2.

Theorem A. *Suppose M is a symplectic manifold with a Hamiltonian action τ of a torus $T^n = G$ and an anti-symplectic involution σ. Let $X = M^\sigma$ denote the real locus of M. Then the group $G_{\mathbb{R}}$ acts on X, and the $G_{\mathbb{R}}$-equivariant cohomology of X with $\mathbb{Z}/2\mathbb{Z}$ coefficients is*

$$H_{G_{\mathbb{R}}}^*(X;\mathbb{Z}/2\mathbb{Z}) = \sum_{i \in I} H_{G_{\mathbb{R}}}^{*-\frac{d_i}{2}}(F_i^\sigma;\mathbb{Z}/2\mathbb{Z}). \tag{1.6}$$

As above, the subset $I \subseteq \{1,\dots,N\}$ is the set for which σ preserves F_i.

The idea of the proof will be to derive (1.6) from (1.4) by a simple trick.

The second of the main theorems concerns the structure of

$$H^*_{G_{\mathbb{R}}} = (X; \mathbb{Z}/2\mathbb{Z})$$

as a module over the ring

$$H^*_{G_{\mathbb{R}}} = H^*_{G_{\mathbb{R}}}(pt) = \mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_n]. \tag{1.7}$$

Let $BG_{\mathbb{R}}$ and $EG_{\mathbb{R}}$ denote the classifying space and classifying bundle of $G_{\mathbb{R}}$. Then by the Borel definition of equivariant cohomology

$$H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z}) = H^*(X \times_{G_{\mathbb{R}}} EG_{\mathbb{R}}; \mathbb{Z}/2\mathbb{Z}).$$

The cohomology on the right-hand side can be computed by the spectral sequence associated with the fibration

$$X \times_{G_{\mathbb{R}}} EG_{\mathbb{R}} \to BG_{\mathbb{R}},$$

and we will deduce from this computation the following theorem.

Theorem B. *The equivariant cohomology $H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z})$ is a free module over $H^*_{G_{\mathbb{R}}}$ generated in dimension zero. Moreover, as an $H^*_{G_{\mathbb{R}}}$ module, $H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z})$ is isomorphic to*

$$H^*_{G_{\mathbb{R}}} \otimes_{\mathbb{Z}/2\mathbb{Z}} H^*(X; \mathbb{Z}/2\mathbb{Z}). \tag{1.8}$$

The idea of the proof is to show that this spectral sequence collapses at its E_2 term, i.e., X is equivariantly formal. In particular, this implies that $H^*_{G_{\mathbb{R}}}(X)$ is a free module over $\mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_n]$.

Isomorphisms (1.1), (1.2), (1.4), and (1.6) are all isomorphisms in *additive* cohomology. The next two sections of this paper will be concerned with the ring structure of $H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z})$. Henceforth, we will assume that the fixed point set M^G is finite and that the one-skeleton of τ,

$$M^{(1)} = \{p \in M \mid \dim(G \cdot p) \leqslant 1\},$$

is of dimension 2. It is not hard to see that these two assumptions imply that

$$M^{(1)} = \bigcup_{i=1}^{N} E_i, \quad E_i \cong \mathbb{C}P^1.$$

Moreover, E_i is point-wise fixed by an $(n-1)$-dimensional torus H_i, and the diffeomorphism

$$E_i \to \mathbb{C}P^1$$

intertwines the action of G/H_i on E_i with the standard S^1 action on $\mathbb{C}P^1$. In particular, E_i contains exactly two G-fixed points. Thus, one can describe the intersection properties of the E_i by a graph Γ with edges e_i corresponding to the spheres E_i and vertices $V_\Gamma = M^G$. Two vertices p and q are joined by the edge e_i if $E_i^G = \{p, q\}$. Furthermore, each edge, e_i is labeled by a weight $\tilde{\alpha}_{e_i}$ of G, the weight associated with the intertwining homomorphism

$$G/H_i \to S^1.$$

Let $i : M^G \to M$ be the inclusion of the fixed points into the manifold, and consider the induced map in equivariant cohomology,

$$i^* : H_G^*(M; \mathbb{C}) \to H_G^*(M^G; \mathbb{C}). \tag{1.9}$$

By a theorem of Kirwan [Ki], this map is injective. Moreover,

$$H_G^*(M^G; \mathbb{C}) = \bigoplus_{p \in M^G} H_G^*(\{p\}; \mathbb{C}), \tag{1.10}$$

and since

$$H_G^*(\{p\}; \mathbb{C}) \cong S(\mathfrak{g}^*),$$

one can regard an element of $H_G^*(M^G; \mathbb{C})$ as a map

$$f : M^G \to S(\mathfrak{g}^*). \tag{1.11}$$

Goresky et al. [GKM] computed the image of i^*, thus determining not just the additive equivariant cohomology of M, but in fact the ring structure of this cohomology.

Theorem 1.1 (Goresky et al. [GKM]). *A map $f : M^G \to S(\mathfrak{g}^*)$ is in the image of i^* if and only if for each edge $e_i = \{p, q\}$ of Γ*

$$f(p) - f(q) \in \tilde{\alpha}_{e_1} \cdot S(\mathfrak{g}^*). \tag{1.12}$$

The third of the four main theorems of this paper will be a $\mathbb{Z}/2\mathbb{Z}$ version of this result for the manifold X. We define the one-skeleton of the real locus to be the set

$$X^{(1)} = \{x \in X \mid \#(G_\mathbb{R} \cdot x) \leqslant 2\}. \tag{1.13}$$

Assume in addition to the above that $M^G = X^{G_\mathbb{R}}$ and the real locus of the one-skeleton is the same as the one-skeleton of the real locus. We will call a manifold with these properties a mod 2 *GKM manifold*. The map

$$i^* : H_{G_\mathbb{R}}^*(X; \mathbb{Z}/2\mathbb{Z}) \to H_{G_\mathbb{R}}^*(X^{G_\mathbb{R}}; \mathbb{Z}/2\mathbb{Z}) \tag{1.14}$$

is injective, and by factoring (1.14) through the map

$$H^*_{G_{\mathbb{R}}}(X;\mathbb{Z}/2\mathbb{Z}) \to H^*_{G_{\mathbb{R}}}(E_i^\sigma;\mathbb{Z}/2\mathbb{Z}), \tag{1.15}$$

where $E_i^\sigma = E_i \cap X \cong \mathbb{R}P^1$, we obtain the following theorem.

Theorem C. *Suppose M is a* mod 2 *GKM manifold. An element*

$$f \in H^*_{G_{\mathbb{R}}}(X^{G_{\mathbb{R}}};\mathbb{Z}/2\mathbb{Z})$$

can be thought of as a map $f: V_\Gamma \to \mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_n]$, and such a map f is in the image of i^* if and only if, for each edge $e = \{p, q\}$ of Γ

$$f_p - f_q \in \alpha_e \cdot \mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_n],$$

where $\alpha_e \in \mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_n]$ is the image of the weight $\tilde{\alpha}_e$.

This completely determines the ring structure of $H^*_{G_{\mathbb{R}}}(X;\mathbb{Z}/2\mathbb{Z})$. This theorem is proved independently by Schmid [S] using different techniques. The main examples of mod 2 GKM manifolds include the real loci of non-singular projective toric varieties and real loci of coadjoint orbits, including Grassmannian and flag varieties. In the case of toric varieties, this equivariant cohomology ring was computed already by Davis and Januszkiewicz [DJ], but our description is quite different from theirs.

In [GH], Goldin and Holm generalize the GKM result to the case where the one-skeleton has dimension at most 4. Assume in addition to the dimension hypothesis that $M^G = X^{G_{\mathbb{R}}}$ and the real locus of the one-skeleton is the same as the one-skeleton of the real locus. We will call a manifold with these properties a mod 2 GH *manifold*. The last of the main theorems is a $\mathbb{Z}/2\mathbb{Z}$ version of the result of Goldin and Holm for the real locus X. For a subgroup $H_{\mathbb{R}}$ of $G_{\mathbb{R}}$, we will let $\pi^*_{H_{\mathbb{R}}}$ will denote the change of coefficient map

$$\pi^*_{H_{\mathbb{R}}} : H^*_{G_{\mathbb{R}}} \to H^*_{H_{\mathbb{R}}}$$

associated with the inclusion $H_{\mathbb{R}} \hookrightarrow G_{\mathbb{R}}$. In Section 6, we will prove the following theorem.

Theorem D. *Suppose that M is a* mod 2 *GH manifold with G fixed points fixed points $M^G = \{p_1, \dots, p_d\}$. Let $f_i \in H^*_{G_{\mathbb{R}}}$ denote the restriction of $f \in H^*_{G_{\mathbb{R}}}(X)$ to the fixed point p_i. The image of the injection $i^*: H^*_{G_{\mathbb{R}}}(X) \to H^*_{G_{\mathbb{R}}}(X^{G_{\mathbb{R}}})$ is the subalgebra of functions $(f_1, \dots, f_d) \in \bigoplus_{i=1}^d H^*_{G_{\mathbb{R}}}$ which satisfy*

$$\begin{cases} \pi^*_{H_{\mathbb{R}}}(f_{i_j}) = \pi^*_{H_{\mathbb{R}}}(f_{i_k}) & \textit{if } \{p_{i_1}, \dots, p_{i_l}\} = Z^{G_{\mathbb{R}}}_{H_{\mathbb{R}}}, \\ \sum_{j=1}^{l} \dfrac{f_{i_j}}{\alpha_1^{i_j}\alpha_2^{i_j}} \in H^*_{G_{\mathbb{R}}} & \textit{if } \{p_{i_1}, \dots, p_{i_l}\} = Z^{G_{\mathbb{R}}}_{H_{\mathbb{R}}} \textit{ and } \dim Z_{H_{\mathbb{R}}} = 2 \end{cases}$$

for all subgroups $H_{\mathbb{R}}$ *of* $G_{\mathbb{R}}$ *of order* $|H_{\mathbb{R}}| = 2^{n-1}$ *and all connected components* $Z_{H_{\mathbb{R}}}$ *of* $X^{H_{\mathbb{R}}}$, *where* $\alpha_1^{i_j}$ *and* $\alpha_2^{i_j}$ *are the* (*linearly dependent*) *weights of the* $G_{\mathbb{R}}$ *action on* $T_{p_{i_j}} Z_{H_{\mathbb{R}}}$.

