Strict stability of impulsive functional differential equations

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Abstract

Strict stability is the kind of stability that can give us some information about the rate of decay
of the solutions. There are some results about strict stability of differential equations. In the present
paper, we shall extend the strict stability to impulsive functional differential equations. By using
Lyapunov functions and Razumikhin technique, we shall get some criteria for the strict stability of
impulsive functional differential equations, and we can see that impulses do contribute to the system’s
strict stability behavior.
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1. Introduction

Impulses can make unstable systems stable, so it has been widely used in many fields,
such as physics, chemistry, biology, population dynamics, industrial robotics, and so on.
The impulsive differential equations represents a more natural framework for mathematici-
ical modelling of many real world phenomena than differential equations. In recent years,
significant progress has been made in the theory of impulsive differential equations [3–13] and references therein.

Strict stability is analogous to Lyapunov’s uniform asymptotic stability. It can give us some information about the rate of decay of the solutions. In [1], the authors have further the definitions of strict stability for differential equations, and have gotten some results. In [2], the authors have gotten some results about the strict practical stability of differential equations. But there does not exist impulses in these systems. In book [4], the author has given us many results about impulsive systems without time delay. But time delay systems are frequently encountered in engineering, biology, economy, and other disciplines, so it is necessary to study these systems with time delay. And there are some results about impulsive functional differential equations and impulsive differential equations with time delay [3,5–7,9,12]. In the present paper, we shall consider the strict stability of impulsive functional differential equations. We can see that impulses do contribute to the system’s strict stability behavior.

This paper is organized as follows. In Section 2, we introduce some basic definitions and notations. In Section 3, we get some criteria for strict stability of impulsive functional differential equations. Finally, concluding remarks are given in Section 4.

2. Preliminaries

Consider the following impulsive functional differential equations:

\[ \dot{x}(t) = f(t, x_t), \quad t \geq t_0, \quad t \neq \tau_k, \]

\[ \Delta x(t) \triangleq x(t_k) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \ldots, \]  

(1)

where \( f \in C[\mathbb{R}^+ \times D, \mathbb{R}^n] \), \( D \) is an open set in \( PC([-\tau, 0], \mathbb{R}^n) \), where \( \tau > 0 \) and \( PC([-\tau, 0], \mathbb{R}^n) = \{ \phi : [-\tau, 0] \to \mathbb{R}^n \mid \phi(t) \) is continuous everywhere except a finite number of points \( t \) at which \( \phi(t^+) \) and \( \phi(t^-) \) exist and \( \phi(t^+) = \phi(t^-) \}. \) \( f(t, 0) = 0 \), for all \( t \in \mathbb{R}. \) \( I_k(0) = 0, \) for all \( k \in \mathbb{Z}, \) \( 0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots, \) \( t_k \to \infty \) for \( k \to \infty, \) and \( x(t^+) = \lim_{s \to t^+} x(s), \) \( x(t^-) = \lim_{s \to t^-} x(s). \) For each \( t \geq t_0, \) \( x_t \in D \) is defined by \( x_t(s) = x(t + s), -\tau \leq s \leq 0. \) For \( \phi \in PC([-\tau, 0], \mathbb{R}^n), \) \( |\phi|_1 \) is defined by \( |\phi|_1 = \sup_{-\tau \leq s \leq 0} \| \phi(s) \|, \) \( |\phi|_2 \) is defined by \( |\phi|_2 = \inf_{-\tau \leq s \leq 0} \| \phi(s) \|, \) where \( \| \cdot \| \) denotes the norm of vector in \( \mathbb{R}^n. \) We can see that \( x(t) \equiv 0 \) is a solution of (1), which we call the zero solution.

For given \( \sigma \geq t_0 \) and \( \varphi \in PC([-\tau, 0], \mathbb{R}^n), \) the initial value problem of Eq. (1) is

\[ \dot{x}(t) = f(t, x_t), \quad t \geq \sigma, \quad t \neq \tau_k, \]

\[ \Delta x(t) \triangleq x(t_k) - x(t_k^-) = I_k(x(t_k^-)), \]

\[ x_{\sigma} = \varphi, \quad k = 1, 2, \ldots. \]  

(2)

Throughout this paper we let the following hypotheses hold:

(H1) For each function \( x(s) : [-\tau, \infty) \to \mathbb{R}^n, \) \( \sigma \geq t_0, \) which is continuous everywhere except the points \( \tau_k \) at which \( x(t_k^+) \) and \( x(t_k^-) \) exist and \( x(t_k^+) = x(t_k^-) = x(t_k), \) \( f(t, x_t) \)
is continuous for all $t \in [\sigma, \infty)$ and at the discontinuous points $f$ is right continuous.

