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Richardson varieties and equivariant K-theory

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Abstract

We generalize Standard Monomial Theory (SMT) to intersections of Schubert varieties and opposite Schubert varieties; such varieties are called Richardson varieties. The aim of this article is to get closer to a geometric interpretation of the standard monomial theory as constructed in (P. Littelmann, J. Amer. Math. Soc. 11 (1998) 551–567). In fact, the construction given here is very close to the ideas in (P. Lakshmibai, C.S. Seshadri, J. Algebra 100 (1986) 462–557). Our methods show that in order to develop a SMT for a certain class of subvarieties in G/B (which includes G/B), it suffices to have the following three ingredients, a basis for $H^0(G/B, \mathcal{L}_\lambda)$, compatibility of such a basis with the varieties in the class, certain quadratic relations in the monomials in the basis elements. An important tool (as in (P. Lakshmibai, C.S. Seshadri, J. Algebra 100 (1986) 462–557)) will be the construction of nice filtrations of the vanishing ideal of the boundary of the varieties above. This provides a direct connection to the equivariant K-theory (products of classes of structure sheaves with classes of line bundles), where the combinatorially defined notion of standardness gets a geometric interpretation.

Introduction

In many respects, the simplest case of the construction of a Standard Monomial basis is the case of the Grassmannian $X = \mathrm{Gr}_{r,n}$ of linear subspaces of dimension r in k^n, embedded into the projective space $\mathbb{P}(\bigwedge^r k^n)$ via the Plücker embedding; let L be the

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corresponding very ample line bundle on X. Then the homogeneous coordinate ring of the embedded variety coincides with the ring $R = \bigoplus_{m=0}^{\infty} H^0(X, L^{\otimes m})$, and this ring admits a nice basis, defined as follows. Let $\{e_1, \dots, e_n\}$ be the standard basis of k^n, then the wedge products $e_{\boldsymbol{i}} = e_{i_1} \wedge \cdots \wedge e_{i_r}$, where $\boldsymbol{i}$ varies over the set I of all r-tuples such that $1 \leqslant i_1 < \cdots < i_r \leqslant n$, form a basis of $\bigwedge^r k^n$.

Let $\mathbb{B} = \{p_{\boldsymbol{i}} \mid \boldsymbol{i} \in I\}$ be the dual basis, the elements $p_{\boldsymbol{i}}$ are called the *Plücker coordinates*. These elements form naturally a generating set of the homogeneous coordinate ring of the embedded variety, and it is a natural question to ask for a description of the ring in terms of relations, and, similarly, for the subvarieties like Schubert varieties, opposite Schubert varieties, unions and intersections of these. To formulate this more precisely, first observe that we have a natural partial order on I: $\boldsymbol{i} \geqslant \boldsymbol{j}$ if $i_1 \geqslant j_1, \ \dots, \ i_r \geqslant j_r$.

A product of Plücker coordinates $p_{\boldsymbol{i}} p_{\boldsymbol{j}} \dots p_{\boldsymbol{k}}$ is called *standard* if the indices are linearly ordered with respect to this partial order, i.e., $\boldsymbol{i} \geqslant \boldsymbol{j} \geqslant \cdots \geqslant \boldsymbol{k}$. These monomials deserve this "special name" because non-standard products can be expressed as linear combinations of standard products. This follows from the well-known *Plücker relations*: The Grassmannian $\mathrm{Gr}_{d,n} \subset \mathbb{P}(\bigwedge^d(V))$ consists of the zeroes of the following quadratic polynomials:

$$\sum_{l=1}^{d+1} (-1)^l p_{i_1,\dots,i_{d-1},j_l} p_{j_1,\dots,\widehat{j_l},\dots,j_{d+1}}, \tag{1}$$

where $i_1, \dots, i_{d-1}$ and $j_1, \dots, j_{d+1}$ are any numbers between 1 and n. Here we set $p_{k_1,\dots,k_d} = 0$ if at least two of the indices are equal, and otherwise $p_{k_1,\dots,k_d}$ is up to sign the Plücker coordinate obtained by rewriting the indeces in increasing order. More precisely, $p_{k_1,\dots,k_d} = \mathrm{sgn}(\sigma) p_{k_{\sigma(1)},\dots,k_{\sigma(d)}}$, where σ is the unique element in the symmetric group S_d such that $k_{\sigma(1)} < \cdots < k_{\sigma(d)}$. If two Plücker coordinates $p_{\boldsymbol{i}}$ and $p_{\boldsymbol{i}'}$ are not comparable in the partial order, by applying the Plücker relation (1) repeatedly, one gets an expression of the following form, called *quadratic straightening relation*

$$p_{\boldsymbol{i}} p_{\boldsymbol{i}'} = \sum_{\boldsymbol{j} \geqslant \boldsymbol{i}, \boldsymbol{i}' \geqslant \boldsymbol{k}} a_{\boldsymbol{j},\boldsymbol{k}} p_{\boldsymbol{j}} p_{\boldsymbol{k}}, \tag{2}$$

where the coefficients are elements of the ground field k and the products $p_{\boldsymbol{j}} p_{\boldsymbol{k}}$ occurring on the right side of the equation are standard. Hodge [8] has already observed that these relations give a full presentation of the ring R. In particular, the standard monomials form a basis of the ring R. Further, Hodge has also noticed that this basis is compatible with Schubert varieties, i.e., for $\boldsymbol{i} \in I$ let $X_{\boldsymbol{i}}$ be the Schubert variety consisting of subspaces $U \in X$ such that for all $1 \leqslant s \leqslant r$, $\dim(U \cap \mathrm{span}(e_1, \dots, e_s))$ is greater or equal to the number of j such that $i_j \leqslant s$. We say that a standard monomial $p_{\boldsymbol{j}} \dots p_{\boldsymbol{k}}$ is standard on $X_{\boldsymbol{i}}$ if $\boldsymbol{i} \geqslant \boldsymbol{j}$. Then Hodge has already shown that the restriction of the standard monomials, not standard on $X_{\boldsymbol{i}}$, vanish identically, and the restrictions of the standard monomials, standard on $X_{\boldsymbol{i}}$, form a basis of $R_{\boldsymbol{i}} = \bigoplus_{m=0}^{\infty} H^0(X_{\boldsymbol{i}}, L^{\otimes m})$. Such a description has some immediate geometric applications, for example the Schubert variety is projectively normal in the embedding, one has a presentation of the homogeneous ideal of the Schubert varieties, etc.