We re-emphasize that Duistermaat's techniques only apply to additive cohomology. Since we are able to obtain results concerning the ring structure of the equivariant cohomology and its relationship to ordinary cohomology, we also obtain statements about the ring structure of the ordinary cohomology as well. Indeed, in many cases, Duistermaat's isomorphism (1.4) turns out to give a ring isomorphism. (See Corollaries 5.7 and 5.8 to Theorem C and Corollaries 6.7 and 6.8 to Theorem D.) When describing these ring isomorphisms, we will make use of the following notation. The symbol

$$H^{2*}(M; \mathbb{Z}/2\mathbb{Z})$$

will denote the subring

$$\bigoplus_i H^{2i}(M; \mathbb{Z}/2\mathbb{Z}) \subseteq H^*(M; \mathbb{Z}/2\mathbb{Z}),$$

endowed with a new grading wherein a class in $H^{2i}(M; \mathbb{Z}/2\mathbb{Z})$ is given degree i (and similarly for equivariant cohomology). Then under suitable hypotheses, the additive isomorphism of Duistermaat becomes an isomorphism of graded rings.

In Section 7, we discuss an application of our main theorems to string theory. The $\mathbb{Z}/2\mathbb{Z}$-equivariant cohomology ring of T^n with $\mathbb{Z}/2\mathbb{Z}$ coefficients classifies all possible orientifold configurations of Type II string theories, compactified on T^n. We explain how to compute this cohomolgy ring.

The last section of the paper contains some applications of these results to elementary problems in combinatorics. A typical such application is the following. Let Γ be the permutahedron, the Cayley graph of the symmetric group S_n with edges generated by transpositions. By definition, the vertices of Γ are elements of S_n and two vertices σ and τ are joined by an edge if $\tau\sigma^{-1}$ is a transposition. Our goal is to attach to each vertex σ a subset S_σ of $\{1, \dots, n\}$ such that, for all pairs σ and τ of adjacent vertices, either $S_\sigma = S_\tau$ or the symmetric difference

$$(S_\sigma - S_\tau) \cup (S_\tau - S_\sigma)$$

is $\{i, j\}$, where $\tau\sigma^{-1} = (ij)$.

Let $\mathscr{F}_{n+1}$ be the real flag variety in $n+1$ dimensions. We will prove that the set of solutions to this problem can be identified with the set

$$H^*_{G_{\mathbb{R}}}(\mathscr{F}_{n+1}; \mathbb{Z}/2\mathbb{Z}),$$

where $G_{\mathbb{R}} = \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$ is the n-fold product of $\mathbb{Z}/2\mathbb{Z}$. The results in this section were inspired by a remark of Ethan Bolker, who pointed out to us that $\mathbb{Z}/2\mathbb{Z}$ representation theory is simply Boolean algebra.

We conclude these prefatory remarks with a few comments about the $\mathbb{Z}/2\mathbb{Z}$ coefficients. We recall Witten's recipe for computing the homology of compact manifolds by Morse theory. Let X be a compact manifold, $f : X \to \mathbb{R}$ a Morse function, and C_f^i the index i critical set of f. Let $\mathscr{C}_i$ be the vector space

$$\mathscr{C}_i = \bigoplus_{p_i \in C_f^i} p_i \mathbb{R}$$

with basis C_f^i.

Equip X with a Riemannian metric and let v be the gradient vector field of f. For generic metrics, the stable and unstable manifolds of v intersect transversally. In particular, for every critical point $p \in C_f^i$, there are a finite number of gradient curves joining p to critical points

$$q_1, \ldots, q_m \in C_f^{i-1}. \tag{1.16}$$

Moreover, each of these points can be assigned an intrinsic orientation $\varepsilon(p, q_j) \in \{\pm 1\}$. Now define a boundary operator

$$\partial : \mathscr{C}_i \to \mathscr{C}_{i-1}$$

by setting

$$\partial p = \sum \varepsilon(p, q_j) \cdot q_j.$$

Witten [W] was the first to explicitly formulate Morse theory in this way; he showed that ∂ *is* a boundary operator, namely $\partial^2 = 0$, and that $H_*(X; \mathbb{R})$ is the homology of the complex $(\mathscr{C}, \partial)$.

In particular, when all the critical points are of even index, ∂ is automatically zero. Thus, one gets

$$\dim(H_i(X; \mathbb{R})) = \begin{cases} 0, & i \text{ odd}, \\ \#\{p \in C_f^i\}, & i \text{ even}. \end{cases} \tag{1.17}$$

This fact is key to Atiyah's result (1.1). He observes that if $\# M^G < \infty$ and if f is a generic component of the moment map, then f is a Morse function with critical points all of even index, so (1.1) is a special case of (1.17).

This recipe for computing homology also works in characteristic two; however, when $\mathbb{Z}/2\mathbb{Z}$ symmetries are present, the gradient curves joining the points $p \in C_f^i$ to the points on list (1.16) often occur in pairs. For instance, for manifolds which satisfy the GKM hypotheses, these pairs of curves correspond to the edges e_i of the graph Γ. That is, each $E_i^\sigma \cong \mathbb{R}P^1$ in (1.15) contains of a pair of gradient curves joining the two vertices of e_i. Hence, the mod 2 version of ∂ is identically zero.

One can obtain Duistermaat's result by exploiting this phenomenon. The goal of this article is to push these ideas forward by systematically applying these techniques to the equivariant setting.

2. The equivariant cohomology of the real locus

Recall M^{2d} is a symplectic manifold with a Hamiltonian action τ of a torus $G = T^n$. Suppose further that there is an anti-symplectic involution $\sigma : M \to M$ with the property that

$$\sigma \circ \tau_g = \tau_{g^{-1}} \circ \sigma.$$

Let $X = M^\sigma$ be the fixed point set of σ. We call X the *real locus* of M. Recall that Duistermaat proved the following equality, computing the ordinary cohomology of the real locus.

$$H^*(X; \mathbb{Z}/2\mathbb{Z}) = \sum_{i \in I} H^{*-\frac{d_i}{2}}(F_i^\sigma; \mathbb{Z}/2\mathbb{Z}), \tag{2.1}$$

where $I \subseteq \{1, \ldots, N\}$ is the set for which σ preserves F_i, and the d_i are the indices of the fixed point sets F_i.

We will prove the equivariant analogue (1.6) to this equality, computing the additive structure of the equivariant cohomology:

$$H^*_{G_\mathbb{R}}(X; \mathbb{Z}/2\mathbb{Z}) = \sum_{i \in I} H^{*-\frac{d_i}{2}}_{G_\mathbb{R}}(F_i^\sigma; \mathbb{Z}/2\mathbb{Z}).$$

Proof of Theorem A. Consider the product action of T^n on

$$M \times \underbrace{(\mathbb{C}^d \times \cdots \times \mathbb{C}^d)}_{n}$$

in which each S^1 factor acts by multiplication on the corresponding factor of $\mathbb{C}^d$. This is a Hamiltonian action. If $(\phi_1, \ldots, \phi_n) = \Phi : M \to \mathbb{R}^n$ is the moment map associated with τ, then the moment map of this product action is $\Psi = (\psi_1, \ldots, \psi_n)$, with

$$\psi_i(m, z_{1,1}, \ldots, z_{1,d}, \ldots, z_{d,d}) = \phi_i(m) + \sum_{j=1}^{d} |z_{i,j}|^2.$$

Let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. If $a_i > \sup(\phi_i)$ for every i, then $\Psi^{-1}(a)$ and $M \times S^{2d-1} \times \cdots \times S^{2d-1}$ are equivariantly diffeomorphic, so the reduced space

$$M_{\text{red}} = M /\!/_a \, T^n = \psi^{-1}(a)/T^n$$

is diffeomorphic to $M \times_{T^n} (S^{2d-1} \times \cdots \times S^{2d-1})$. Moreover, there is another action of T^n on $M \times \mathbb{C}^d \times \cdots \times \mathbb{C}^d$, namely τ coupled with the trivial action on $(\mathbb{C}^d)^n$. Since this commutes with the product action, it induces a Hamiltonian action of T^n on M_{red}. In addition, one gets from σ an involution

$$(m, z_1, \dots, z_d) \mapsto (\sigma(m), \overline{z_1}, \dots, \overline{z_d})$$

of $M \times \mathbb{C}^d \times \cdots \times \mathbb{C}^d$. This induces an anti-symplectic involution $\tilde{\sigma}$ on M_{red}. Thus, one can apply Duistermaat's theorem to M_{red} to get a formula for the cohomology of the space

$$M_{\text{red}}^{\tilde{\sigma}} = X \times_{G_{\mathbb{R}}} (S^{d-1} \times \cdots \times S^{d-1})$$

in terms of the cohomology of the spaces

$$Z_i^d \coloneqq F_i^\sigma \times_{G_{\mathbb{R}}} (S^{d-1} \times \cdots \times S^{d-1}) = F_i^\sigma \times (\mathbb{R}P^{d-1} \times \cdots \times \mathbb{R}P^{d-1}).$$

Now $F_i^\sigma \times BG_{\mathbb{R}}$ is obtained from Z_i^d by attaching cells of dimension d and higher. So, for fixed k, the sequence $H^k(Z_i^d; \mathbb{Z}/2\mathbb{Z})$ stabilizes as d grows large, and moreover is equal to the equivariant cohomology of X. Thus one obtains from (2.1) the following real analogue:

$$H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z}) = \sum H^{*-\frac{d_i}{2}}_{G_{\mathbb{R}}}(F_i^\sigma; \mathbb{Z}/2\mathbb{Z}),$$

where $G_{\mathbb{R}} = \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$. $\square$

3. A spectral sequence

The goal of this section is to determine the structure of $H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z})$ as a module over the ring $H^*_{G_{\mathbb{R}}} = \mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_n]$. We do this by calculating the E_2-term of the Leray–Serre spectral sequence converging to $H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z})$ and deducing, by dimensional considerations obtained in the previous section, that the spectral sequence must collapse. This then gives us the desired statement about the E_∞-term and hence $H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z})$.

Recall that by definition, we have

$$H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z}) = H^*(X \times_{G_{\mathbb{R}}} EG_{\mathbb{R}}; \mathbb{Z}/2\mathbb{Z}),$$

where $EG_{\mathbb{R}}$ is the total space of the universal $G_{\mathbb{R}}$-bundle. Denote this fiber product by E. We then have a fibration $p : E \to BG_{\mathbb{R}}$ with fiber X. Let $\mathscr{H}^*X$ denote the local coefficient system on $BG_{\mathbb{R}}$ associated to this fibration. The E_2-term we would like to compute is then

$$E_2 = H^*(BG_{\mathbb{R}}; \mathscr{H}^*X).$$

The computation takes place in two steps. The first step, which is the technical heart of the argument, consists of carrying out the computation in the one-dimensional case. The remainder of the proof consists of a relatively straightforward exercise in bookkeeping.