(H2) $f(t, \phi)$ is Lipschitzian in $\phi$ in each compact set in $PC([-\tau, 0], \mathbb{R}^n)$.

(H3) The functions $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \ldots$, are such that for any $H > 0$, there exists a $\rho > 0$ such that if $x \in S(\rho) = \{x \in \mathbb{R}^n : \|x\| < \rho\}$ implies that $\|x + I_k(x)\| < H$.

Under the hypotheses (H1)–(H3), there is a unique solution of problem (2) throughout $(\sigma, \phi)$.

Let $K = \{a \in C[\mathbb{R}^+, \mathbb{R}^+] : a(t)$ is monotone strictly increasing and $a(0) = 0\}$,

$K_1 = \{w \in C[\mathbb{R}^+, \mathbb{R}^+] : w(t) \in K$ and $0 < w(s) < s$, $s > 0\}$,

$PC_1(\rho) = \{\phi \in PC([-\tau, 0], \mathbb{R}^n) : |\phi_1| < \rho\}$,

$PC_2(\theta) = \{\phi \in PC([-\tau, 0], \mathbb{R}^n) : |\phi_2| > \theta\}$.

We have the following definitions.

**Definition 1.** The zero solution of (1) is said to be

(A1) strictly stable, if for any $\sigma \geq t_0$ and $\varepsilon_1 > 0$, there exists a $\delta_1 = \delta_1(\sigma, \varepsilon_1) > 0$ such that $\phi \in PC_1(\delta_1)$ implies $\|x(t; \sigma, \phi)\| < \varepsilon_1$, $t \geq \sigma$, and for every $0 < \delta_2 \leq \delta_1$, there exists an $0 < \varepsilon_2 < \delta_2$ such that

$\phi \in PC_2(\delta_2)$ implies $\varepsilon_2 < \|x(t; \sigma, \phi)\|$, $t \geq \sigma$;

(A2) strictly uniformly stable, if $\delta_1, \delta_2$ and $\varepsilon_2$ is independent of $\sigma$;

(A3) strictly attractive, if given $\sigma \geq t_0$ and $\alpha_1 > 0, \varepsilon_1 > 0$, for every $\alpha_2 \leq \alpha_1$, there exists $\varepsilon_2 < \varepsilon_1$ and $T_1 = T_1(\sigma, \alpha_1), T_2 = T_2(\sigma, \alpha_2)$ such that

$\phi \in PC_1(\alpha_1) \cap PC_2(\alpha_2)$ implies $\varepsilon_2 < \|x(t; \sigma, \phi)\| < \varepsilon_1$, for $\sigma + T_1 \leq t \leq \sigma + T_2$;

(A4) strictly uniformly attractive if $T_1, T_2$ in (A3) are independent of $\sigma$;

(A5) strictly asymptotically stable if (A3) holds and the trivial solution is stable;

(A6) strictly uniformly asymptotically stable if (A4) holds and the trivial solution is uniformly stable.

It is obvious that (A1) and (A3), or (A2) and (A4) cannot hold at the same time.

**Definition 2** [3]. The function $V : [t_0, \infty) \times S(\rho) \to \mathbb{R}^+$ belongs to class $v_0$ if

1. the function $V$ is continuous on each of the sets $[\tau_k - 1, \tau_k)$ and for all $t \geq t_0$, $V(t, 0) = 0$;
2. $V(t, x)$ is locally Lipschitzian in $x \in S(\rho)$;
3. for each $k = 1, 2, \ldots$, there exist finite limits
\[
\lim_{(t,y)\to(t_k^-,x)} V(t,y) = V(t_k^-,x),
\]
\[
\lim_{(t,y)\to(t_k^+,x)} V(t,y) = V(t_k^+,x)
\]

with \( V(t_k^+,x) = V(t_k,x) \) satisfied.

**Definition 3** [4]. Let \( V \in \mathcal{V}_0 \), for \( (t,x) \in [t_{k-1}, t_k) \times S(\rho) \), \( D^+ V \) is defined as
\[
D^+ V(t,x(t)) = \lim_{\delta \to 0^+} \sup_{\delta} \frac{1}{\delta} \{ V(t + \delta, x(t + \delta)) - V(t,x(t)) \}.
\]

### 3. Main results

Now we consider the strict stability of the impulsive functional differential equation (1).

We have the following results.