The purpose of *Standard Monomial Theory* (SMT) is to generalize this kind of description to all embeddings $G/P \hookrightarrow \mathbb{P}(H^0(G/P, \mathcal{L}_\lambda)^*)$, where G is a semi-simple algebraic group over an algebraically closed field, P is a parabolic subgroup, and $\mathcal{L}_\lambda$ is an ample line bundle on G/P corresponding to a dominant weight of G.

This paper has two aims: The first aim is to generalize SMT to intersections of Schubert varieties and opposite Schubert varieties; such varieties will be called Richardson varieties, since Richardson looked at these first (cf. [27]). The second aim of this paper is to make SMT easier generalizable to other classes of varieties by relying mostly on geometric (K-theory) and combinatorial (character formula) arguments, and reducing quantum-Frobenius splitting arguments used for example in [20] as much as possible.

To be more precise, fix a Borel subgroup $B \subset G$ and a maximal torus T, and let B^- denote the Borel subgroup of G opposite to B (it is the unique Borel subgroup of G with the property $B \cap B^- = T$). Let $P \supset B$ be a parabolic subgroup, let W_P be its Weyl group, we identify W_P with a subgroup of the Weyl group W of G. For $\tau \in W/W_P$, let $e_\tau \in G/P$ be the corresponding coset and let X_τ denote the *Schubert variety*, the Zariski closure of the B-orbit Be_τ in G/P. Similarly, let X^τ denote the *opposite Schubert variety*, the Zariski closure of the B^--orbit B^-e_τ in G/P. The *Richardson variety* X_τ^κ is then the intersection $X_\tau \cap X^\kappa$. For $P = B$, such double coset intersections $B\,\tau\,B \cap B^-\kappa B$ first appear in [5,9,10,27,28]; in fact, it is shown in [27] that $B\,\tau\,B \cap B^-\,\kappa\,B$ is dense in $\overline{B\,\tau\,B} \cap \overline{B^-\,\kappa\,B}$. Recently, such (as well as similar) double coset intersections have appeared in the context of total positivity (cf. [22]), but this connection is not relevant for the context of this paper. Richardson varieties also appear in the context of K-theory of flag varieties as explained below.

The starting point of our approach will be as in the example above. We start with:

(a) a basis $\mathbb{B}(\lambda)$ of $H^0(G/P, \mathcal{L}_\lambda)$ which will be indexed by a partially ordered set $B(\lambda)$;
(b) $\mathbb{B}(\lambda)$ is compatible with Schubert and opposite Schubert varieties Z (i.e., the set $\{b|_Z \mid b \in \mathbb{B}(\lambda), b|_Z \not\equiv 0\}$ is linearly independent);
(c) $\mathbb{B}(\lambda)$ satisfies certain quadratic relations similar to the *quadratic straightening relations* (2) described above.

The main step in the construction of SMT is the following: let X_τ^κ be a Richardson variety. By the boundary $\partial^+ X_\tau^\kappa$ we mean the subvariety of X_τ^κ consisting of the union of all Richardson varieties $X_{\tau'}^{\kappa'}$ such that $\kappa \leqslant \kappa' \leqslant \tau' < \tau$. Let $\partial^+\mathcal{I}_\tau^\kappa$ be the ideal sheaf of this subvariety. Using the global basis $\mathbb{B}(\lambda) \subset H^0(G/P, \mathcal{L}_\lambda)$ and the straightening relations, we show that the twisted sheaf $\partial^+\mathcal{I}_\tau^\kappa \otimes \mathcal{L}_\lambda$ admits a filtration as a coherent $\mathcal{O}_X$-sheaf such that the associated graded $\mathcal{O}_X$-sheaf is a direct sum of structure sheaves of Richardson varieties, and for the coefficients we have a combinatorial formula involving the combinatorics of the indexing set $B(\lambda)$. Once we have such a *Pieri–Chevalley type formula*, we can proceed in a straight forward way (using induction on the dimension) and establish a standard monomial theory compatible with all unions of Richardson varieties.

Summarizing, we conclude the introduction with a short description of the development of SMT and its present status. It was developed by Lakshmibai, Musili, and Seshadri in a series of papers, culminating in [16] where it is established for all $G/P \hookrightarrow \mathbb{P}(H^0(G/P, \mathcal{L}_\lambda)^*)$, G a classical group. Further results concerning certain exceptional and Kac–Moody groups led to conjectural formulations of a general SMT, see [17]. The combinatorial conjectures have been proved by the second author in [18], where he introduced a new combinatorial tool, the path model. Another development was the introduction of the dual of Lusztig's Frobenius map for quantum groups at roots of unity (see [20]; see also [12,13]). This Frobenius map turns out to provide exactly the right tool to give a construction of SMT in the most general setting: $G/P \hookrightarrow \mathbb{P}(H^0(G/P, \mathcal{L}_\lambda)^*)$, where G is an arbitrary symmetrizable Kac–Moody group and λ is an integral dominant weight such that $\mathcal{L}_\lambda$ is ample on G/P.

To be able to generalize SMT to other types of varieties (compactifications of symmetric spaces or, more generally, spherical varieties for example), it would be very helpful to get a better algebraic geometric description or characterization of the SMT. A first step in this direction is the observation of the connection between SMT and equivariant K-theory as described in [21], and this article can be described as an effort to reduce the input from quantum group methods (which are not available in the examples above) in [20] as much as possible and to use the Pieri–Chevalley type formula as part of the construction instead of proving it as a consequence of SMT (as in [21]). In fact, once one has a good candidate for SMT, the inductive procedure and a Pieri–Chevalley type formula reduces the proof to the counting of global sections while the proof of the vanishing theorem for higher cohomology has reminiscences of the corresponding proofs using characteristic $p > 0$ methods (see Theorem 20).