Lemma 3.1. *Let* $G_{\mathbb{R}} = \mathbb{Z}/2\mathbb{Z}$, *so that* $BG_{\mathbb{R}} = K(\mathbb{Z}/2\mathbb{Z}, 1) = \mathbb{R}P^{\infty}$. *Then* $H^*(BG_{\mathbb{R}}; \mathscr{H}^*X)$ *is generated over* $H^*_{G_{\mathbb{R}}} = (\mathbb{Z}/2\mathbb{Z})[x]$ *in degree zero by* $H^*(X)^{G_{\mathbb{R}}}$. *Moreover, the only relation is given by* $x \cdot (\alpha + v(\alpha)) = 0$ *for* $\alpha \in H^*(X)$ *and* $v \in G_{\mathbb{R}}$.

Proof. By definition, the cohomology $H^*(BG_{\mathbb{R}}; \mathscr{H}^*X)$ that we would like to compute is the group cohomology $H^*(G_{\mathbb{R}}; H^*(X))$ with respect to the natural action of $G_{\mathbb{R}}$ on $H^*(X)$; we will henceforth pass back and forth between these two notations without comment. Our goal is to understand the cohomology as a module over the cohomology ring $H^*(G_{\mathbb{R}}; \mathbb{Z}/2\mathbb{Z})$ corresponding to the trivial action of $G_{\mathbb{R}}$ on $\mathbb{Z}/2\mathbb{Z}$.

Denote the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ by v. Consider the $G_{\mathbb{R}}$-module $H^*(X) \oplus H^*(X)$ with $G_{\mathbb{R}}$-action defined by the equation $v(\alpha, \beta) = (v(\beta), v(\alpha))$ for all $\alpha, \beta \in H^*(X)$. We then get a short exact sequence of $G_{\mathbb{R}}$-modules

$$0 \to H^*(X) \xrightarrow{f} H^*(X) \oplus H^*(X) \xrightarrow{g} H^*(X) \to 0, \tag{3.1}$$

where $f(\alpha) = (\alpha, \alpha)$ and $g(\alpha, \beta) = \alpha + \beta$. Of course, it is completely essential to this identification of the cokernel of f that we work over $\mathbb{Z}/2\mathbb{Z}$; in order for this sequence to be exact, we must have $g(\alpha, \beta) = \alpha - \beta$, but this would not be a map of $\mathbb{Z}/2\mathbb{Z}$-modules if we did not also have $g(\alpha, \beta) = \alpha + \beta$.

We would like to consider the long exact cohomology sequence associated with (3.1). Observe that we have isomorphisms

$$\begin{aligned} H^*(X) \oplus H^*(X) &\cong \mathbb{Z}/2\mathbb{Z}[G_{\mathbb{R}}] \otimes_{\mathbb{Z}/2\mathbb{Z}} H^*(X) \\ &\cong \mathbb{Z}[G_{\mathbb{R}}] \otimes_{\mathbb{Z}} H^*(X) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G_{\mathbb{R}}], H^*(X)) \end{aligned}$$

of $G_{\mathbb{R}}$-modules. The first isomorphism is more or less the definition of the left-hand side, the second follows from the fact that $H^*(X)$ is 2-torsion, and the third follows from the finiteness of $G_{\mathbb{R}}$. Thus, the module $H^*(X) \oplus H^*(X)$ is co-induced and its higher cohomology vanishes. Moreover, the map $\alpha \mapsto (\alpha, v(\alpha))$ provides an isomorphism of $H^*(X)$ with $(H^*(X) \oplus H^*(X))^{G_{\mathbb{R}}}$, so we have

$$H^0(G_{\mathbb{R}}, H^*(X) \oplus H^*(X)) \cong H^*(X)$$

and, of course, $H^0(G_{\mathbb{R}}, H^*(X)) = H^*(X)^{G_{\mathbb{R}}}$. The long exact sequence in question, therefore, takes the following form:

$$0 \longrightarrow H^*(X)^{G_{\mathbb{R}}} \longrightarrow H^*(X) \longrightarrow H^*(X)^{G_{\mathbb{R}}} \longrightarrow H^1(G_{\mathbb{R}}; H^*(X)) \longrightarrow 0 \longrightarrow H^1(G_{\mathbb{R}}; H^*(X)) \longrightarrow H^2(G_{\mathbb{R}}; H^*(X)) \longrightarrow 0 \longrightarrow \cdots$$

The first map in this sequence is the natural inclusion, and the second sends α to $\alpha + v(\alpha)$. Therefore, $H^1(BG_{\mathbb{R}}; \mathscr{H}^*X)$ is the sought-after quotient of $H^*(X)^{G_{\mathbb{R}}}$ by the subgroup of all elements of the form $\alpha + v(\alpha)$; as a result, so is each $H^n(BG_{\mathbb{R}}; \mathscr{H}^*X)$ for $n \geqslant 1$. Recall that $H^*(BG_{\mathbb{R}}) = (\mathbb{Z}/2\mathbb{Z})[x]$ where x is a class of degree 1. The desired result then follows from the fact that the connecting homomorphisms are multiplication by x. $\square$

The remainder of the game consists in playing the results of Lemma 3.1 and Theorem A off of one another.

Corollary 3.2. *The action of $G_{\mathbb{R}}$ on $H^*(X)$ is trivial. Thus, in the case that $G_{\mathbb{R}} = \mathbb{Z}/2\mathbb{Z}$, Theorem B holds.*

Proof. Lemma 3.1 computes the E_2-term of the Leray–Serre spectral sequence converging to $H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z})$ in the case $G_{\mathbb{R}} = \mathbb{Z}/2\mathbb{Z}$. Now, the dimensions of the graded pieces of this E_2-term are maximized precisely when $G_{\mathbb{R}}$ acts trivially on $H^*(X)$. Moreover, the results of the previous section tell us that the graded pieces of $H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z})$ have exactly these maximal dimensions. Since the E_∞-term of a spectral sequence can only be as large as its E_2-term, this tells us that the action must be trivial and further that

$$H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z}) = E_\infty = E_2.$$

In this case, $H^*(X)^{G_{\mathbb{R}}} = H^*(X)$, and the relation $x \cdot (\alpha + v(\alpha)) = 0$ is automatically satisfied, so Lemma 3.1 tells us that $H^*(BG_{\mathbb{R}}, \mathscr{H}^*X)$ is a free module over $\mathbb{Z}/2\mathbb{Z}[x]$ generated in degree zero by $H^*(X)$. This completes the proof of Theorem B in the case $G_{\mathbb{R}} = \mathbb{Z}/2\mathbb{Z}$.

The triviality of the action of $G_{\mathbb{R}}$ on $H^*(X)$ for higher-dimensional $G_{\mathbb{R}}$ follows by restricting to arbitrary one-dimensional subtori. $\square$

The fact that $G_{\mathbb{R}}$ acts trivially on $H^*(X)$ is not new; it can also be derived from Duistermaat's original argument. Indeed, Duistermaat's isomorphism can be seen to be $G_{\mathbb{R}}$-equivariant, and the result then follows from the connectedness of the torus. Using this fact would have somewhat simplified our argument, but we chose to give the above proof so as to avoid appealing to unpublished modifications of the literature.

We now have the technical input to handle the general case.

Lemma 3.3. *Let $G_{\mathbb{R}} \cong (\mathbb{Z}/2\mathbb{Z})^n$, for any positive integer n. Then $H^*(BG_{\mathbb{R}}; \mathscr{H}^*X)$ is a free module over $H^*_{G_{\mathbb{R}}} = (\mathbb{Z}/2\mathbb{Z})[x_1, \ldots, x_n]$ generated by $H^0(BG_{\mathbb{R}}; H^*_{G_{\mathbb{R}}}) \cong H^*(X)$.*

Proof. This is now completely classical. By Corollary 3.2, the action of $G_{\mathbb{R}}$ on $H^*(X)$ is trivial, so that $\mathscr{H}^*X$ is actually the constant sheaf $H^*(X)$. Therefore, we have

$$H^*(BG_{\mathbb{R}}; \mathscr{H}^*X) = H^*(BG_{\mathbb{R}}; \mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}/2\mathbb{Z}} H^*(X) = \mathbb{Z}/2\mathbb{Z}[x_1, \ldots, x_n] \otimes_{\mathbb{Z}/2\mathbb{Z}} H^*(X).$$

This completes the proof. □

Our goal is now entirely within reach; we need only combine the results we have proven so far as in the proof of Corollary 3.2 to establish the $H^*_{G_{\mathbb{R}}}$-module isomorphism

$$H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z}) \cong H^*_{G_{\mathbb{R}}} \otimes_{\mathbb{Z}/2\mathbb{Z}} H^*(X; \mathbb{Z}/2\mathbb{Z}). \tag{3.2}$$

Proof of Theorem B. Lemma 3.3 tells us that the E_2-term of the Leray–Serre spectral sequence converging to $H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z})$ takes precisely the form that $H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z})$ itself is asserted to have. However, the results of the previous section tell us that the graded pieces of $H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z})$ have the same dimension as those of this E_2-term, and hence that the spectral sequence collapses. Therefore,

$$H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z}) = E_\infty = E_2,$$

so $H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z})$ is a free module over $H^*_{G_{\mathbb{R}}}$ generated in dimension zero. Thus, its additive structure is as given by (3.2). □

4. Chang–Skjelbred in $\mathbb{Z}/2\mathbb{Z}$

As a result of the collapse of the spectral sequence proved in the previous section, the map

$$i^* : H^*_{G_{\mathbb{R}}}(X; \mathbb{Z}/2\mathbb{Z}) \to H^*_{G_{\mathbb{R}}}(X^{G_{\mathbb{R}}}; \mathbb{Z}/2\mathbb{Z})$$

is an injection. In the case of the original manifold M, the Chang–Skjelbred theorem [CS] identifies the image of this map. We prove a $\mathbb{Z}/2\mathbb{Z}$ version of that theorem here.

As usual, let $G_{\mathbb{R}} = (\mathbb{Z}/2\mathbb{Z})^n$ be the n-dimensional "real torus," M a $G_{\mathbb{R}}$-manifold, and $M^{G_{\mathbb{R}}}$ the fixed point set of the action; we let $i : M^{G_{\mathbb{R}}} \hookrightarrow M$ denote the inclusion.