**Theorem 1. Assume that**

(i) There exists \( V_1 \in \mathcal{V}_0 \) such that
\[
b_1(\|x\|) \leq V_1(t,x) \leq a_1(\|x\|), \quad a_1, b_1 \in K;
\]
(ii) For any solution \( x(t) \) of (1), \( V_1(t+s, x(t+s)) \leq V_1(t,x(t)) \) for \( s \in [-\tau, 0] \), implies that \( D^+ V(t,x(t)) \leq 0 \).

Also, for all \( k \in \mathbb{Z}^+ \) and \( x \in S(\rho) \),
\[
V_1(t_k, x(t_k^-) + I_k(x(t_k^-))) \leq (1 + d_k)V_1(t_k^-, x(t_k^-)),
\]
where \( d_k \geq 0 \) and \( \sum_{k=1}^{\infty} d_k < \infty \).

(iii) There exists \( V_2 \in \mathcal{V}_0 \) such that
\[
b_2(\|x\|) \leq V_2(t,x) \leq a_2(\|x\|), \quad a_2, b_2 \in K;
\]
(iv) For any solution \( x(t) \) of (1), \( V_2(t+s, x(t+s)) \geq V_2(t,x(t)) \) for \( s \in [-\tau, 0] \), implies that \( D^+ V_2(t,x(t)) \geq 0 \).

Also, for all \( k \in \mathbb{Z}^+ \) and \( x \in S(\rho) \),
\[
V_2(t_k, x(t_k^-) + I_k(x(t_k^-))) \geq (1 - c_k)V_2(t_k^-, x(t_k^-)),
\]
where \( 0 \leq c_k < 1 \) and \( \sum_{k=1}^{\infty} c_k < \infty \).

Then the trivial solution of (1) is strictly uniformly stable.
Proof. Since \( \sum_{k=1}^{\infty} d_k < \infty \), \( \sum_{k=1}^{\infty} \varepsilon_k < \infty \), it follows that \( \prod_{k=1}^{\infty} (1 + d_k) = M \) and \( \prod_{k=1}^{\infty} (1 - d_k) = N \), obviously \( 1 \leq M < \infty \), \( 0 < N \leq 1 \).

Let \( 0 < \varepsilon_1 < \rho \) and \( \sigma \geq t_0 \) be given, and \( \sigma \in [\tau_{k-1}, \tau_k) \) for some \( k \in \mathbb{Z}^+ \). Choose \( \delta_1 = \delta_1(\varepsilon_1) > 0 \) such that \( M\alpha_1(\delta_1) < \beta_1(\varepsilon_1) \).

Then we claim that \( \varphi \in PC_1(\delta_1) \) implies \( \|x(t)\| < \varepsilon_1 \), \( t \geq \sigma \).

Obviously for any \( t \in [\sigma - \tau, \sigma] \), there exists a \( \theta \in [-\tau, 0] \) such that

\[
V_1(t, x(t)) = \alpha_1 \left( \|x(t)\| \right) = a_1 \left( \|x_\sigma(\theta)\| \right) = a_1 \left( \|\varphi(\theta)\| \right) \leq a_1(\delta_1).
\]

Then we claim that

\[
V_1(t, x(t)) \leq a_1(\delta_1), \quad \sigma \leq t < \tau_k.
\] (3)

If inequality (3) does not hold, then there is a \( \hat{t} \in (\sigma, \tau_k) \) such that

\[
V_1(\hat{t}, x(\hat{t})) > a_1(\delta_1) \geq V_1(\sigma, x(\sigma))
\]

which implies that there is a \( \tilde{t} \in (\sigma, \hat{t}] \) such that

\[
D^+ V_1(\tilde{t}, x(\tilde{t})) > 0
\] (4)

and

\[
V_1(\tilde{t} + s, x(\tilde{t} + s)) \leq V_1(\tilde{t}, x(\tilde{t})), \quad s \in [-\tau, 0].
\]

By condition (ii), which implies that \( D^+ V_1(\tilde{t}, x(\tilde{t})) \leq 0 \). This contradicts inequality (4), so inequality (3) holds.