Another step in this direction has been proved recently by M. Brion and the first author (cf. [2]). They provide a geometric construction of a (SMT) basis for $H^0(G/P, \mathcal{L}_\lambda)$, compatible with unions of Richardson varieties. The main tool introduced in [2] is a flat T-invariant degeneration of the diagonal in $X \times X$ (X being G/P) to the union of all products $X_\tau \times X^\tau$, where the X_τ are the Schubert varieties and the X^τ are the corresponding opposite Schubert varieties; we also obtain this degeneration as a consequence of the standard monomial basis, see Section 11. It is interesting to note that while in [2], the above degeneration is the starting point towards developing a SMT for Richardson varieties, in this paper such a degeneration is obtained as a consequence of SMT! In fact, by a result of [2], the fact that the special fibre of the degeneration of the diagonal is the union of all products $X_\tau \times X^\tau$ is equivalent to the fact that standard monomials form a basis (Theorem 20(ii)). In [2], for any Richardson variety X_τ^κ, the authors first construct a basis of the space of sections of $\mathcal{L}_\lambda$ on X_τ^κ vanishing on both of the boundaries $\partial^\pm X_\tau^\kappa$ (here, $\partial^- X_\tau^\kappa$ is the subvariety of X_τ^κ consisting of the union of all Richardson varieties $X_{\tau'}^{\kappa'}$ such that $\kappa < \kappa' \leqslant \tau' \leqslant \tau$). This basis is then lifted to global sections of $\mathcal{L}_\lambda$ that vanish on all Richardson varieties not containing X_τ^κ, which in turn give raise to a basis for $H^0(G/P, \mathcal{L}_\lambda)$; this basis for $H^0(G/P, \mathcal{L}_\lambda)$ is then shown to have compatibility with unions of Richardson varieties. In this article, we show the compatibility of the basis of [20] with unions of Richardson varieties (cf. Theorem 20), and the vanishing of a typical basis vector on $\partial^\pm X_\tau^\kappa$ for suitable κ and τ. Thus the basis as constructed in [20] may be considered as a specially nice basis of the type constructed in [2]. In spite of this relationship between the bases of [2] and [20], there are still some missing links; to be very precise, let $\pi = (\boldsymbol{\tau}, \boldsymbol{a})$ be an L–S path of shape λ, where $\boldsymbol{\tau} = (\tau_0, \ldots, \tau_r)$ is a strictly decreasing sequence in W/W_P. The first and last element describe the smallest Richardson variety $X_{\tau_0}^{\tau_r}$ such that the restriction of the path vector p_π does not vanish identically. The other terms come up naturally in the quantum group construction and have a suggestive algebraic geometric interpretation in [20], but they do not have yet an interpretation in the construction in [2].

The paper is organized as follows: In the first three sections, we recall the combinatorics of the path model and the construction of the path vector basis: the indexing set I in the Grassmannian case will be replaced by the set of L–S paths $B(\lambda)$ of shape λ (Section 2), the basis $\{p_{\boldsymbol{i}}, \boldsymbol{i} \in I\}$ will be replaced by the basis $\mathbb{B}(\lambda)$ consisting of path vectors $\{p_\pi, \pi \in B(\lambda)\}$ (Section 3), and the Plücker relations will be replaced by the quadratic straightening relations (Section 5). In Section 4 we recall the connection between coherent sheaves and graded finitely generated modules over the homogeneous coordinate ring $k[G/P]$ given by the embedding. Using the properties (a)–(c), we show in Section 6 that the ideal I_τ^κ has a basis given by standard monomials. This is used in Section 7 to prove the Pieri–Chevalley type formula for the ideal sheaf $\partial^+\mathcal{I}_\tau^\kappa$, and, as a consequence, we obtain in Section 8 the construction of SMT and the vanishing of higher cohomology groups. In Sections 9 and 10 we discuss the case where the bundle $\mathcal{L}_\lambda$ is only base point free but not necessarily ample. In the last Section 11, the relationship between K-theory and SMT is brought out, especially, the computation of the coefficients of the classes of structure sheaves appearing in the product of the class of the structure sheaf of a Schubert variety with the class of a line bundle.

1. Notation

In this section we fix some standard notation which will be used throughout the paper. The ground field k is supposed to be algebraically closed of arbitrary characteristic. The group G is a semisimple, simply connected algebraic group defined over k. We fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. We denote by B^- the opposite Borel subgroup, i.e., B^- is the unique Borel subgroup of G such that $B \cap B^- = T$. The unipotent radical of B is denoted by U, and the unipotent radical of B^- is denoted by U^-.

We use the same notation (but with Gothic letters) for the corresponding Lie algebras, i.e., $\mathfrak{g}$ is the Lie algebra of G, $\mathfrak{b}$ the Lie algebra of B, $\mathfrak{b}^-$ is the Lie algebra of Lie B^- and $\mathfrak{n}^\pm$ denotes the Lie algebra of $U^\pm$. The corresponding enveloping algebras are denoted by $U(\mathfrak{g})$, $U(\mathfrak{b}^\pm)$ and $U(\mathfrak{n}^\pm)$.

Let $Q \supset B$ be a parabolic subgroup of G containing B. The projective variety G/Q admits a finite number of B-orbits (as well as a finite number of B^--orbits), they are indexed by the T-fixed points in G/Q. Let W be the Weyl group of G and let W_Q be the Weyl group of Q. The group W_Q can be canonically identified with the subgroup of W generated by the simple reflections s_α such that $-\alpha$ is a root of the root system of Q. For $\tau \in W/W_Q$ let $e_\tau \in G/Q$ be the corresponding T-fixed point. The closure of the B-orbit: $X_\tau = \overline{Be_\tau} \subset G/Q$ is the *Schubert variety* corresponding to τ, and the closure of the B^--orbit: $X^\tau = \overline{B^- e_\tau} \subset G/Q$, the *opposite Schubert variety* corresponding to τ.

Let $\tau, \sigma \in W/W_Q$. The *Richardson variety* corresponding to the pair (τ, σ) is the (set theoretic) intersection $X_\tau^\sigma = X_\tau \cap X^\sigma$ (with the induced reduced structure). Note that if

$\overline{w}_0 \in W/W_Q$ is the class of the longest element w_0 in W, then $X^{\sigma}_{\overline{w}_0} = X^{\sigma}$. Similarly, one has $X^{id}_{\tau} = X_{\tau}$.

