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A Convenient Computational Form for the Adomian Polynomials*

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Recent important generalizations by G. Adomian ("Stochastic Systems," Academic Press, 1983) have extended the scope of his decomposition method for nonlinear stochastic operator equations (see also iterative method, inverse operator method, symmetrized method, or stochastic Green's function method) very considerably so that they are now applicable to differential, partial differential, delay, and coupled equations which may be strongly nonlinear and/or strongly stochastic (or linear or deterministic as subcases). Thus, for equations modeling physical problems, solutions are obtained rapidly, easily, and accurately. The methodology involves an analytic parametrization in which certain polynomials A_n, dependent on the nonlinearity, are derived. This paper establishes simple symmetry rules which yield Adomian's polynomials quickly to high orders.

INTRODUCTION

A series of publications since 1976 by Adomian (a few are given in Refs. [2–6]) have focussed on the approximate solution of nonlinear (deterministic or stochastic) differential equations for physical problems. These equations may be ordinary or partial differential equations or systems of equations and can even include delays. Solutions are obtained in series form for deterministic equations, and in the case of stochastic equations for *statistical measures* (mean, correlation, etc.) of the solution process.

A very important advantage of Adomian's basic method is the elimination of a number of restrictive and generally unsatisfactory assumptions on the nature of stochastic processes, the magnitude of fluctuations, or on the nonlinearities which are inherent in other methods. No linearization or closure approximations are necessary. One doesn't require "weak" nonlinearities or "small" fluctuations, stationarity, gaussian or white noise behavior, etc. Thus, the physical system is not forced into a nice

* Dedicated to Dr. George Adomian on his birthday, March 21, at the end of two decades of accomplishment since his 1963 dissertation.

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mathematical mold for which solutions are readily available. As a consequence it appears certain that solutions will conform far more closely to physical problems. The objective throughout Adomian's work has been the solution of real dynamical systems which are almost invariably nonlinear and stochastic. Linear stochastic, nonlinear deterministic, linear deterministic subcases are readily obtained from the method as special simple cases.

One might quite well have surmised that methodology developed for stochastic cases was unnecessary and cumbersome for deterministic cases or for linear deterministic partial differential equations. However, this is not true. Whether the equations are deterministic or stochastic, solutions are obtained very easily and with surprising accuracy, often with only a few terms.

The work has been generalized rapidly by Adomian since his first solution of (deterministic or stochastic) differential equations with polynomial nonlinearities [2, 3] to now include exponential or trigonometric nonlinearities, products of derivatives of the dependent variable, radicals, negative powers, or even decimal powers. In addition to increasing the classes of now tractable nonlinearities, he has extended the methodology to include nonlinear delay equations, coupled nonlinear differential and partial differential equations, and other general systems. The method allows inclusion of randomness in the differential operators as well as in the initial or boundary conditions and can even handle coupled boundary conditions.

The procedure is called the decomposition method* by Adomian [1] because it involves an assumed *decomposition* of the solution into components to be determined. The nonlinear terms are dealt with by an analytic parametrization in which certain polynomials A_n dependent on the nonlinearity and order of the components are derived. These represent expansion coefficients for the nonlinear terms. (We will henceforth refer to the A_n as Adomian polynomials.)

The objective in this paper is the establishment of simple symmetry rules which yield the Adomian polynomials quickly to high orders. This symmetry rule adds to the convenience of the computations and is easier to remember and derive. It, in turn, was inspired by the basic method, of course, and also by the elegant "symmetrized method" of Adomian and Sibul (also discussed in [1]) which is an alternative version of the decomposition method convenient for polynomial nonlinearities. Hopefully, this method of calculating the A_n will be useful in applications of this powerful method which has provided a new and very significant potential for the solution of numerous frontier applications involving dynamical systems [7, 8].

* Adomian's book refers extensively to "Decomposition Method." While the name is very appropriate, it is also used in other fields such as Galois theory. It is therefore preferable to call it Adomian's method as it is uniquely his creation.