Theorem 4.1. *Suppose that* $H^*_{G_\mathbb{R}}(M;\mathbb{Z}/2\mathbb{Z})$ *is a free* $H^*_{G_\mathbb{R}}$*-module. For a subgroup* $H_\mathbb{R} < G^*_\mathbb{R}$, let $i_{H_\mathbb{R}} : M^{G_\mathbb{R}} \hookrightarrow M^{H_\mathbb{R}}$ *denote the inclusion. Then we have*

$$i^* H^*_{G_\mathbb{R}}(M;\mathbb{Z}/2\mathbb{Z}) = \bigcap_{\substack{H_\mathbb{R} < G^*_\mathbb{R} \\ |H_\mathbb{R}| = 2^{n-1}}} i^*_{H_\mathbb{R}} H^*_{G_\mathbb{R}}(M^{H_\mathbb{R}};\mathbb{Z}/2\mathbb{Z}).$$

Our proof closely models the argument given in [GS], with appropriate modifications.

First of all, recall that

$$H^*_{G_\mathbb{R}} = H^*((\mathbb{R}P^\infty)^n;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x_1,\dots,x_n],$$

with $\deg(x_i) = 1$. Moreover, we may view each x_i as a linear functional $x_i : G_\mathbb{R} \to \mathbb{Z}/2\mathbb{Z}$, that is, an element of $G^*_\mathbb{R}$. This allows us to identify $H^*_{G_\mathbb{R}}$ with the symmetric algebra $S(G^*_\mathbb{R})$, a fact analogous to the ordinary identification $H^*_G \cong S(\mathfrak{g}^*)$ when G is a torus with Lie algebra $\mathfrak{g}$. This allows us to view elements of $H^*_{G_\mathbb{R}}$ as polynomial functions on $G_\mathbb{R}$.

Lemma 4.2. *Let* $K_\mathbb{R} < G_\mathbb{R}$ *be a subgroup and* $\phi : M \to G_\mathbb{R}/K_\mathbb{R}$ *a* $G_\mathbb{R}$*-equivariant map. If* $\rho \in S(G^*_\mathbb{R})$ *annihilates* $K_\mathbb{R}$*, then it must also kill* $H^*_{G_\mathbb{R}}(M;\mathbb{Z}/2\mathbb{Z})$.

Proof. We have a sequence of $G_\mathbb{R}$-equivariant maps

$$M \to G_\mathbb{R}/K_\mathbb{R} \to pt. \tag{4.1}$$

Note that

$$\begin{aligned} H^*_{G_\mathbb{R}}(G_\mathbb{R}/K_\mathbb{R}) &= H^*(G_\mathbb{R}/K_\mathbb{R} \times_{G_\mathbb{R}} EG_\mathbb{R};\mathbb{Z}/2\mathbb{Z}) \\ &= H^*(BK_\mathbb{R};\mathbb{Z}/2\mathbb{Z}) \\ &= S(K^*_\mathbb{R}). \end{aligned}$$

Sequence (4.1) of $G_\mathbb{R}$-spaces therefore gives rise to the following diagram of algebras:

$$\begin{array}{ccccc} H^*_{G_\mathbb{R}}(M;\mathbb{Z}/2\mathbb{Z}) & \longleftarrow & H^*_{G_\mathbb{R}}(G_\mathbb{R}/K_\mathbb{R}) & \longleftarrow & H^*_{G_\mathbb{R}} \\ & & \| & & \| \\ & & S(K^*_\mathbb{R}) & & S(G^*_\mathbb{R}) \end{array}$$

Therefore, the map $S(G^*_\mathbb{R}) \to H^*_{G_\mathbb{R}}(M;\mathbb{Z}/2\mathbb{Z})$ defining the module structure factors as

$$S(G^*_\mathbb{R}) \to S(K^*_\mathbb{R}) \to H^*_{G_\mathbb{R}}(M;\mathbb{Z}/2\mathbb{Z}),$$

and the proof is complete. □

This observation furnishes us with the fundamental tool in proving localization theorems for equivariant cohomology.

Proposition 4.3. *Let X be a closed $G_{\mathbb{R}}$-invariant submanifold of M. For some positive integer L, there exist subgroups $(K_{\mathbb{R}})_1, \ldots, (K_{\mathbb{R}})_L$ of G, each of which is an isotropy subgroup of some point $p \in M \setminus X$, such that for any $\alpha_1, \ldots, \alpha_L \in G^*_{\mathbb{R}}$ with $\alpha_i|_{(K_{\mathbb{R}})_i} = 0$, the product $\alpha_1 \alpha_2^2 \cdots \alpha_L^2 \in H^*_{G_{\mathbb{R}}}$ kills $H^*_{G_{\mathbb{R}}}(M \setminus X)$.*

Proof. Let U be a $G_{\mathbb{R}}$-invariant tubular neighborhood of X; it suffices to prove the desired result for the module $H^*_{G_{\mathbb{R}}}(M \setminus U; \mathbb{Z}/2\mathbb{Z})$. Now, given any orbit X_i in $M \setminus U$ with isotropy subgroup $(K_{\mathbb{R}})_i$, we may find a $G_{\mathbb{R}}$-invariant open neighborhood U_i of X_i admitting a $G_{\mathbb{R}}$-equivariant map $U_i \to G_{\mathbb{R}}/(K_{\mathbb{R}})_i$. By compactness, we may cover $M \setminus U$ by finitely many such sets $U_1, \ldots, U_L$. We now show by induction that for all $r \leqslant L$, if $\alpha_1, \ldots, \alpha_r$ are elements of $G^*_{\mathbb{R}}$ with $\alpha_i|_{(K_{\mathbb{R}})_i} = 0$, then $\alpha_1 \alpha_2^2 \cdots \alpha_r^2$ annihilates $H^*_{G_{\mathbb{R}}}(U_1 \cup \cdots \cup U_r; \mathbb{Z}/2\mathbb{Z})$.

The case $r = 1$ is simply a restatement of Lemma 4.2. For the inductive step, consider the Meyer–Vietoris sequence associated to the cover $U_1 \cup \cdots \cup U_r = (U_1 \cup \cdots \cup U_{r-1}) \cup U_r$. Denoting $U_1 \cup \cdots \cup U_{r-1}$ by V, we find an exact sequence

$$H^k_{G_{\mathbb{R}}}(V \cap U_r) \to H^{k+1}(V \cup U_r) \to H^{k+1}_{G_{\mathbb{R}}}(V) \oplus H^{k+1}_{G_{\mathbb{R}}}(U_r).$$

Now, since $V \cap U_r \subset U_r$, we have a $G_{\mathbb{R}}$-equivariant map $V \cap U_r \to X_r$, and so the left-hand term of the sequence is killed by α_r. Meanwhile, by induction, the right-hand term is annihilated by $\alpha_1 \alpha_2^2 \cdots \alpha_{r-1}^2 \alpha_r$, and so the product $\alpha_1 \alpha_2^2 \cdots \alpha_r^2$ kills the middle term. □

We will also need a relative version of the same result.

Proposition 4.4. *Under the hypotheses of Proposition* 4.3, *the module $H^*_{G_{\mathbb{R}}}(M, X)$ is annihilated by the element $\alpha_1^2 \alpha_2^4 \cdots \alpha_L^4 \in H^*_{G_{\mathbb{R}}}$.*

Proof. Of course, the map of pairs $(M, X) \to (M/X, X)$ is an equivalence, so it suffices to compute $H^*_{G_{\mathbb{R}}}(M/X, X)$. Once again, let U be a $G_{\mathbb{R}}$-equivariant tubular neighborhood of X. We cover M/X by two open sets U/X and $M \setminus X$; since, the projection map $U/X \to X/X$ is an equivalence, so we may identify $H^*_{G_{\mathbb{R}}}(M, X)$ with the kernel of the map $H^*_{G_{\mathbb{R}}}(M) \to H^*_{G_{\mathbb{R}}}(U)$. Now, let us write the Meyer–Vietoris sequence for this cover:

$$H^{k-1}_{G_{\mathbb{R}}}(U \setminus X) \to H^k_{G_{\mathbb{R}}}(M/X) \to H^k_{G_{\mathbb{R}}}(U/X) \oplus H^k_{G_{\mathbb{R}}}(M \setminus X).$$

But by the above discussion, this gives rise to the following exact sequence:

$$H^{k-1}_{G_{\mathbb{R}}}(U \setminus X) \to H^k_{G_{\mathbb{R}}}(M, X) \to H^k_{G_{\mathbb{R}}}(M \setminus X).$$

Now, Proposition 4.3 applies to both ends of this sequence, so the middle term is killed by $(\alpha_1\alpha_2^2\cdots\alpha_L^2)^2 = \alpha_1^2\alpha_2^4\cdots\alpha_L^4$. $\square$

Proposition 4.4 gives us the basic localization results we will need.

Corollary 4.5. *In the setting of Proposition* 4.3, *the kernel and cokernel of the map*

$$i^* : H^*_{G_\mathbb{R}}(M; \mathbb{Z}/2\mathbb{Z}) \to H^*_{G_\mathbb{R}}(X; \mathbb{Z}/2\mathbb{Z})$$

are annihilated by the same element $\alpha_1^2\alpha_2^4\ldots\alpha_L^4 \in H^*_{G_\mathbb{R}}$.

Proof. Simply apply Proposition 4.4 to the exact sequence

$$H^k_{G_\mathbb{R}}(M, X) \to H^k_{G_\mathbb{R}}(M) \to H^k_{G_\mathbb{R}}(X) \to H^{k+1}_{G_\mathbb{R}}(M, X). \qquad \square$$

Corollary 4.6. *The kernel of the map* $i^* : H^*_{G_\mathbb{R}}(M) \to H^*_{G_\mathbb{R}}(M^{G_\mathbb{R}})$ *is torsion, and hence trivial when* $H^*_{G_\mathbb{R}}(M; \mathbb{Z}/2\mathbb{Z})$ *is a free module.*

Corollary 4.7. *For every subgroup* $K_\mathbb{R} < G_\mathbb{R}$, *there exists a monomial* $p = \alpha_1\ldots\alpha_N$ *annihilating the cokernel of the map* $i^* : H^*_{G_\mathbb{R}}(M) \to H^*_{G_\mathbb{R}}(M^{K_\mathbb{R}})$ *such that no* α_i *vanishes on* $K_\mathbb{R}$.