If $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_s)$ and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_s)$ are sequences of elements in W/W_Q, then we denote by $X^{\boldsymbol{\sigma}}_{\boldsymbol{\kappa}}$ the union $X^{\sigma_1}_{\kappa_1} \cup \cdots \cup X^{\sigma_s}_{\kappa_s}$ of Richardson varieties with the induced reduced structure.

Let Λ be the character group of T. We denote by Λ^+ the dominant weights and by Λ^{++} the regular dominant weights. For the parabolic subgroup Q, let Λ_Q be the subgroup of weights which can be (trivially) extended to characters of Q, let $\Lambda^+_Q \subset \Lambda_Q$ be the dominant ones and denote by Λ^{++}_Q the Q-regular dominant weights, i.e., those dominant weights that cannot be extended to characters of a parabolic subgroup containing Q properly.

For $\lambda \in \Lambda$, let $\mathcal{L}_\lambda = G \times_B k_{-\lambda}$ be the line bundle on G/B associated to the B-character $(-\lambda)$. A geometric way to characterize $\Lambda_Q \subset \Lambda$ is to say that λ is an element of Λ_Q if and only if $\mathcal{L}_\lambda$ "goes down" to a line bundle $\mathcal{L}_\lambda = G \times_Q k_{-\lambda}$ on G/Q, and $\lambda \in \Lambda^+_Q$ if only if $\mathcal{L}_\lambda$ is base point free on G/Q, and $\lambda \in \Lambda^{++}_Q$ if only if $\mathcal{L}_\lambda$ is an ample line bundle on G/Q.

2. The path model and some partial orders

An important combinatorial tool will be the L–S paths of shape λ, we recall quickly the most important features of the path model. Let $\lambda \in \Lambda^+$ be a dominant weight. Fix a *total order* "$\succeq$" on W/W_λ refining the *Bruhat order* "$\geqslant$". Let $\boldsymbol{\tau} = (\tau_0 \succ \cdots \succ \tau_r)$ be a strictly decreasing sequence of elements of W/W_λ and let $\boldsymbol{a} = (0 < a_1 < \cdots < a_r < 1)$ be strictly increasing sequence of rational numbers. The pair $\pi = (\boldsymbol{\tau}, \boldsymbol{a})$ is called a *convex subset of shape* λ of the orbit $W.\lambda$. To motivate the name, set $a_0 = 0$ and $a_{r+1} = 1$ and set $x_i = a_i - a_{i-1}$ for $1 \leqslant i \leqslant r+1$. Then x_i is called the *weight of* τ_{i-1}, and the sum $\sum_{i=0}^{r} x_{i+1}\tau_i(\lambda)$ is a convex linear combination, called the *weight of* π, and is denoted $\pi(1)$.

Such a *convex subset* π is called a L–S path of shape λ if the sequence $\boldsymbol{\tau} = (\tau_0, \ldots, \tau_r)$ is strictly decreasing in the Bruhat order (on W/W_λ), and if the pair satisfies the following integrality condition. For all $i = 1, \ldots, r$:

- Set $s_i = l(\tau_{i-1}) - l(\tau_i)$. There exists a sequence $\beta_1, \ldots, \beta_{s_i}$ of positive roots joining τ_{i-1} and τ_i by the corresponding reflections, i.e.,

$$\tau_{i-1} > s_{\beta_1}\tau_{i-1} > s_{\beta_2}s_{\beta_1}\tau_{i-1} > \cdots > s_{\beta_{s_i}} \cdots s_{\beta_1}\tau_{i-1} = \tau_i,$$

  and $a_i\langle \tau_{i-1}(\lambda), s_{\beta_1} \cdots s_{\beta_{j-1}}(\beta_j^\vee)\rangle \in \mathbb{Z}$ for all $j = 1, \ldots, s_i$. Here $\beta^\vee$ denotes the coroot of β, and $\langle \mu, \beta^\vee\rangle$ stands for the evaluation of $\mu \in \Lambda$ on the coroot.

For more details on the combinatorics of L–S paths we refer the reader to [18]. Let $B(\lambda)$ be the set of L–S paths of shape λ. The character of the Weyl module $V(\lambda)$ of highest weight λ can be calculated using the L–S paths:

Theorem 1 [18]. Char $V(\lambda) = \sum_{\pi \in B(\lambda)} \mathrm{e}^{\pi(1)}$.

Let $\pi = (\boldsymbol{\tau}, \boldsymbol{a})$ be an L–S path of shape λ, where $\boldsymbol{\tau} = (\tau_0, \ldots, \tau_r)$. We call $i(\pi) = \tau_0$, the *initial direction* and $e(\pi) = \tau_r$, the *final direction* of the path.

Definition 2. Let $B(\lambda)$ be the set of L–S paths of shape λ. We say that $\pi \in B(\lambda)$ is *standard on a Richardson variety* X_τ^κ if $\tau \geqslant i(\pi)$ and $e(\pi) \geqslant \kappa$, and we denote by $B_\tau^\kappa(\lambda)$ the set of all L–S paths of shape λ, standard on X_τ^κ. If τ (respectively κ) is the class of the longest word in W (respectively *id*), then τ (respectively κ) will be omitted and we will write just $B^\kappa(\lambda)$ (respectively $B_\tau(\lambda)$).

A sequence $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$ of L–S paths of shape λ is called *standard* if

$$e(\pi_1) \geqslant i(\pi_2) \geqslant \cdots \geqslant e(\pi_{m-1}) \geqslant i(\pi_m). \tag{3}$$

The notion of initial and final direction generalizes to standard sequences of length m as follows: We set $i(\boldsymbol{\pi}) = i(\pi_1)$ and $e(\boldsymbol{\pi}) = e(\pi_m)$. The notion of a standard sequence on a Richardson variety generalizes in the obvious way.

Let $\boldsymbol{\pi}(1) = \pi_1(1) + \cdots + \pi_m(1)$ be the *weight of such a standard sequence*. The theorem above generalizes to standard sequences:

Theorem 3 [18]. Char $V(m\lambda) = \sum \mathrm{e}^{\boldsymbol{\pi}(1)}$, *where the sum runs over all standard sequences of length m of L–S paths of shape λ.*

We need several orderings on the set of convex subsets of $W.\lambda$. Let $\pi = (\boldsymbol{\tau}, \boldsymbol{a})$ and $\eta = (\boldsymbol{\kappa}, \boldsymbol{b})$ (where $\boldsymbol{\kappa} = (\kappa_0, \ldots, \kappa_s)$ and $\boldsymbol{b} = (b_1, \ldots, b_s)$) be two convex subsets of $W.\lambda$. Induced by the Bruhat order on W/W_λ, we have two types of partial orders on such convex subsets.