Discussion

For convenience, we summarize briefly our understanding of the methodology (although the reader is strongly recommended to read the referenced book and the papers referenced there). Consider an operator equation of the form $\mathcal{F}u = g$ which Adomian uses consistently and we can refer to it conveniently as *Adomian's equation* in this paper as necessary. Here g represents a stochastic process (e.g., $g(t, \omega)$ or $g(x, y, z, t, \omega)$) appropriately defined on a probability space, and $\mathcal{F}u$ is assumed in the form $Lu + \mathcal{R}u + Nu + \mathcal{M}u$ where L is a linear deterministic operator, $\mathcal{R}$ is a linear *stochastic operator* [1], N is a nonlinear deterministic operator, and $\mathcal{M}$ is a nonlinear stochastic operator. We will concern ourselves here with differential operators. By letting any one, two, or three of the operators L, $\mathcal{R}$, N, $\mathcal{M}$ be zero, one considers the possible linear, nonlinear, deterministic, or stochastic subcases. The solution is formally $u = \mathcal{F}^{-1}g$ which is assumed decomposed into a sum of components u_0, u_1,... or $\mathcal{F}_0^{-1}g$, $\mathcal{F}_1^{-1}g$,... to be determined. Let's assume $L \neq 0$ for simplicity of discussion here (if it is, and we have nonzero initial conditions, the method still works—see [1]) and further that L is invertible. (Again, if finding the Green's function is difficult a procedure exists to decompose L into $L_1 + L_2$ where L_1 is easily or trivially invertible and L_2 is included with the other operators.)

The term $L^{-1}g$ and terms which we need not discuss here—see [1]—including given initial or boundary conditions are taken as u_0 in the decomposition. The nonlinear terms are essentially sums of Adomian polynomials $\sum A_n$.

More precisely the polynomials A_n symbolize expansion coefficients for analytic expansions in a parameter λ for nonlinear terms of the form $N(u)$. The λ is not in any sense a perturbation parameter but a convenient device for collecting terms and is dropped at the end of the calculation. We will assume in *this* paper that N is a nondifferential operator and only u appears in the operand rather than cases such as $N(u, u',...)$. These more complex cases are discussed elsewhere. In the earlier work, the polynomials were found by implicit differentiations,

$$A_n = (1/n!)(d^n/d\lambda^n) f(u(\lambda))|_{\lambda=0} \tag{1}$$

for $N(u) = f(u)$ and we parametrize u as $u(\lambda) = \sum_{n=0}^{\infty} u_n \lambda^n$ and $f(u)$ as $f(u(\lambda)) = \sum_{n=0}^{\infty} A_n \lambda^n$. Later, direct procedures resulted in

$$A_n = \sum_{v=1}^{n} C(v, n)\, H_v(u_0) \tag{2}$$

where the $C(v, n)$ were specified by a recurrence rule. Our procedure now can be viewed as a convenient symmetry rule for generation of the Adomian polynomials which we present heuristically.

For any n in A_n, compute $H_v(u_0)$ for $v = 1, 2, \ldots, n$ by differentiating $f(u)$ v times with respect to u and evaluating at $\lambda = 0$.

Thus A_3, for example, involves $H_1(u_0)$, $H_2(u_0)$, and $H_3(u_0)$. Since $A_3 = C(1,3)\,H_1(u_0) + C(2,3)\,H_2(u_0) + C(3,3)\,H_3(u_0)$ from Eq. (2), we must calculate $C(v,3)$ for $v = 1, 2, 3$. Consider the case of $v = 2$, $n = 3$ or $C(2,3)$. We ask how many ways can the sum of v integers $= n$, i.e., $\sum_{i=1}^{v} i_i = n$. We consider only combinations, not permutations. Since in this case $v = 2$, we are asking how many combinations of two integers will equal 3, i.e., $i_1 + i_2 = 3$. Consequently, we take $i_1 = 1$, $i_2 = 2$. Then $C(2,3)$ is written $u_{i_1} u_{i_2} = u_1 u_2$. In other words we simply take a product of the components of u with v factors in the product with subscripts adding to n.