Proof. Let $q \in M \setminus M^{K_\mathbb{R}}$, and let $K'_\mathbb{R}$ be its isotropy subgroup. Since obviously $K'_\mathbb{R} \not\supset K_\mathbb{R}$, there is an $\alpha' : G_\mathbb{R} \to \mathbb{Z}/2\mathbb{Z}$ with $\alpha'|_{K'_\mathbb{R}} = 0$ and $\alpha'|_{K_\mathbb{R}} \neq 0$. But by Corollary 4.5, we can find a monomial which is a product of elements of the form α' and annihilates coker i^*. $\square$

Finally, we are ready to prove our main result.

Proof of Theorem 4.1. First of all, since the map i factors as

$$M^{G_\mathbb{R}} \overset{i_{H_\mathbb{R}}}{\hookrightarrow} M^{H_\mathbb{R}} \hookrightarrow M,$$

we know that for all $H_\mathbb{R}$, the inclusion $im(i^*) \subset im(i^*_{H_\mathbb{R}})$ holds.

For the other direction, recall first that $H^*_{G_\mathbb{R}}(M; \mathbb{Z}/2\mathbb{Z})$ is free; therefore, by Corollary 4.6 the map i^* is injective and we may consequently view $H^*_{G_\mathbb{R}}(M; \mathbb{Z}/2\mathbb{Z})$ as a submodule of $H^*_{G_\mathbb{R}}(M^{G_\mathbb{R}}; \mathbb{Z}/2\mathbb{Z})$. Suppose $\{e_1, \ldots, e_k\}$ is an $S(G^*_\mathbb{R})$-basis for $H^*_{G_\mathbb{R}}(M; \mathbb{Z}/2\mathbb{Z})$. By Corollary 4.7, there is a monomial $p = \alpha_1\cdots\alpha_N$ with $pe \in H^*_{G_\mathbb{R}}(M; \mathbb{Z}/2\mathbb{Z})$ for every $e \in H^*_{G_\mathbb{R}}(M^{G_\mathbb{R}}; \mathbb{Z}/2\mathbb{Z})$. Thus, we may write

$$pe = f_1e_1 + \cdots + f_ke_k$$

for unique $f_i \in S(G_{\mathbb{R}}^*)$. Now, since $S(G_{\mathbb{R}}^*)$ is a unique factorization domain, we may divide both sides of this identity by p and cancel common factors to obtain the formula

$$e = \frac{g_1}{p_1} e_1 + \cdots + \frac{g_k}{p_k} e_k, \tag{4.2}$$

where the g_i are uniquely determined elements of $S(G_{\mathbb{R}}^*)$ and the p_i are uniquely determined divisors of p such that g_i and p_i are relatively prime.

Suppose now that e were actually in $im(i_{H_{\mathbb{R}}}^*)$. We may find a subset $\{j_1, \ldots, j_R\} \subset \{1, \ldots, N\}$ such that no α_{j_i} kills $H_{\mathbb{R}}$, and $q = \alpha_{j_1} \cdots \alpha_{j_R}$ annihilates the cokernel of the map

$$H_{G_{\mathbb{R}}}^*(M; \mathbb{Z}/2\mathbb{Z}) \to H_{G_{\mathbb{R}}}^*(M^{H_{\mathbb{R}}}; \mathbb{Z}/2\mathbb{Z}).$$

Therefore, multiplying both sides of (4.2) by q, we find that

$$\alpha_{j_1} \cdots \alpha_{j_R} e = h_1 e_1 + \cdots + h_k e_k$$

with $h_i \in S(G_{\mathbb{R}}^*)$. Thus, in (4.2), none of the weights $\alpha : G \to \mathbb{Z}/2\mathbb{Z}$ that divide the denominators p_i vanish on $H_{\mathbb{R}}$. Hence, if $e \in im(i_{H_{\mathbb{R}}}^*)$ for all $H_{\mathbb{R}}$, then $p_i = 1$ for all i, and so (4.2) tells us that

$$e = g_1 e_1 + \cdots + g_k e_k \in H_{G_{\mathbb{R}}}^*(M; \mathbb{Z}/2\mathbb{Z})$$

and the proof is complete. □

Now suppose that $Z_{H_{\mathbb{R}}}$ is a connected component of $M^{H_{\mathbb{R}}}$ for some subgroup $H_{\mathbb{R}}$ of $G_{\mathbb{R}}$ of order $|H_{\mathbb{R}}| = 2^{n-1}$. Let $i_{Z_{H_{\mathbb{R}}}}$ be the inclusion

$$i_{Z_{H_{\mathbb{R}}}} : Z_{H_{\mathbb{R}}}^{G_{\mathbb{R}}} \to Z_{H_{\mathbb{R}}}$$

of the fixed points of $Z_{H_{\mathbb{R}}}$ into $Z_{H_{\mathbb{R}}}$. Let $r_{Z_{H_{\mathbb{R}}}}$ be the inclusion

$$r_{Z_{H_{\mathbb{R}}}} : Z_{H_{\mathbb{R}}}^{G_{\mathbb{R}}} \to M^{G_{\mathbb{R}}}$$

of the fixed points of $Z_{H_{\mathbb{R}}}$ into all of the fixed points. Then, we have the following corollary of Theorem 4.1.

Corollary 4.8. *Suppose that $H_{G_{\mathbb{R}}}^*(M; \mathbb{Z}/2\mathbb{Z})$ is a free $H_{G_{\mathbb{R}}}^*$-module. A class*

$$f \in H_{G_{\mathbb{R}}}^*(M^{G_{\mathbb{R}}}; \mathbb{Z}/2\mathbb{Z})$$

is in the image of i^* if and only if

$$r_{Z_{H_{\mathbb{R}}}}^*(f) \in i_{Z_{H_{\mathbb{R}}}}^*(H_{G_{\mathbb{R}}}^*(Z_{H_{\mathbb{R}}}; \mathbb{Z}/2\mathbb{Z}))$$

for every subgroup $H_{\mathbb{R}}$ of $G_{\mathbb{R}}$ of order $|H_{\mathbb{R}}| = 2^{n-1}$ and every connected component $Z_{H_{\mathbb{R}}}$ of $M^{H_{\mathbb{R}}}$.

Proof. The proof is analogous to the proof of Theorem 1 in [GH]. It follows directly from Theorem 4.1. □

5. Real GKM

The goal of this section is to prove an analogue of Theorem 1.1 for the real locus X of M. The proof will require two hypotheses on X, namely

$$X^{G_{\mathbb{R}}} = M^G \tag{5.1}$$

and

$$X^{(1)} = X \cap M^{(1)}, \tag{5.2}$$

where $M^{(1)}$ is the one-skeleton of M and $X^{(1)}$ the one-skeleton of X. We will begin by analyzing these conditions and their implications. We first note that the analogues for M of the conditions (5.1) and (5.2), namely

$$\# M^G < \infty \tag{5.3}$$

and

$$\dim(M^{(1)}) \leqslant 2, \tag{5.4}$$

have a very simple and elegant interpretation in terms of the isotropy representations of G at fixed points of M.

Theorem 5.1. *The conditions $\# M^G < \infty$ and $\dim(M^{(1)}) \leqslant 2$ are satisfied if and only if, for every $p \in M^G$, the weights $\tilde{\alpha}_{i,p}$, $i = 1, \dots, d$ of the isotropy representation of G on T_pM are pair-wise linearly independent, that is for $i \neq j$, $\tilde{\alpha}_{i,p}$ is not a multiple of $\tilde{\alpha}_{j,p}$.*

For the proof of this, see [GZ]. When M satisfies the two conditions (5.1) and (5.2), we say that M is a *GKM manifold*. Let $\mathbb{Z}_G^*$ be the weight lattice of G. By the mod 2 reduction of a weight $\tilde{\alpha} \in \mathbb{Z}_G^*$, we mean its image α in $\mathbb{Z}_G^*/2\mathbb{Z}_G^*$. We will prove a real analogue of Theorem 5.1.

Theorem 5.2. *Suppose M satisfies the hypotheses of Theorem* 5.1. *Then the conditions $X^{G_{\mathbb{R}}} = M^G$ and $X^{(1)} = X \cap M^{(1)}$ are satisfied if and only if, for every $p \in M^G$, the* mod 2 *reduced weights, $\alpha_{i,p}$, are all distinct and non-zero.*

Proof. Let Y be a connected component of $M^{G_{\mathbb{R}}}$. Then Y is a G-invariant symplectic submanifold of M, and the action of G on it is Hamiltonian, so it contains at least one G-fixed point p. However, the hypotheses above imply that the linear isotropy action of $G_{\mathbb{R}}$ on T_pM has no fixed points other than the origin. Hence, $\dim(Y) = 0$ and $Y = \{p\}$. This argument applies to all the connected components of $M^{G_{\mathbb{R}}}$, hence the connected components are just the fixed points of G, and thus $X^{G_{\mathbb{R}}} = M^G$.

The proof that $X^{(1)} = X \cap M^{(1)}$ is similar. Let $H_{\mathbb{R}}$ be a subgroup of $G_{\mathbb{R}}$ of index 2, and let Y be a connected component of $M^{H_{\mathbb{R}}}$. Then Y is a G-invariant submanifold of M, and because $\sigma \circ \tau_g = \tau_{g^{-1}} \circ \sigma$, it is also σ-invariant. Let $p \in Y$ be a G-fixed point, and let

$$T_pM = V_1 \oplus \cdots \oplus V_d$$

be the decomposition of T_pM into the two-dimensional weight spaces corresponding to the $\tilde{\alpha}_{i,p}$. By the hypotheses on the reduced weights $\alpha_{i,p}$, either

$$(T_pM)^{H_{\mathbb{R}}} = \{0\},$$

in which case $Y = \{p\}$ or

$$(T_pM)^{H_{\mathbb{R}}} = V_i = T_pY \tag{5.5}$$

for some i. Let χ_i be the character of G associated with the representation of G on V_i and let $H = \ker(\chi_i)$. Then $H_{\mathbb{R}} \subset H$ and

$$(T_pM)^H = V_i.$$

Thus, by (5.5), Y is the connected component of M^H containing p, and in particular, Y is contained in $M^{(1)}$. Thus,

$$Y^{\sigma} \subseteq X \cap M^{(1)}.$$

Applying this argument to all index 2 subgroups $H_{\mathbb{R}}$ of $G_{\mathbb{R}}$ and all connected components of the fixed point sets of these groups, one obtains the inclusion

$$X^{(1)} \subseteq X \cap M^{(1)}.$$

The reverse inclusion is obvious. This completes the proof. □

The hypotheses of Theorem 5.2 impose some rather severe restrictions on the manifold M. For instance, the cardinality of the set of mod 2 reduced weights, $\mathbb{Z}_G^*/2\mathbb{Z}_G^*$, is 2^n. Therefore, since the reduced weights $\alpha_{i,p}$ are distinct and non-zero for $i = 1, \ldots, d$, we must have that $d \leq 2^n - 1$. Hence,

$$\dim(M) = 2d \leqslant 2^{n+1} - 2. \tag{5.6}$$

For example, if $n = 2$, then $\dim(M) \leqslant 6$. This leads us to make the following definition.