- We say $\pi \geqslant \eta$ if π is greater than η in the weighted lexicographic sense: $\tau_0 > \kappa_0$, or $\tau_0 = \kappa_0$ and $a_1 > b_1$, or $\tau_0 = \kappa_0, a_1 = b_1$ and $\tau_1 > \kappa_1$, etc.
- We say $\pi \geqslant^r \eta$ if π is greater than η in the reverse weighted lexicographic sense: $\tau_r > \kappa_s$ or $\tau_r = \kappa_s$ and $1 - a_r > 1 - b_s$ or $\tau_r = \kappa_s, 1 - a_r = 1 - b_s$ and $\tau_{r-1} > \kappa_{s-1}$, etc.

For a total order $\succeq$ on W/W_λ we define in the same way total orders $\succeq$ and $\succeq^r$ on the set of convex subset of elements of W/W_λ.

We extend these (partial) orders in the obvious way to sequences. Let $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_t)$ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_t)$ be two sequences of weighted subsets.

- We say $\boldsymbol{\pi} \geqslant \boldsymbol{\eta}$ if $\pi_1 > \eta_1$ or $\pi_1 = \eta_1$ and $\pi_2 > \eta_2$ etc.
- We say $\boldsymbol{\pi} \geqslant^r \boldsymbol{\eta}$ if $\pi_t > \eta_t$ or $\pi_t = \eta_t$ and $\pi_{t-1} > \eta_{t-1}$ etc.

The total orders $\succeq$ and $\succeq^r$ are extended to sequences in the same way.

3. The path vector basis

Suppose $\lambda \in \Lambda_Q^+$, then the line bundle $\mathcal{L}_\lambda$ is base point free. Let $V(\lambda)$ be the Weyl module of highest weight λ. The parabolic subgroup Q stabilizes the line through the highest weight vector v_λ and we have a corresponding map

$$G/Q \to \mathbb{P}\big(V(\lambda)\big), \quad gQ \mapsto [gv_\lambda].$$

Further, $H^0(G/Q, \mathcal{L}_\lambda) = V(\lambda)^*$ is the dual space of $V(\lambda)$. Associated to the combinatorial path model given by the L–S paths $B(\lambda)$ of shape λ, we have the basis

$$\mathbb{B}(\lambda) = \big\{p_\pi \mid \pi \in B(\lambda)\big\} \subset H^0(G/Q, \mathcal{L}_\lambda)$$

given by the path vectors p_π as constructed in [20].

Remark 4. The reader not acquainted with the construction should think of these sections in the following way: Let $\pi = (\tau_0, \ldots, \tau_r, a_1, \ldots, a_r)$ be an L–S path of shape λ. Fix $\ell \in \mathbb{N}$ minimal such that $\ell a_i \in \mathbb{N}$ for all i, and fix weight vectors $p_{\tau_i} \in H^0(G/Q, \mathcal{L}_\lambda)$ of weight $-\tau_i(\lambda)$ (these weight spaces are one-dimensional). Then p_π can be thought of as an algebraic approximation

$$p_\pi \sim \sqrt[\ell]{p_{\tau_0}^{\ell x_1} p_{\tau_1}^{\ell x_2} p_{\tau_3}^{\ell x_3} \cdots p_{\tau_r}^{\ell x_{r+1}}}, \quad x_i = a_i - a_{i-1},\ 1 \leqslant i \leqslant r+1.$$

Note that in the framework of quantum groups at roots of unity the expression above "makes sense". Indeed, $\sqrt[\ell]{p_{\tau_0}^{\ell x_1} \cdots}$ is then the Frobenius splitting map at an ℓth root of unity applied to the product, for details see [20].

The path vectors are T-weight vectors of weight $-\pi(1)$. Further, the partial orders on L–S paths introduced in the section before are closely related to B-stable (respectively B^--stable) submodules spanned by path vectors.

More precisely, we call a subset $S^+ \subset B(\lambda)$ of L–S paths of shape λ *positive saturated* if for all $\pi, \pi' \in S^+$ the following holds:

- if $\eta \in B(\lambda)$ is such that $\pi' > \eta > \pi$, then $\eta \in S^+$.

We say that $S^+ \subset B(\lambda)$ is *maximally positive saturated* if for $\pi \in S^+$ and $\eta \in B(\lambda)$ the relation $\eta > \pi$ implies $\eta \in S^+$.

Similarly, we call a subset $S^- \subset B(\lambda)$ of L–S paths of shape λ *negative saturated* if for all $\pi, \pi' \in S^-$ the following holds:

- if $\eta \in B(\lambda)$ is such that $\pi' >^r \eta >^r \pi$, then $\eta \in S^-$.

We say that $S^- \subset B(\lambda)$ is *maximally negative saturated* if for $\pi \in S^-$ and $\eta \in B(\lambda)$ the relation $\pi >^r \eta$ implies $\eta \in S^-$.

Let S^+ be a maximally positive saturated subset and let S^- be a maximally negative saturated subset of $B(\lambda)$. The corresponding path vectors span T-submodules of $H^0(G/Q, \mathcal{L}_\lambda)$:

$$M\big(S^+\big) = \big\langle p_\pi \mid \pi \in S^+\big\rangle \quad \text{and} \quad M(S^-) = \big\langle p_\pi \mid \pi \in S^-\big\rangle.$$

Theorem 5 [14,20]. *The T-submodule $M(S^+)$ is B-stable and the T-submodule $M(S^-)$ is a B^--stable submodule of $H^0(G/Q, \mathcal{L}_\lambda)$.*

Corollary 6. *Let $S^+ \subset B(\lambda)$ be a positive saturated subset and let $M(S^+)$ be the T-submodule $\langle p_\pi \mid \pi \in S^+\rangle$ of $H^0(G/Q, \mathcal{L}_\lambda)$ spanned by the corresponding path vectors. Then $M(S^+)$ admits a B-module structure which is isomorphic to a subquotient of a B-stable filtration of $H^0(G/Q, \mathcal{L}_\lambda)$.*