Similarly, $C(1,3)$, the coefficient of $H_1(u_0)$, requires a single component whose subscript must be 3, i.e., $C(1,3) = u_3$. For the coefficient $C(3,3)$ of $H_3(u_0)$ we need three subscripts adding to 3 thus $i_1 + i_2 + i_3 = 3$. Hence we get $u_1 u_1 u_1$ or u_1^3 and we state the rule that whenever we have such a repetition of factors, we must divide by the factorial of the number of repetitions. Thus $C(3,3) = (1/3!)\,u_1^3$. Now we can write

$$A_3 = H_1(u_0)\,u_3 + H_2(u_0)\,u_1 u_2 + H_3(u_0)(1/3!)\,u_1^3.$$

In general then we have $\sum_{i_i} \prod_{i=1}^{v} u_{i_i} = \sum_{i_i} u_{i_1} u_{i_2} \cdots u_{i_v}$ with $\sum_{i=1}^{v} i_i = n$ and if we have a subscript repeated k times, we divide by $k!$.

By application of the above easy symmetry rule for writing the Adomian polynomials, general solutions can be written by the decomposition method for nonlinear operator equations (Adomian's equation). We now generalize the preceding heuristic definition to include the case of a multiplicity of repeated factors within any product of solution components.

Symmetry Rule for Adomian Polynomials

The Adomian polynomials A_n are generated by

$$A_n = \sum_{v=1}^{n} H_v(u_0)\,C(v,n)$$

where

$$C(v,n) = \sum_{i_i} \prod_{i=1}^{v} (1/k_i!)\,u_{i_i}^{k_i}$$

where $\sum_{i=1}^{v} k_i i_i = n$ and k_i is the number of repetitions in the u_{i_i}, $n > 0$, $0 \leqslant i \leqslant n$, the values of the indices i_i are greater or equal to 1 and less than or equal to $(n - v + 1)$, the numerical values of i_i are selected from this specified range by combinations without repetition, and $H_v(u_0)$ is obtained by differentiating $f(u)$, the nonlinear term, v times with respect to u and evaluating at $\lambda = 0$.

Specific examples of the polynomials A_n are listed in Ref. [1] and in many papers for various Nu such as u^2, u^3, $\sin u$, e^u, $1/u^2$, etc., e.g., [8]. Computational experience has shown that one can obtain remarkably accurate results usually with something on the order of a half dozen terms quite painlessly. When sufficiently high precision is not obtained by using a few of the A_n, we have two alternatives. One is to compute additional terms by any of the available procedures. The second approach is to use the Adomian–Malakian "convergence acceleration" procedure [10]. This unique approach conveniently yields the error-damping effect of calculating many more terms of the A_n to determine whether further calculation is justified.

REFERENCES

1. G. ADOMIAN, "Stochastic Systems," Academic Press, New York, 1983.
2. G. ADOMIAN, The solution of general linear and nonlinear stochastic system, *in* "Norbert Wiener Memorial Volume," Modern Trends in Cybernetics and Systems" (J. Rose, Ed.), pp. 160–170, Editura Technica, Bucharest, 1976.
3. G. ADOMIAN, Nonlinear stochastic differential equations, *J. Math. Anal. Appl.* **55** (1976), 441–542.
4. G. ADOMIAN, On the modeling and Analysis of nonlinear stochastic systems, *in* "Proceedings, International Conference on Mathematical Modeling" (Avula, Bellman, Luke, and Rigler, Eds.), Vol. 1, pp. 29–40, Univ. of Missouri, Rolla, 1980.
5. G. ADOMIAN, Stochastic systems analysis, *in* "Applied Stochastic Processes" (G. Adomian, Ed.), pp. 1–17, Academic Press, New York, 1980.
6. G. ADOMIAN, Solution of nonlinear stochastic physical problems, *Rend. Sem. Mat. Univ. Politec. Torino*, Numero Speciale (1982), 1–22.
7. G. ADOMIAN, "Applications of Stochastic Systems Theory to Physics and Engineering," Academic Press, New York, in press.
8. G. ADOMIAN AND R. E. BELLMAN, "New Methods in Partial Differential Equations," Reidel, Dordrecht, in press.
9. G. ADOMIAN AND R. RACH, Inversion of nonlinear stochastic operators, *J. Math. Anal. Appl.* **91** (1983), 39–46.
10. G. ADOMIAN AND K. MALAKIAN, Self-correcting approximate solutions by the iterative method for nonlinear stochastic differential equations, *J. Math. Anal. Appl.* **76** (1980), 309–327.