Definition 5.3. If M is a GKM manifold, and if for every $p \in M^G$, the mod 2 reduced weights, $\alpha_{i,p}$, are all distinct and non-zero, we will say that M is a mod 2 *GKM manifold*.

Next, we show that relatively few compact *homogeneous* symplectic manifolds (e.g. coadjoint orbits) are mod 2 GKM manifolds. Consider coadjoint orbits of the classical compact simple Lie groups associated with the Dynkin diagrams A_n, B_n, C_n and D_n. Let ε_i, for $i = 1, \dots, n$, be the standard basis vectors of $\mathbb{R}^n$. The positive roots associated to the Dynkin diagram A_n consist of

$$\varepsilon_i - \varepsilon_j, \quad i<j,$$

so their mod 2 reductions are distinct and non-zero. However, for B_n, C_n, and D_n, this list of positive roots contains

$$\varepsilon_i - \varepsilon_j \quad \text{and} \quad \varepsilon_i + \varepsilon_j, \; i<j,$$

so we conclude

Theorem 5.4. *Each coadjoint orbit of $SU(n)$ is a* mod 2 *GKM space. However, for other compact simple Lie groups,* no *maximal coadjoint orbit can be a* mod 2 *GKM space.*

On the other hand, on a more positive note, one has

Theorem 5.5. *If M is a non-singular projective toric variety, then it is a* mod 2 *GKM space.*

Proof. If M is a non-singular toric variety, the weights $\tilde{\alpha}_{i,p}$, $i = 1, \dots, n$, are a $\mathbb{Z}$-basis for $\mathbb{Z}_G^*$, so their images in $\mathbb{Z}_G^*/2\mathbb{Z}_G^*$ are a $\mathbb{Z}/2\mathbb{Z}$ basis of $\mathbb{Z}_G^*/2\mathbb{Z}_G^*$. $\square$

This theorem, combined with Theorem C, gives us a new description of the equivariant cohomology of a real toric variety. The ordinary and $G_{\mathbb{R}}$-equivariant cohomology of these real loci has been computed by Davis and Januszkiewicz [DJ]. Their description of these rings is analogous to Danilov's description of the ordinary and G-equivariant cohomology of the original toric varieties [Da].

We will now prove a real locus version of the GKM theorem with $\mathbb{Z}/2\mathbb{Z}$ coefficients. Recall from Section 1 that Theorem 1.1 of GKM characterizes the image of $i^* : H_G^*(M;\mathbb{C}) \to H_G^*(M^G;\mathbb{C})$ in terms of the weights of the isotropy representations of G on the tangent spaces at the fixed points.

To prove an analogue of this for the real locus of a symplectic manifold, we must first compute the $\mathbb{Z}/2\mathbb{Z}$-equivariant cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients of $\mathbb{R}P^1$. Recall that S^1 acts on $\mathbb{C}P^1$ by $\theta \cdot [z_0 : z_1] = [z_0 : e^{i\theta} z_1]$. This is a Hamiltonian action, with respect to the Fubini-Study symplectic form on $\mathbb{C}P^1$. Furthermore, complex conjugation is an anti-symplectic involution on $\mathbb{C}P^1$, with fixed point set $\mathbb{R}P^1$. There is a residual action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{R}P^1 \cong S^1$ which reflects S^1 about the y-axis.

Lemma 5.6. *Let N and S denote the fixed points of the $\mathbb{Z}/2\mathbb{Z}$ action on $\mathbb{R}P^1$. Then the image of the map*

$$i^* : H^*_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{R}P^1; \mathbb{Z}/2\mathbb{Z}) \to H^*_{\mathbb{Z}/2\mathbb{Z}}(N; \mathbb{Z}/2\mathbb{Z}) \oplus H^*_{\mathbb{Z}/2\mathbb{Z}}(S; \mathbb{Z}/2\mathbb{Z})$$

is the set of pairs (f_N, f_S) such that

$$f_N + f_S \in x \cdot \mathbb{Z}/2\mathbb{Z}[x].$$

Proof. It is clear that the constant functions are equivariant classes in

$$H^0_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{R}P^1; \mathbb{Z}/2\mathbb{Z}).$$

Furthermore, we know that $\dim H^0_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{R}P^1; \mathbb{Z}/2\mathbb{Z}) = 1$, and so these are the only equivariant classes. Finally, $\dim H^i_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{R}P^1; \mathbb{Z}/2\mathbb{Z}) = 2$ for $i > 0$, and so indeed, the condition stated is the only condition of pairs $(f_N, f_S) \in H^*_{\mathbb{Z}/2\mathbb{Z}}(N; \mathbb{Z}/2\mathbb{Z}) \oplus H^*_{\mathbb{Z}/2\mathbb{Z}}(S; \mathbb{Z}/2\mathbb{Z})$. □

Theorem C identifies the image of the map

$$i^* : H^*_{G_\mathbb{R}}(X; \mathbb{Z}/2\mathbb{Z}) \to H^*_{G_\mathbb{R}}(X^{G_\mathbb{R}}; \mathbb{Z}/2\mathbb{Z})$$

in terms of weights of isotropy representations of $G_\mathbb{R}$ on the tangent spaces at the fixed points.

Proof of Theorem C. The result follows immediately from Corollary 4.8 and Lemma 5.6. □

The results of this section and the previous section have been proved independently by Schmid [S]. Schmid uses an equivariant Morse theoretic approach, and consequently the proofs are quite different.

As a result of equivariant formality, we get two corollaries of Theorem C concerning the relation between the ring structure of the cohomology of M and the cohomology of X.

Corollary 5.7. *Suppose that M is a GKM manifold and a* mod 2 *GKM manifold. Then there is a graded ring isomorphism*

$$H^{2*}_G(M; \mathbb{Z}/2\mathbb{Z}) \cong H^*_{G_\mathbb{R}}(X; \mathbb{Z}/2\mathbb{Z}).$$

Corollary 5.8. *Suppose that M is a GKM manifold and a* mod 2 *GKM manifold. Then there is a graded ring isomorphism*

$$H^{2*}(M;\mathbb{Z}/2\mathbb{Z}) \cong H^*(X;\mathbb{Z}/2\mathbb{Z}).$$

Note that this last corollary strengthens Duistermaat's original result from an isomorphism of vector spaces to an isomorphism of rings.

Remark 5.9. Some of the results of this section, most importantly Theorem C, are valid not only for the real locus X of a Hamiltonian G-manifold, but more generally for any compact $G_{\mathbb{R}}$-manifold X which satisfies the following properties:

(1) X is equivariantly formal;
(2) $X^{G_{\mathbb{R}}}$ is finite; and
(3) the weights of X satisfy the properties of a mod 2 GKM manifold.

In this situation, we may still characterize the structure of the one-skeleton. Theorem C still follows from injectivity and the Chang–Skjelbred theorem.

6. Real GH

Goldin and Holm generalize Theorem 1.1 to the case where the one-skeleton has dimension at most 4. The goal of this section is to prove a real version of the Goldin–Holm theorem with $\mathbb{Z}/2\mathbb{Z}$ coefficients. Again, we require the hypotheses that the $(\mathbb{Z}/2\mathbb{Z})^n$-fixed points of the real locus are the same as the G-fixed points of M as in (5.1); and that the real locus of the one-skeleton is the same as the one-skeleton of the real locus, as in (5.2). Finally, we require

$$\#M^G < \infty$$

and

$$\dim(M^{(1)}) \leqslant 4.$$

If a manifold satisfies these last two hypotheses, we will say that it is a *GH manifold*. These hypotheses have a nice interpretation in terms of the isotropy representations of G at the fixed points of M.

Theorem 6.1. *The conditions* $\#M^G < \infty$ *and* $\dim(M^{(1)}) \leqslant 4$ *are satisfied if and only if the weights* $\alpha_{i,p}$ *of the isotropy representation of* G *on* T_pM *have the property that every three span a vector subspace of dimension at least two.*

These hypotheses on M have real analogues, namely that $\#X < \infty$ and the one-skeleton $X^{(1)}$ of X is at most two-dimensional. We will state without proof the following real analogue of Theorem 6.1.

Theorem 6.2. *Suppose that M satisfies the hypotheses of Theorem* 6.1. *If the conditions $X^{G_{\mathbb{R}}} = M^G$ and $X^{(1)} = M^{(1)} \cap X$ are satisfied, then for every $p \in M^G$, the* mod 2 *reduced weights $\alpha_{i,p}^{\#}$ are all non-zero, and each element of $S((G_{\mathbb{R}})^*) = \mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_n]$ appears no more than twice.*

The proof of this theorem is nearly identical to that of Theorem 5.2. The hypotheses of this theorem, although weaker than those of Theorem 5.2, still impose restrictions on the manifold M. The cardinality of the set of mod 2 reduced weights is 2^n. Since the weights are non-zero, and each weight can appear at most twice,

$$d \leqslant 2 \cdot (2^n - 1),$$

and so

$$\dim(M) = 2d \leqslant 2 \cdot (2 \cdot (2^n - 1)) = 2^{n+2} - 4.$$

For instance, if $n = 2$, $\dim(M) \leqslant 12$. We will now show an example where the condition that the reduced weights be non-zero is *not* satisfied.

Example. Consider $\mathbb{C}P^2$ with homogeneous coordinates $[z_0 : z_1 : z_2]$. Let $T = S^1$ act on $\mathbb{C}P^2$ by

$$e^{i\theta} \cdot [z_0 : z_1 : z_2] = [e^{-i\theta} z_0 : z_1 : e^{i\theta} z_2].$$

This action has three fixed points: $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$.