From condition (ii), we have

\[
V_1(\tau_k, x(\tau_k)) \leq V_1(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \leq (1 + d_k) V_1(\tau_k^-, x(\tau_k^-)) \\
\leq (1 + d_k) a_1(\delta_1).
\]

Next, we claim that

\[
V_1(t, x(t)) \leq (1 + d_k) a_1(\delta_1), \quad \tau_k \leq t < \tau_{k+1}.
\] (5)

If inequality (5) does not hold, then there is an \( \hat{s} \in (\tau_k, \tau_{k+1}) \) such that

\[
V_1(\hat{s}, x(\hat{s})) \geq (1 + d_k) a_1(\delta_1) \geq V_1(\tau_k, x(\tau_k))
\]

which implies that there is an \( \tilde{s} \in (\tau_k, \hat{s}) \) such that

\[
D^+ V_1(\tilde{s}, x(\tilde{s})) > 0
\] (6)

and

\[
V_1(\tilde{s} + s, x(\tilde{s} + s)) \leq V_1(\tilde{s}, x(\tilde{s})), \quad s \in [-\tau, 0].
\]

By condition (ii), which implies that \( D^+ V_1(\tilde{s}, x(\tilde{s})) \leq 0 \). This contradicts inequality (6), so inequality (5) holds.

And from condition (ii), we have

\[
V_1(\tau_{k+1}, x(\tau_{k+1})) = V_1(\tau_{k+1}, x(\tau_{k+1}^-) + I_{k+1}(x(\tau_{k+1}^-))) \\
\leq (1 + d_{k+1}) V_1(\tau_{k+1}^-, x(\tau_{k+1}^-)) \leq (1 + d_{k+1})(1 + d_k) a_1(\delta_1).
\]
By a simple induction, we can easily prove that, in general, for \( m = 0, 1, 2, \ldots \),

\[ V_1(t, x(t)) \leq (1 + d_k + m) \ldots (1 + d_k) a_1(\delta_1), \quad \tau_{k+m} \leq t < \tau_{k+m+1}. \]

which together with inequality (3) and condition (i) we have

\[ b_1\left( \| x(t) \| \right) \leq V_1(t, x(t)) \leq M a_1(\delta_1) < b_1(\varepsilon_1), \quad t \geq \sigma. \]

Thus, we have

\[ \| x(t) \| < \varepsilon_1, \quad t \geq \sigma. \]

Now, let \( 0 < \delta_2 \leq \delta_1 \) and choose \( 0 < \varepsilon_2 < \delta_2 \) such that \( a_2(\varepsilon_2) \leq N b_2(\delta_2) \).

Next we claim that \( \varphi \in PC_2(\delta_2) \) implies \( \| x \| > \varepsilon_2, \quad t \geq \sigma \). If this holds, \( \varphi \in PC_1(\delta_1) \cap PC_2(\delta_2) \) implies \( \varepsilon_2 < \| x \| < \varepsilon_1, \quad t \geq \sigma \).

Obviously for any \( t \in [\sigma - \tau, \sigma] \), there exists a \( \theta \in [\tau, 0] \) such that

\[ V_2(t, x(t)) = V_2(\tau_k + \theta, x(\tau_k + \theta)) \geq b_2\left( \| x(\tau_k + \theta) \| \right) = b_2\left( \| x_x(\theta) \| \right) \geq b_2(\delta_2). \]

Then we claim that

\[ V_2(t, x(t)) \geq b_2(\delta_2), \quad \sigma \leq t < \tau_k. \quad (7) \]

If inequality (7) does not hold, then there is a \( \bar{t} \in (\sigma, \tau_k) \) such that

\[ V_2(\bar{t}, x(\bar{t})) < b_2(\delta_2) \leq V_2(\sigma, x(\sigma)) \]

which implies that there exists a \( t_1 \in (\sigma, \bar{t}) \) such that

\[ D^+ V_2(t_1, x(t_1)) < 0 \quad (8) \]

and

\[ V_2(t_1 + s, x(t_1 + s)) \geq V_2(t_1, x(t_1)), \quad s \in [-\tau, 0]. \]

By condition (iv), this implies that \( D^+ V_2(t_1, x(t_1)) \geq 0 \), which contradicts inequality (8), so inequality (7) holds.

From condition (iv), we have

\[ V_2(\tau_k, x(\tau_k)) = V_2(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \geq (1 - c_k) V_2(\tau_k, x(\tau_k^-)) \geq (1 - c_k) b_2(\delta_2). \]

Next, we claim that

\[ V_2(t, x(t)) \geq (1 - c_k) b_2(\delta_2), \quad \tau_k \leq t < \tau_{k+1}. \quad (9) \]

If inequality (9) does not hold, then there is an \( \bar{r} \in (\tau_k, \tau_{k+1}) \) such that

\[ V_2(\bar{r}, x(\bar{r})) < (1 - c_k) b_2(\delta_2) \leq V_2(\tau_k, x(\tau_k)) \]

which implies that there is an \( \tilde{r} \in (\tau_k, \bar{r}) \) such that

\[ D^+ V_2(\tilde{r}, x(\tilde{r})) < 0 \quad (10) \]
and
\[ V_2(\tilde{r} + s, x(\tilde{r} + s)) \geq V_2(\tilde{r}, x(\tilde{r})), \quad s \in [-\tau, 0]. \]

By condition (iv), this implies that \( D^+ V_2(\tilde{r}, x(\tilde{r})) \geq 0 \), which contradicts inequality (10). So inequality (9) holds.