Proof. Let $S_1^+ = \{\eta \in B(\lambda) \mid \exists \pi \in S^+\colon\ \eta \geqslant \pi\}$ be the "closure" of S^+ with respect to $>$, i.e., S_1^+ consists of all paths which are greater or equal to some element of S^+. The set S_1^+ is a maximally positive saturated subset and hence $M(S_1^+)$ is a B-submodule of $H^0(G/Q, \mathcal{L}_\lambda)$. The set $S_2^+ := S_1^+ - S^+$ is again a maximally positive saturated subset because: suppose $\pi \in S_2^+$ and $\eta \in B(\lambda)$ is such that $\eta > \pi$. By the definition of S_1^+ and S_2^+, there exists an element $\pi_1 \in S^+$ such that $\pi > \pi_1$, so $\eta > \pi > \pi_1$ implies $\eta \in S_1^+$. So either $\eta \in S_2^+$ or $\eta \in S^+$. The latter is not possible because $\eta, \pi_1 \in S^+$ would imply $\pi \in S^+$, in contradiction to the assumption $\pi \in S_2^+$, which finishes the proof of the claim.

Now $M(S_2^+)$ is hence a B-submodule, and the quotient $M(S_1^+)/M(S_2^+)$ is a B-module which, as T-module, is isomorphic to $M(S^+)$. □

The corresponding version for negative saturated subsets holds also, the proof is left to the reader.

Corollary 7. *Let $S^- \subset B(\lambda)$ be a negative saturated subset and let $M(S^-)$ be the T-submodule $\langle p_\pi \mid \pi \in S^+\rangle$ of $H^0(G/Q, \mathcal{L}_\lambda)$ spanned by the corresponding path vectors. Then $M(S^-)$ admits a B^--module structure which is isomorphic to a subquotient of a B^--stable filtration of $H^0(G/Q, \mathcal{L}_\lambda)$.*

For $\tau \in W/W_\lambda$ let $v_\tau \in V(\lambda)$ be a weight vector of weight $\tau(\lambda)$, v_τ is a so-called *extremal weight vector*. The B-submodule spanned by the orbit $B.v_\tau$ is called the *Demazure module* associated to τ and is denoted $V_\tau(\lambda)$. Note that set theoretically we have $X_\tau = G/Q \cap \mathbb{P}(V_\tau(\lambda))$. Similarly, the B^- submodule spanned by the orbit $B^-.v_\tau$ is called the *opposite Demazure module* associated to τ and is denoted $V^\tau(\lambda)$.

Theorem 8 [14,20]. *The path vector basis is compatible with the Demazure submodules, i.e., the restrictions $\{p_\pi|_{V_\tau(\lambda)} \mid \pi \in B_\tau(\lambda)\}$ form a basis of $V_\tau(\lambda)^*$ and the restrictions of the other path vectors vanish on the submodule. Similarly, the restrictions $\{p_\pi|_{V^\tau(\lambda)} \mid \pi \in B^\tau(\lambda)\}$ form a basis of $V^\tau(\lambda)^*$ and the restrictions of the other path vectors vanish.*

Let $\mathbb{B}(\lambda)^* = \{u_\pi \in V(\lambda) \mid \pi \in B(\lambda)\}$ be the basis of $V(\lambda)$ *dual* to the path vector basis of $H^0(G/Q, \mathcal{L}_\lambda)$.

Corollary 9. *The vectors $\{u_\pi \mid \pi \in B_\tau(\lambda)\}$ form a basis of the Demazure module $V_\tau(\lambda)$, the vectors $\{u_\pi \mid \pi \in B^\tau(\lambda)\}$ form a basis of the opposite Demazure module $V^\tau(\lambda)$, and the vectors $\{u_\pi \mid \pi \in B_\tau^\sigma(\lambda)\}$ form a basis of the intersection $V_\tau^\sigma(\lambda) = V_\tau(\lambda) \cap V^\sigma(\lambda)$.*

Proof of the corollary. Theorem 8 implies that $V_\tau(\lambda)$ is the subspace of $V(\lambda)$ orthogonal to $\langle p_\pi \mid i(\pi) \nleqslant \tau\rangle$. Hence $\{u_\pi \mid i(\pi) \leqslant \tau\} \subset V_\tau(\lambda)$, and, again by Theorem 8, $\langle u_\pi \mid i(\pi) \leqslant \tau\rangle = V_\tau(\lambda)$. The proof for the opposite Demazure module is similar and is left to the reader. The statement for the intersection is then an immediate consequence of the fact that the basis is compatible with Demazure and opposite Demazure modules. $\square$

4. Some graded rings and modules

Suppose $\lambda \in \Lambda_Q^{++}$, so the line bundle $\mathcal{L}_\lambda$ is very ample on G/Q and we have a corresponding embedding $G/Q \hookrightarrow \mathbb{P}(V(\lambda))$. Consider the two rings

$$R = \bigoplus_{m\geqslant 0} H^0(G/Q, \mathcal{L}_{m\lambda}) \quad \text{and} \quad k[G/Q] = \bigoplus_{m\geqslant 0} k[G/Q]_m,$$

where $k[G/Q]$ denotes the homogeneous coordinate ring of the embedding $G/Q \hookrightarrow \mathbb{P}(V(\lambda))$ with the usual grading. Since λ is ample and G/Q is smooth, one knows that $H^0(G/Q, \mathcal{L}_{m\lambda}) = k[G/Q]_m$ for $m \gg 0$. (Actually, by standard monomial theory [20] or Frobenius splitting [25], one knows that they coincide for all m, but we do not need this later.) Correspondingly we denote for a union of Richardson varieties $X_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}$ the associated rings by $R_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}$ and $k[X_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}]$, i.e.,

$$R_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}} = \bigoplus_{m\geqslant 0} H^0\big(X_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}, \mathcal{L}_{m\lambda}\big) \quad \text{and} \quad k\big[X_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}\big] = \bigoplus_{m\geqslant 0} k\big[X_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}\big]_m.$$

Let $\mathcal{I}_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}} \subset \mathcal{O}_{G/Q}$ be the ideal sheaf and let $I_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}} \subset k[G/Q]$ be the homogeneous ideal of $X_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}} \subset G/Q \subset \mathbb{P}(V(\lambda))$. We consider also the graded R-module, respectively the graded $k[G/Q]$-module:

$$J_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}} = \bigoplus_{m\geqslant 0} H^0\big(G/Q, \mathcal{I}_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}} \otimes \mathcal{L}_{m\lambda}\big), \quad \text{respectively} \quad I_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}} = \bigoplus_{m\geqslant 0} \big(I_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}\big)_m.$$

For Schubert varieties and opposite Schubert varieties we still use the notation X_τ, I_τ, X^τ, I^τ, etc., instead of X_τ^{id}, I_τ^{id}, $X_{\overline{w}_0}^\tau$, $I_{\overline{w}_0}^\tau$ etc.