The weights at these fixed points are as follows:

Fixed point	Weights
$p_1 = [1 : 0 : 0]$	$x, 2x$
$p_2 = [0 : 1 : 0]$	$-x, x$
$p_3 = [0 : 0 : 1]$	$-2x, -x$

where we have identified t^* with degree one polynomials in $\mathbb{C}[x]$. As cohomology elements, these are assigned degree two. Using Theorem 6.5 below, we can compute the S^1 equivariant cohomology of $\mathbb{C}P^2$ as follows. The image of the equivariant cohomology $H^*_{S^1}(\mathbb{C}P^2)$ in

$$H^*_{S^1}(\{p_1, p_2, p_3\}) \cong \bigoplus_{i=1}^{3} \mathbb{C}[x]$$

is the subalgebra generated by the triples of functions (f_1, f_2, f_3) such that

$$f_i - f_j \in x \cdot \mathbb{C}[x] \text{ for every } i \text{ and } j$$

and

$$\frac{f_1}{2x^2} - \frac{f_2}{x^2} + \frac{f_3}{2x^2} \in \mathbb{C}[x].$$

However, when we try to compute the $\mathbb{Z}/2\mathbb{Z}$ equivariant cohomology of $\mathbb{R}P^2$, the real locus of $\mathbb{C}P^2$, we run into a problem. The mod 2 reduced weights are as follows:

Fixed point	Weights
$p_1 = [1:0:0]$	$x, 0$
$p_2 = [0:1:0]$	x, x
$p_3 = [0:0:1]$	$0, x$

The problem with this $\mathbb{Z}/2\mathbb{Z}$ action on $\mathbb{R}P^2$ is that it no longer has isolated fixed points. There is an entire $\mathbb{R}P^1$ which is fixed by this $\mathbb{Z}/2\mathbb{Z}$ action. Thus, we cannot compute the $\mathbb{Z}/2\mathbb{Z}$ equivariant cohomology of $\mathbb{R}P^2$ using these methods.

We make the following definition, analogous to the definition of mod 2 GKM manifolds given in Section 5.

Definition 6.3. Suppose that M is a GH manifold, and furthermore that $X^{G_{\mathbb{R}}} = M^G$ and $X^{(1)} = M^{(1)} \cap X$. In this case, we will say that M is a mod 2 *GH space*.

Recall the following properties about the G-equivariant cohomology of manifolds with one-skeleta of dimension at most 4. These are proved in [GH], although the reader is cautioned to the different notation used there. First, we compute the S^1 equivariant cohomology of a 4-manifold, and then we use this computation to determine the equivariant cohomology of any manifold with one skeleton of dimension at most 4.

Lemma 6.4. *Let X be a compact, connected symplectic 4-manifold with an effective Hamiltonian S^1 action with isolated fixed points $X^{S^1} = \{p_1, \dots, p_d\}$. The map $i^* : H^*_{S^1}(X) \to H^*_{S^1}(X^{S^1})$ induced by inclusion is an injection with image*

$$\left\{ (f_1, \dots, f_d) \in \bigoplus_{i=1}^{d} S(\mathfrak{s}^*) \,\middle|\, f_i - f_j \in x \cdot \mathbb{C}[x],\ \sum_{i=1}^{d} \frac{f_i}{\alpha_1^i \alpha_2^i} \in S(s^*) \right\},$$

where α_1^i and α_2^i are the (linearly dependent) weights of the $S = S^1$ isotropy action on $T_{p_i}X$.

Theorem 6.5. *Let M be a compact, connected symplectic manifold with an effective Hamiltonian G-action. Suppose further that the G-action has only isolated fixed points $M^G = \{p_1, \dots, p_d\}$ and that the one skeleton has dimension at most 4. Let $f_i \in H^*_G$ denote the restriction of $f \in H^*_G(M)$ to the fixed point p_i. The image of the injection $i^* : H^*_G(M) \to H^*_G(M^G)$ is the subalgebra of functions*

$(f_1, \ldots, f_d) \in \bigoplus_{i=1}^{d} S(\mathfrak{g}^*)$ *which satisfy*

$$\begin{cases} \pi_H^*(f_{i_j}) = \pi_H^*(f_{i_k}) & \text{if } \{p_{i_1}, \ldots, p_{i_\ell}\} = Z_H^G, \\ \sum_{j=1}^{l} \dfrac{f_{i_j}}{\alpha_1^{i_j}\alpha_2^{i_j}} \in S(\mathfrak{g}^*) & \text{if } \{p_{i_1}, \ldots, p_{i_\ell}\} = Z_H^G \text{ and } \dim Z_H = 4 \end{cases}$$

for all $H \subset G$ codimension-1 tori and all connected components Z_H of M^H, where $\alpha_1^{i_j}; \alpha_2^{i_j}$ are the (linearly dependent) weights of the G action on $T_{p_{i_j}} Z_H$; and $\pi_H : \mathfrak{h} \hookrightarrow \mathfrak{g}$ is inclusion.

We can use these computations to compute the $(\mathbb{Z}/2\mathbb{Z})^n$ equivariant cohomology of a mod 2 GH manifold.

Lemma 6.6. *Let M be a compact, connected symplectic 4-manifold with an effective Hamiltonian S^1 action with isolated fixed points $M^{S^1} = \{p_1, \ldots, p_d\}$. Suppose further that M is a* mod 2 *GH manifold with real locus X. The map*

$$i^* : H^*_{\mathbb{Z}/2\mathbb{Z}}(X; \mathbb{Z}/2\mathbb{Z}) \to H^*_{\mathbb{Z}/2\mathbb{Z}}(X^{\mathbb{Z}/2\mathbb{Z}}; \mathbb{Z}/2\mathbb{Z})$$

induced by inclusion is an injection with image

$$\left\{ (f_1, \ldots, f_d) \in \bigoplus_{i=1}^{d} \mathbb{Z}/2\mathbb{Z}[x] \;\middle|\; \begin{array}{l} f_i - f_j \in x \cdot \mathbb{Z}/2\mathbb{Z}[x], \\ \sum_{i=1}^{d} \dfrac{f_i}{\alpha_1^i \alpha_2^i} \in \mathbb{Z}/2\mathbb{Z}[x], \end{array} \right\}, \tag{6.1}$$

where α_1^i and α_2^i are the linearly dependent weights of the $\mathbb{Z}/2\mathbb{Z}$ isotropy representation on $T_{p_i}X$. (In this case, $\alpha_1^i = \alpha_2^i = x$.)

Proof. The map i^* is injective because X is equivariantly formal. We know that the f_i must satisfy the first condition because the functions constant on all the vertices are the only equivariant classes in degree 0, as $\dim H^0_{\mathbb{Z}/2\mathbb{Z}}(X; \mathbb{Z}/2\mathbb{Z}) = 1$. The second condition is necessary as a direct result of the $\mathbb{Z}/2\mathbb{Z}$ version of the localization theorem proved in Section 4. Note that this condition gives us one relation in degree 1 cohomology. A dimension count shows us that these conditions are sufficient. As an $S((\mathbb{Z}/2\mathbb{Z})^*)$-module, $H^*_{\mathbb{Z}/2\mathbb{Z}}(X; \mathbb{Z}/2\mathbb{Z}) \cong H^*(X; \mathbb{Z}/2\mathbb{Z}) \otimes H^*_{\mathbb{Z}/2\mathbb{Z}}(pt; \mathbb{Z}/2\mathbb{Z})$. Thus, the equivariant Poincaré polynomial is

$$\begin{aligned} P_t^{\mathbb{Z}/2\mathbb{Z}}(X) &= (1 + (d-2)t + t^2) \cdot (1 + t + t^2 + \cdots) \\ &= 1 + (d-1)t + dt^2 + \cdots + dt^n + \cdots. \end{aligned}$$

As $H^*_{\mathbb{Z}/2\mathbb{Z}}(X;\mathbb{Z}/2\mathbb{Z})$ is generated in degree 1, the $d-1$ degree 1 classes given by the $(f_1,\dots,f_d)$ subject to the localization condition generate the entire cohomology ring. Thus, we have found all the conditions. □

We now prove Theorem D, computing the cohomology of any mod 2 GH manifold. We will show that the image of $i^*: H^*_{G_\mathbb{R}}(X) \to H^*_{G_\mathbb{R}}(X^{G_\mathbb{R}})$ is the subalgebra of functions $(f_1,\dots,f_d) \in \bigoplus_{i=1}^d S(G^*_\mathbb{R})$ which satisfy

$$\begin{cases} \pi^*_{H_\mathbb{R}}(f_{i_j}) = \pi^*_{H_\mathbb{R}}(f_{i_k}) & \text{if } \{p_{i_1},\dots,p_{i_l}\} = Z^{G_\mathbb{R}}_{H_\mathbb{R}}, \\ \sum_{j=1}^{l} \frac{f_{i_j}}{\alpha_1^{i_j}\alpha_2^{i_j}} \in S(G^*_\mathbb{R}) & \text{if } \{p_{i_1},\dots,p_{i_l}\} = Z^{G_\mathbb{R}}_{H_\mathbb{R}} \text{ and } \dim Z_{H_\mathbb{R}} = 2 \end{cases}$$

for all subgroups $H_\mathbb{R}$ of $G_\mathbb{R}$ of order $|H_\mathbb{R}| = 2^{n-1}$ and all connected components $Z_{H_\mathbb{R}}$ of $X^{H_\mathbb{R}}$.

Proof of Theorem D. This follows immediately from Corollary 4.8 and Lemma 6.6. □

There are two immediate corollaries in this setting, analogous to Corollaries 5.7 and 5.8.

Corollary 6.7. *Suppose that M is a GH manifold, and that $M^G = X^{G_\mathbb{R}}$ and $M^{(1)} \cap X = X^{(1)}$. Then there is a graded ring isomorphism*

$$H^{2*}_G(M;\mathbb{Z}/2\mathbb{Z}) \cong H^*_{G_\mathbb{R}}(X;\mathbb{Z}/2\mathbb{Z}).$$

Corollary 6.8. *Suppose that M is a GH manifold, and that $M^G = X^{G_\mathbb{R}}$ and $M^{(1)} \cap X = X^{(1)}$. Then there is a graded ring isomorphism*

$$H^{2*}(M;\mathbb{Z}/2\mathbb{Z}) \cong H^*(X;\mathbb{Z}/2\mathbb{Z}).$$

7. Applications to string theory

Consider the $\mathbb{Z}/2\mathbb{Z}$ action on T^n, which reflects each copy of S^1. Then the equivariant cohomology ring

$$H^*_{\mathbb{Z}/2\mathbb{Z}}(T^n;\mathbb{Z}/2\mathbb{Z})$$

classifies all possible orientifold configurations of Type II string theories, compactified on T^n. See Section 3 and Appendix C of [dB] for more details.