And from condition (iv), we have
\[ V_2(\tau_{k+1}, x(\tau_{k+1})) = V_2(\tau_{k+1}, x(\tau_{k+1} + I_{k+1} x(\tau_{k+1}))) \geq (1 - c_{k+1}) V_2(\tau_{k+1}, x(\tau_{k+1})) \geq (1 - c_{k+1})(1 - c_k)b_2(\delta_2). \]

By a simple induction, we can easily prove that, in general, for \( m = 0, 1, 2, \ldots, \)
\[ V_2(\tau_{k+m}, x(\tau_{k+m})) \geq (1 - c_{k+m}) \ldots (1 - c_k)b_2(\delta_2), \quad \tau_{k+m} \leq t < \tau_{k+m+1}, \]
which together with inequality (7) and condition (iii) we have
\[ a_2(\|x(t)\|) \geq V_2(t, x(t)) \geq Nb_2(\delta_2) > a_2(\varepsilon_2), \quad t \geq \sigma. \]

Thus, we have
\[ \|x(t)\| > \varepsilon_2, \quad t \geq \sigma. \]

Thus, the zero solution of (1) is strictly uniformly stable.

The proof of Theorem 1 is complete. \( \square \)

**Theorem 2.** Assume that

(i) There exists \( V_1 \in v_0 \) such that
\[ b_1(\|x\|) \leq V_1(t, x(t)) \leq a_1(\|x\|), \quad a_1, b_1 \in K. \]

(ii) For any solution \( x(t) \) of (1), there exists a \( \psi_1 \in K_1 \), such that \( V_1(t + s, x(t + s)) \leq \psi_1^{-1}(V_1(t, x(t))) \) for \( s \in [-\tau, 0] \), implies that \( D^+ V_1(t, x(t)) \leq g(t) w(V_1(t, x)), g, w : C[R^+, R^+] \), locally integrable.

And for all \( k \in Z^+ \), \( V_1(\tau_k, x(\tau_k)) \leq \psi_1(V_1(\tau_k, x(\tau_k))). \)

(iii) There exists a constant \( A > 0 \) such that
\[ \int_{\tau_{k-1}}^{\tau_k} g(s) \, ds < A, \quad k \in Z^+. \]

Also, for any \( u > 0 \),
\[ \int_{u}^{\psi_1^{-1}(u)} \frac{ds}{w(s)} \geq A. \]
(iv) There exists $V_2 \in v_0$ such that
\[ b_2(\|x\|) \leq V_2(t, x(t)) \leq a_2(\|x\|), \quad a_2, b_2 \in K. \]

(v) For any solution $x(t)$ of (1), there exists a $\psi_2 \in K_1$, such that $V_2(t+s, x(t+s)) \geq \psi_2(V_2(t, x(t)))$ for $s \in [-\tau, 0]$, implies that $D^+ V_2(t, x(t)) \leq h(t) p(V_2(t, x))$, $h, p : C[\mathbb{R}^+, \mathbb{R}^+]$, locally integrable.

And for all $k \in \mathbb{Z}^+$, $V_2(\tau_k, x(\tau_k)) \geq \psi_2^{-1}(V_2(\tau_{k-1}, x(\tau_{k-1})))$.

(vi) There exists a constant $B > 0$ such that
\[ \tau \int_{\tau_{k-1}}^{\tau_k} h(s) \, ds < B, \quad k \in \mathbb{Z}^+. \]

Also, for any $u > 0$,
\[ \frac{\psi_2(u) \int_0^u p(s) \, ds}{p(s)} \geq B. \]

Then the zero solution of (1) is strictly uniformly stable.

**Proof.** For given $0 < \varepsilon_1 < \rho, \sigma \geq 0$. Choose $\delta_1 = \delta_1(\varepsilon_1) > 0$, such that $\psi_1^{-1}(a_1(\delta_1)) < b_1(\varepsilon_1)$. Let $\sigma \in [\tau_{k-1}, \tau_k)$ for some $k \in \mathbb{Z}^+$.