Our aim is to show in the next *four sections that the rings $R_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}$ and $k[X_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}]$ and the modules $J_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}$ and $I_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}$ coincide and have a basis by standard monomials*.

5. Standard monomials and quadratic relations

Let $\lambda \in \Lambda_Q^{++}$, we use the same notation as in Section 4. We analyze the structure of the homogeneous coordinate ring $k[G/Q]$. We view $k[G/Q]$ as the subalgebra of R generated by $H^0(G/Q, \mathcal{L}_\lambda) = k[G/Q]_1$. A monomial of path vectors $p_{\boldsymbol{\pi}} = p_{\pi_1} \cdots p_{\pi_m}$ is called *standard* if the sequence $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$ is standard. The monomial $p_{\boldsymbol{\pi}}$ is called *standard on a Richardson variety* X_τ^σ if in addition $\tau \geqslant i(\boldsymbol{\pi}) \geqslant e(\boldsymbol{\pi}) \geqslant \sigma$ for the initial and final directions of $\boldsymbol{\pi}$. The monomial is called *standard on a union of Richardson varieties* $X_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}$ if it is standard on at least one of the $X_{\tau}^{\sigma} \subset X_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}$.

The following relations provide an algorithm to express a non-standard monomial as a linear combination of standard monomials. To be more precise, we need to introduce the wedge product of two convex subsets of W/W_λ.

Let $\succeq$ be a the fixed total order refining the Bruhat order. Given two convex subsets $\pi = (\boldsymbol{\tau}, \boldsymbol{a})$ and $\pi' = (\boldsymbol{\sigma}, \boldsymbol{b})$ of shape λ, let $\boldsymbol{\kappa}$ be the sequence obtained from $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$

$$\{\kappa_0, \kappa_1, \ldots\} = \{\tau_0, \ldots, \tau_r\} \cup \{\sigma_0, \ldots, \sigma_s\}$$

by writing the elements κ_j in strictly decreasing order with respect to $\succ$.

Next let $\boldsymbol{c}$ be the strictly increasing sequence of rational numbers obtained from $\boldsymbol{a}$ and $\boldsymbol{b}$ as follows: c_i is half the sum of the weights of the τ_j and σ_j which are smaller or equal to κ_i. More precisely: set $c_0, a_0, b_0 := 0$. If $\kappa_{i-1} = \tau_{j-1}$ and not equal to one of the σ_m, then $c_i := c_{i-1} + (a_j - a_{j-1})/2$; if $\kappa_{i-1} = \sigma_{j-1}$, and not equal to one of the τ_m, then $c_i := c_{i-1} + (b_j - b_{j-1})/2$; if $\kappa_{i-1} = \tau_{j-1} = \sigma_{m-1}$, then $c_i := c_{i-1} + ((a_j - a_{j-1}) + (b_m - b_{m-1}))/2$.

Definition 10. The *wedge product* $\pi \wedge \pi'$ of two convex subsets of shape λ is the convex subset $(\boldsymbol{\kappa}, \boldsymbol{c})$ of shape 2λ.

Note that if π, π' are L–S paths, then $\pi \wedge \pi'$ is in general not an L–S path. Let now π, π' be L–S paths of shape λ. We say that π and π' have the *same support* if the sequence $\boldsymbol{\kappa}$ in $\pi \wedge \pi'$ is strictly decreasing with respect to the Bruhat ordering $>$. One checks easily the following properties:

Lemma 11. (i) *The map $(\pi, \pi') \to \pi \wedge \pi'$ induces a bijection between the set of standard sequences of length two of L–S paths of shape λ and the set of L–S paths of shape 2λ.*

(ii) *If π, π' have the same support, then $\pi \wedge \pi'$ is an L–S path of shape 2λ.*

Remark 12. The wedge product of convex sequences generalizes in the obvious way to m-tuples: We define $(\boldsymbol{\kappa}, \boldsymbol{c}) = \pi_1 \wedge \cdots \wedge \pi_m$ as follows: The sequence $\boldsymbol{\kappa}$ is the union of the Weyl group cosets occurring in the π_j, rewritten in strictly decreasing order. Every coset κ_i in $\boldsymbol{\kappa}$ occurs in at least one path; set $w_i^k =$ the weight of the coset κ_i, if κ_i occurs in π_k, and set $w_i^k = 0$ if κ_i does not occur in π_k. The rational number c_i in $\boldsymbol{c}$ corresponding to the coset κ_i is

$$c_i = \frac{1}{m} \sum_{1 \leqslant \ell \leqslant i} \left(\sum_{1 \leqslant k \leqslant m} w_\ell^k \right),$$

the sum over all paths of all weights of the cosets smaller or equal to κ_i, divided by m. One checks easily as above: If the π_i are L–S paths of shape λ and have the same support, then $\bigwedge \boldsymbol{\pi} = \pi_1 \wedge \cdots \wedge \pi_m$ is an L–S path of shape $m\lambda$. Further, the map

$$\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m) \to \bigwedge \boldsymbol{\pi} = \pi_1 \wedge \cdots \wedge \pi_m$$

induces a bijection between the standard sequences of length m of L–S paths of shape λ, and the set of L–S paths of shape $m\lambda$. By construction, one has $i(\boldsymbol{\pi}) = i(\bigwedge \boldsymbol{\pi})$ and $e(\boldsymbol{\pi}) = e(\bigwedge \boldsymbol{\pi})$, and also the weights coincide: $\boldsymbol{\pi}(1) = \bigwedge \boldsymbol{\pi}(1)$.