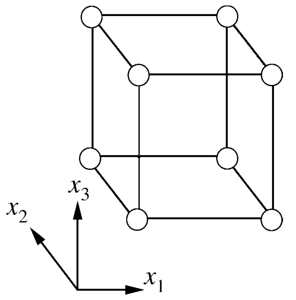

Fig. 1. This shows the GKM graph and the weights for $(\mathbb{C}P^1)^3$.

Yang-Hui He pointed out this example to us. Using the results of Section 5, we can now compute this equivariant cohomology.

First, we recognize T^n as the real locus of $M = \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1 = (\mathbb{C}P^1)^n$. This space M has a natural T^n action, where the ith copy of S^1 acts in the standard fashion on the ith copy of $\mathbb{C}P^1$. We can compute the $(\mathbb{Z}/2\mathbb{Z})^n$-equivariant cohomology of this space quite easily. The GKM graph associated to $(\mathbb{C}P^1)^n$ with the T^n action described above is the n-dimensional hypercube. The vertices correspond to the binary words of length n. Two binary words are connected by an edge if they differ in exactly one bit. Suppose v and w differ in exactly the ith bit. Then the weight associated to the edge (v, w) is x_i. Thus, when $n = 3$, the GKM graph and weights are shown in the figure above (Fig. 1).

Note that the reduced weights are all non-zero and are distinct in $\mathbb{Z}_G/2\mathbb{Z}_G$. Thus, we can apply Theorem C to compute

$$H_{(\mathbb{Z}/2\mathbb{Z})^n}(T^n; \mathbb{Z}/2\mathbb{Z}).$$

That is, the equivariant cohomology is the set of functions $f : V \to \mathbb{Z}/2\mathbb{Z}[x_1, \ldots, x_n]$ such that for every edge $(v, w) \in E$, we have

$$f(v) + f(w) \in x_i \cdot \mathbb{Z}/2\mathbb{Z}[x_1, \ldots, x_n].$$

We can now consider the copy of $\mathbb{Z}/2\mathbb{Z}$ sitting diagonally inside $(\mathbb{Z}/2\mathbb{Z})^n$. This copy of $\mathbb{Z}/2\mathbb{Z}$ acts on T^n, and this is the action that originally interested physicists. We can now compute the $\mathbb{Z}/2\mathbb{Z}$-equivariant cohomology simply by projecting

$$\pi : S(((\mathbb{Z}/2\mathbb{Z})^n)^*) = \mathbb{Z}/2\mathbb{Z}[x_1, \ldots, x_n] \to \mathbb{Z}/2\mathbb{Z}[x] = S((\mathbb{Z}/2\mathbb{Z})^*)$$

where x_i gets sent to x. Then

$$H^*_{\mathbb{Z}/2\mathbb{Z}}(T^n; \mathbb{Z}/2\mathbb{Z}) = \pi(H_{(\mathbb{Z}/2\mathbb{Z})^n}(T^n; \mathbb{Z}/2\mathbb{Z})).$$

8. Recreational applications

We will describe below a (somewhat idealized) real world application of the results of Section 5. Let $S = \{1, \dots, n\}$ be a list of companies whose stocks are being traded in the stock market. Let A_i for $i = 1, \dots, d$ be investors, and let $S_i \subseteq S$ be the portfolio of A_i. Suppose that for certain pairs of investors A_i and A_j, the symmetric difference

$$(S_i - S_j) \cup (S_j - S_i) = S_e, \quad e = (i,j), \tag{8.1}$$

i.e. the relative status of the portfolios of A_i and A_j, is given. To what extent does this information determine the portfolios S_i? It is easy to see that it cannot *uniquely* determine the S_i's. If $S_1, \dots, S_n$ is one solution to (8.1), one gets another solution by taking a fixed subset S_0 of S and replacing the S_i's by the symmetric difference

$$S_i' = (S_i - S_0) \cup (S_0 - S_i). \tag{8.2}$$

Therefore, we will slightly rephrase this question. Let E be a collection of two-element subsets of S, and suppose that for every $e \in E$, one is given a subset S_e of S. List all solutions $S_1, \dots, S_n$ of the Boolean identities (8.1). We will say two solutions are identical if they satisfy (8.2). Note, by the way, that (8.2) can be rewritten as

$$S_i = ((S_i' - S_0) \cup S_0) - S_i'.$$

Hence, we have defined an equivalence relation.

One can inject an element of randomness into this problem by positing that, for $e = (i,j) \in E$,

$$(S_i - S_j) \cup (S_j - S_i) \in \{S_e, \emptyset\}. \tag{8.3}$$

In other words, either the symmetric difference is given by (8.1) or $S_i = S_j$. Again, the problem is to list all possibilities for the S_i's. Clearly, the solutions of (8.3) contain the solutions to (8.1); so by solving (8.3), one gets an upper bound on the number of solutions to (8.1). Moreover, there are a lot of trivial solutions of (8.3), namely,

$$S_i = S_0, \quad i = 1, \dots, n, \tag{8.4}$$

where S_0 is, as above, a fixed subset of S. These can immediately be discarded as potential solutions of (8.1).

There is an elegant way of reformulating (8.1) and (8.3) in the language of mod 2 arithmetic. Let Γ be the graph whose vertices are the A_i's and whose edges are the members of E. For every edge $e \in E$, let α_e be the element of $\mathbb{Z}/2\mathbb{Z}^n$ whose kth coordinate is 1 if and only if $k \in S_e$ and α_i the element of $\mathbb{Z}/2\mathbb{Z}^n$ whose kth coordinate is 1 if and only if $k \in S_i$. Then (8.1) is equivalent to

$$\alpha_i + \alpha_j = \alpha_e \tag{8.5}$$

and (8.3) is equivalent to

$$\alpha_i + \alpha_j = \lambda\alpha_e, \quad \lambda \in \mathbb{Z}/2\mathbb{Z}. \tag{8.6}$$

In particular, (8.3) becomes an identity of the type described in Theorem C, the multiple of α_e on the right being in the degree zero component of $\mathbb{Z}/2\mathbb{Z}[x_1, \ldots, x_n]$.

Now let M be a GKM manifold acted on by an n-torus T, and let Γ be its associated graph. If the α_e's in (8.6) are the weights of $G_{\mathbb{R}}$ which were defined in Theorem C the results of Section 5 tell us that solutions of (8.6) can be identified with elements of

$$H^1_{G_{\mathbb{R}}}(M^\sigma; \mathbb{Z}/2\mathbb{Z}).$$

We will use this observation to determine the solutions of (8.6) in a couple simple, but interesting, examples.

Example (The complete graph on n vertices). Consider $M = \mathbb{C}P^{n-1}$ as a T^n manifold (ignoring the fact that the diagonal subgroup of T^n acts trivially). The corresponding graph is the complete graph on n vertices: its vertices are $A_1, \ldots, A_n$ and every pair of vertices is joined by an edge. The weights $\alpha_{(i,j)}$ from Theorem C are just $x_i + x_j$, so conditions (8.1) become

$$(S_i - S_j) \cup (S_j - S_i) = \{i, j\}. \tag{8.7}$$

The solutions of (8.7) are in one-to-one correspondence with the elements of

$$H^1_{G_{\mathbb{R}}}(\mathbb{R}P^{n-1}; \mathbb{Z}/2\mathbb{Z})$$

or, alternatively, of

$$(H^0(\mathbb{R}P^{n-1}; \mathbb{Z}/2\mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z}^n) \oplus H^1(\mathbb{R}P^{n-1}; \mathbb{Z}/2\mathbb{Z}). \tag{8.8}$$

The elements of the first summand correspond to the trivial solutions of (8.7). So if we identify solutions which are equivalent in the sense of (8.2), the non-trivial solutions of (8.7) correspond to non-zero elements of $H^1(\mathbb{R}P^{n-1}; \mathbb{Z}/2\mathbb{Z})$. However, $H^1(\mathbb{R}P^{n-1}; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, so there is just one non-trivial solution up to equivalence, and it is given by

$$S_i = \{i\};$$

this is also the unique solution of (8.1) up to equivalence.

Example (The permutahedron). Let M be the complex flag variety $U(n)/T^n$, and consider the T^n action on M by left multiplication. The graph associated with M is the *permutahedron*. Its vertices are the elements of the symmetric group, S_n, and two vertices σ and τ are joined by and edge if $\tau\sigma^{-1}$ is a transposition. If e is the edge joining σ to τ and $\tau\sigma^{-1}$ is the transposition switching i and j, then as in the previous example, $\alpha_e = x_i + x_j$, so condition (8.1) becomes

$$(S_\sigma - S_\tau) \cup (S_\tau - S_\sigma) = \{i,j\}.$$

As in the previous example, the non-trivial solutions to (8.3) can be identified with the non-zero elements of $H^1(M^\sigma; \mathbb{Z}/2\mathbb{Z})$, and since M^σ is the real flag variety,

$$H^1(M^\sigma; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}^{n-1}.$$

If one thinks of $\mathbb{Z}/2\mathbb{Z}^{n-1}$ as the quotient of $\mathbb{Z}/2\mathbb{Z}^n$ by the diagonal subgroup $(\lambda, \ldots, \lambda)$, with $\lambda \in \mathbb{Z}/2\mathbb{Z}$, the solutions corresponding to $\alpha \in \mathbb{Z}/2\mathbb{Z}^n \bmod (\lambda, \ldots, \lambda)$ is given by

$$S_\sigma = \sigma(S_0), \tag{8.9}$$

where $k \in S_0$ if and only if the kth coordinate of α is 1. Note, by the way, if we replace α by $\alpha + (1, \ldots, 1)$, (8.9) becomes

$$S_\sigma^c = \sigma(S_0^c), \tag{8.10}$$

where S_σ^c and S_0^c are the complements of S_σ and S_0 in S, so (8.9) and (8.10) are equivalent. It is easy to see by inspection that none of the solutions (8.9) of (8.3) are also solutions of (8.1).

An interesting special case of this example is the complete bipartite graph $K_{3,3}$. In this case, the vertices of Γ are A_1, A_2, A_3 and B_1, B_2, and B_3; the edges are all pairs (A_i, B_j); and the sets S_e are

$$S_{A_i,B_j} = \begin{cases} \{i,j\}, & i \neq j, \\ \{1,2,3\} - \{i\}, & i = j. \end{cases}$$

In this example, (8.3) has one non-trivial solution, up to equivalence, namely $S_{A_i} = S_{B_i} = \{i\}$.

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