We claim that $\psi \in PC_1(\delta_1)$ implies $\|x\| < \varepsilon_1$, $t \geq \sigma$.

First, we claim that
\[ V_1(t, x(t)) \leq \psi_1^{-1}(a_1(\delta_1)), \quad \sigma \leq t < \tau_k. \quad (11) \]

Obviously for any $t \in [\sigma - \tau, \sigma]$, there exists a $\theta \in [-\tau, 0]$ such that
\[ V_1(t, x(t)) = V_1(\sigma + \theta, x(\sigma + \theta)) \leq a_1(\|x(\sigma + \theta)\|) = a_1(\|x(\sigma)\|) \leq a_1(\|x(\theta)\|) \leq a_1(\|\psi(\theta)\|) \leq a_1(\|x(\theta)\|) = a_1(\|x(\sigma)\|) = a_1(\|x(\sigma + \theta)\|) \leq a_1(\|x(\sigma)\|) \leq a_1(\|x(\theta)\|) \leq a_1(\|x(\sigma + \theta)\|) \leq \psi_1^{-1}(a_1(\delta_1)). \]

So if inequality (11) does not hold, then there is an $\hat{s} \in (\sigma, \tau_k)$ such that
\[ V_1(\hat{s}, x(\hat{s})) > \psi_1^{-1}(a_1(\delta_1)) \geq a_1(\delta_1) \geq V_1(\sigma, x(\sigma)). \]

From the continuity of $V_1(t, x(t))$ at $[\sigma, \tau_k)$, it follows that there exists an $s_1 \in (\sigma, \hat{s})$ such that
\[ V_1(s_1, x(s_1)) = \psi_1^{-1}(a_1(\delta_1)), \]
\[ V_1(t, x(t)) \leq \psi_1^{-1}(a_1(\delta_1)), \quad \sigma - \tau \leq t \leq s_1, \]
and also, there exists an $s_2 \in [\sigma, s_1)$ such that
\[ V_1(s_2, x(s_2)) = a_1(\delta_1), \]
\[ V_1(t, x(t)) \geq a_1(\delta_1), \quad t \in [s_2, s_1]. \]
Therefore, for $t \in [s_2, s_1]$ and $-\tau \leq s \leq 0$, we have

$$V_1(t + s, x(t + s)) \leq \psi_1^{-1}(a_1(\delta_1)) \leq \psi_1^{-1}(V_1(t, x(t))).$$

In view of condition (ii), we get

$$D^+ V_1(t, x(t)) \leq g(t) w(V_1(t, x)), \quad s_2 \leq t \leq s_1. \tag{12}$$

And integrate the inequality (12) over $(s_2, s_1)$, we have by condition (iii),

$$\int_{V_1(s_2, x(s_2))} \frac{du}{w(u)} \leq \int_{s_2}^{s_1} g(s) ds \leq \int_{s_1 - 1}^{s_1} g(s) ds < A.$$ 

On the other hand,

$$\int_{V_1(s_1, x(s_1))} \frac{du}{w(u)} = \int_{a_1(\delta_1)}^{\psi_1^{-1}(a_1(\delta_1))} \frac{du}{w(u)} \geq A,$$

a contradiction. So inequality (11) holds.

From condition (ii) and inequality (11), we have

$$V_1(t_k, x(t_k)) = V_1(t_k, x(t_k^-) + I_k(x(t_k^-))) \leq \psi_1(V_1(t_k^-, x(t_k^-))) \leq a_1(\delta_1). \tag{13}$$

Next, we prove that

$$V_1(t, x(t)) \leq \psi_1^{-1}(a_1(\delta_1)), \quad \tau_k \leq t < \tau_{k+1}. \tag{13}$$

Since $a_1(\delta_1) < \psi_1^{-1}(a_1(\delta_1))$, if inequality (13) does not hold, then there is an $\hat{t} \in (\tau_k, \tau_{k+1})$ such that

$$V_1(\hat{t}, x(\hat{t})) > \psi_1^{-1}(a_1(\delta_1)) > a_1(\delta_1) \geq V_1(t_k, x(t_k)).$$

From the continuity of $V_1(t, x(t))$ at $[\tau_k, \tau_{k+1})$, it follows that there is an $r_1 \in (\tau_k, \hat{t})$ such that

$$V_1(r_1, x(r_1)) = \psi_1^{-1}(a_1(\delta_1)),$$

$$V_1(t, x(t)) \leq \psi_1^{-1}(a_1(\delta_1)), \quad \rho - \tau \leq t \leq r_1,$$

and also, there exists an $r_2 \in [\tau_k, r_1)$ such that

$$V_1(r_2, x(r_2)) = a_1(\delta_1),$$

$$V_1(t, x(t)) \geq a_1(\delta_1), \quad t \in [r_2, r_1].$$

Therefore, for $t \in [r_2, r_1]$ and $-\tau \leq s \leq 0$, we have

$$V_1(t + s, x(t + s)) \leq \psi_1^{-1}(a_1(\delta_1)) \leq \psi_1^{-1}(V_1(t, x(t))).$$

In view of condition (ii), we get

$$D^+ V_1(t, x(t)) \leq g(t) w(V_1(t, x)), \quad r_2 \leq t \leq r_1. \tag{14}$$

Similar as before, we integrate the inequality (14) over $(r_2, r_1)$, by condition (iii) we can obtain a contradiction, so inequality (13) holds.
From condition (ii) and inequality (13),

\[ V_1(t_{k+1}, x(t_{k+1})) = V_1(t_{k+1}, x(t_{k+1}^{-})) + I_{k+1}(x(t_{k+1}^{-})) \leq \psi_1(V_1(t_{k+1}, x(t_{k+1}^{-}))) \leq a_1(\delta_1). \]

By simple induction, we can prove, in general, that for \( i = 0, 1, 2, \ldots \),

\[ V_1(t, x(t)) \leq \psi_1^{-1}(a_1(\delta_1)), \quad t_{k+i} \leq t < t_{k+i+1}, \]

\[ V_1(t_{k+i+1}, x(t_{k+i+1})) \leq a_1(\delta_1). \quad (15) \]

Since \( a_1(\delta_1) < \psi_1^{-1}(a_1(\delta_1)) \), by condition (i) and inequalities (11), (15), we have

\[ b_1(\|x\|) \leq V_1(t, x(t)) \leq \psi_1^{-1}(a_1(\delta_1)) \leq b_1(\varepsilon_1), \quad t \geq \sigma. \]

Thus \( \|x\| < \varepsilon_1, t \geq \sigma \).

Now, let \( 0 < \delta_2 \leq \delta_1 \) and choose \( 0 < \varepsilon_2 < \delta_2 \) such that \( a_2(\varepsilon_2) < \psi_2(b_2(\delta_2)) \).

Next we claim that \( \varphi \in PC_2(\delta_2) \) implies \( \|x\| > \varepsilon_2, t \geq \sigma \). If this holds, \( \varphi \in PC_1(\delta_1) \cap PC_2(\delta_2) \) implies \( \varepsilon_2 < \|x\| < \varepsilon_1, t \geq \sigma \).

First, we claim that

\[ V_2(t, x(t)) \geq \psi_2(b_2(\delta_2)), \quad \sigma \leq t < t_k. \quad (16) \]

Obviously for any \( t \in [\sigma - \tau, \sigma] \), there exists a \( \theta \in [-\tau, 0] \) such that

\[ V_2(t, x(t)) = V_2(\sigma + \theta, x(\sigma + \theta)) \geq b_2(\|x(\sigma + \theta)\|) = b_2(\|x(\sigma)\|) = b_2(\|\varphi(\theta)\|) \geq b_2(\delta_2) > \psi_2(b_2(\delta_2)). \]

So if inequality (16) does not hold, then there is a \( \bar{t} \in (\sigma, t_k) \) such that

\[ V_2(\bar{t}, x(\bar{t})) < \psi_2(b_2(\delta_2)) \leq V_2(\sigma, x(\sigma)). \]

From the continuity of \( V_2(t, x(t)) \) at \( [\sigma, t_k] \) there exists a \( t_1 \in (\sigma, \bar{t}) \) such that

\[ V_2(t_1, x(t_1)) = \psi_2(b_2(\delta_2)), \]

\[ V_2(t, x(t)) \geq \psi_2(b_2(\delta_2)), \quad \sigma - \tau \leq t \leq t_1. \]

And also, there exists a \( t_2 \in [\sigma, t_1) \) such that

\[ V_2(t_2, x(t_2)) = b_2(\delta_2), \]

\[ V_2(t, x(t)) \leq b_2(\delta_2), \quad t \in [t_2, t_1]. \]

Therefore, for \( t \in [t_2, t_1] \), and \( -\tau \leq s \leq 0 \), we have

\[ V_2(t + s, x(t + s)) \geq \psi_2(b_2(\delta_2)) \geq \psi_2(V_2(t, x(t))). \]

In view of condition (vi), we get

\[ D^+V_2(t, x(t)) \leq h(t)p(V_2(t, x(t))), \quad t_2 \leq t \leq t_1. \quad (17) \]

Integrate inequality (17) over \( (t_2, t_1) \), by condition (vi) we have

\[ \int_{V_2(t_2, x(t_2))}^{V_2(t_1, x(t_1))} \frac{d\mu}{p(u)} \leq \int_{t_2}^{t_1} h(s) \, ds \leq \int_{t_2}^{t_1} h(s) \, ds < B \]
If inequality (18) does not hold, then there is a \( \hat{q} \) and at the same time \( q \) and also, there exists a contradiction. So inequality (16) holds.

From the continuity of \( V(q) \), we integrate the inequality (19) over \( [\tau_k, \tau_{k+1}] \). Therefore, we have

\[
V_2(t, x(t)) \geq \psi_2^{-1}(V_2(\tau_k^-, x(\tau_k^-))) \geq b_2(\delta_2).
\]

Next, we prove

\[
V_2(t, x(t)) \geq \psi_2(b_2(\delta_2)), \quad \tau_k \leq t < \tau_{k+1}.
\] (18)

If inequality (18) does not hold, then there is a \( \hat{q} \in (\tau_k, \tau_{k+1}) \) such that

\[
V_2(\hat{q}, x(\hat{q})) < \psi_2(b_2(\delta_2)) < b_2(\delta_2) \leq V_2(\tau_k, x(\tau_k)).
\]

From the continuity of \( V_2(t, x(t)) \) at \( [\tau_k, \tau_{k+1}] \), there is a \( q_1 \in (\tau_k, \hat{q}) \) such that

\[
V_2(q_1, x(q_1)) = \psi_2(b_2(\delta_2)),
\]

\[
V_2(t, x(t)) \geq \psi_2(b_2(\delta_2)), \quad \sigma - \tau \leq t \leq q_1,
\]

and also, there exists a \( q_2 \in [\tau_k, q_1] \) such that

\[
V_2(q_2, x(q_2)) = b_2(\delta_2),
\]

\[
V_2(t, x(t)) \leq b_2(\delta_2), \quad t \in [q_2, q_1].
\]

Therefore, for \( t \in [q_2, q_1] \), and \(-\tau \leq s \leq 0\), we have

\[
V_2(t + s, x(t + s)) \geq \psi_2(b_2(\delta_2)) \geq \psi_2(V_2(t, x(t))).
\]

In view of condition (vi), we get

\[
D^+ V_2(t, x(t)) \leq h(t)p(V_2(t, x)), \quad q_2 \leq t \leq q_1.
\] (19)

Similar as before, we integrate the inequality (19) over \( [q_2, q_1] \), by condition (vi) we can obtain a contradiction, so inequality (18) holds.

From condition (v) and inequality (18), we have

\[
V_2(\tau_{k+1}, x(\tau_{k+1})) = V_2(\tau_{k+1}, x(\tau_{k+1}^-) + I_{k+1}(x(\tau_{k+1})) \geq \psi_2^{-1}(V_2(\tau_{k+1}^-, x(\tau_{k+1}^-))) \geq b_2(\delta_2).
\]

By simple induction, we can prove, in general, that for \( i = 0, 1, 2, \ldots \),

\[
V_2(t, x(t)) \geq \psi_2(b_2(\delta_2)), \quad \tau_{k+i} \leq t < \tau_{k+i+1},
\]

\[
V_2(\tau_{k+i+1}, x(\tau_{k+i+1})) \geq b_2(\delta_2).
\] (20)

Since \( b_2(\delta_2) > \psi_2(b_2(\delta_2)) \), from inequality (16) and (20) we have

\[
V_2(t, x(t)) \geq \psi_2(b_2(\delta_2)), \quad t \geq \sigma,
\]
which together with condition (iv), we have

\[ a_2(\|x\|) \geq V_2(t, x(t)) \geq \psi_2(b_2(\delta_2)) > a_2(\varepsilon_2), \quad t \geq \sigma. \]

So \( \|x\| > \varepsilon_2 \), for \( t \geq \sigma \).

Thus the zero solution of (1) is strictly uniformly stable.

The proof of Theorem 2 is complete. \( \square \)

4. Conclusion

In this paper, we have extended the notion of strict stability to impulsive functional differential equations. By using Lyapunov functions and Razumikhin technique, we have gotten some results for the strict uniform stability of this equation. We can see that impulses do contribute to system’s strict stability behavior.

References