If $\pi \wedge \pi'$ is an L–S path of shape 2λ, then (i) implies there exists a standard sequence (η, η') such that $\eta \wedge \eta' = \pi \wedge \pi'$. In the following we write just $\pi \wedge \pi' = (\eta, \eta')$ to indicate the corresponding standard sequence.

We use the notation $(\eta_1, \eta_2) \succeq \pi_1 \wedge \pi_2$ if either $\eta_1 \wedge \eta_2 \succ \pi_1 \wedge \pi_2$, or $\eta_1 \wedge \eta_2 = \pi_1 \wedge \pi_2$ and $\eta_1 \succeq \pi_1$. The notation $\pi_1 \wedge \pi_2 \succeq^r (\eta_1, \eta_2)$ is defined in the same way: either $\pi_1 \wedge \pi_2 \succeq^r \eta_1 \wedge \eta_2$, or $\pi_1 \wedge \pi_2 = \eta_1 \wedge \eta_2$ and $\pi_2 \succeq^r \eta_2$.

Theorem 13 [14]. *If neither $p_{\pi_1} p_{\pi_2}$ nor $p_{\pi_2} p_{\pi_1}$ is standard, then there exist standard monomials $p_{\eta_1} p_{\eta_2} \in H^0(G/Q, \mathcal{L}_{2\lambda})$ such that*

$$p_{\pi_1} p_{\pi_2} = \sum a_{\eta_1,\eta_2} p_{\eta_1} p_{\eta_2}$$

where the coefficient $a_{\eta_1,\eta_2} \neq 0$ only if $(\eta_1, \eta_2) \succeq \pi_1 \wedge \pi_2 \succeq^r (\eta_1, \eta_2)$. Further, $\pi_1 \wedge \pi_2 = \eta_1 \wedge \eta_2$ is possible only if π and π' have the same support, and then $a_{\eta_1,\eta_2} = 1$ for the standard sequence $(\eta_1, \eta_2) = \pi_1 \wedge \pi_2$.

This relation should be seen as a generalization of the quadratic straightening relation for the Grassmann varieties (2) mentioned in the introduction. Sometimes it is more convenient to formulate the relation as follows:

Corollary 14. *If neither the monomial $p_{\pi_1} p_{\pi_2}$ nor the monomial $p_{\pi_2} p_{\pi_1}$ is standard, then, in the quadratic relation above, the coefficient of $p_{\eta_1} p_{\eta_2}$ is non-zero only if $\eta_1 > \pi_1, \pi_2$ and $\pi_1, \pi_2 >^r \eta_2$.*

Proof. This is a consequence of the proof of the theorem above in [14]. The main point we will use from the proof is that $a_{\eta_1,\eta_2} \neq 0$ only if $\eta_1 \wedge \eta_2 > \pi_1 \wedge \pi_2$ in the partial order on convex subsets. Now on the one hand, the pairs (η_1, η_2) are standard sequences, so the wedge product $\eta_1 \wedge \eta_2$ is actually independent of the chosen ordering. On the other hand, $\pi_1 \wedge \pi_2$ depends on the choice of the total order on W/W_λ, so we are free to choose an appropriate total ordering.

Let $\pi_1 = (\boldsymbol{\tau}, \boldsymbol{a})$, $\pi_2 = (\boldsymbol{\sigma}, \boldsymbol{b})$ and $\eta_1 = (\boldsymbol{\kappa}, \boldsymbol{c})$. Suppose first τ_0 and σ_0 are not comparable in the Bruhat order. We can choose a total order $\succ_1$ such that $\tau_0 \succ_1 \sigma_0$ and a total order $\succ_2$ such that $\sigma_0 \succ_2 \tau_0$. Now $\eta_1 \wedge \eta_2 > \pi_1 \wedge \pi_2$ implies for the first choice of the total order that $\kappa_0 \geqslant \tau_0$, for the second choice we get as a consequence $\kappa_0 \geqslant \sigma_0$. Since τ_0, σ_0 are not comparable, we must have strict inequality in both cases, in particular $\eta_1 > \pi_1, \pi_2$.

If $\tau_0 = \sigma_0$, then $\eta_1 \wedge \eta_2 > \pi_1 \wedge \pi_2$ implies by the definition of the wedge product and the partial order that $\eta_1 \geqslant \pi_1, \pi_2$. So without loss of generality we can assume in the following $\tau_0 > \sigma_0$. Since $\eta_1 \wedge \eta_2 > \pi_1 \wedge \pi_2$ implies $\kappa_0 \geqslant \tau_0 > \sigma_0$, we see that $\eta_1 > \pi_2$. It remains to show: $\eta_1 > \pi_1$.

If there exists a $j \leqslant r$ such that $\sigma_0 \geqslant \tau_j$ but $\sigma_0 \not\geqslant \tau_{j-1}$, then we can choose a total order such that the sequence of Weyl group cosets in $\pi_1 \wedge \pi_2$ is of the form $(\ldots, \tau_{j-1}, \sigma_0, \tau_j, \ldots)$, so $\eta_1 \wedge \eta_2 > \pi_1 \wedge \pi_2$ implies by the partial lexicographic ordering that necessarily $\eta_1 > \pi_1$. Otherwise the sequence of Weyl group cosets in $\pi_1 \wedge \pi_2$ is of the form $(\ldots, \tau_r, \sigma_0, \ldots)$, so $\eta_1 \wedge \eta_2 > \pi_1 \wedge \pi_2$ implies $\eta_1 \geqslant \pi_1$. But note that $\eta_1 = \pi_1$ and $\eta_1 \wedge \eta_2 > \pi_1 \wedge \pi_2$ implies $\eta_2 \geqslant \pi_2$ and hence $\tau_r \geqslant \sigma_0$, which is not possible by the assumption that $p_{\pi_1} p_{\pi_2}$ is not standard. It follows hence also in this case: $\eta_1 > \pi_1$.

By replacing $\geqslant$ and $\succ$ by $\geqslant^r$ and $\succ^r$, the same arguments show that $\eta_2 >^r \pi_1, \pi_2$. □

These quadratic relations and the combinatorial character formula are already sufficient to prove:

Proposition 15. *The homogeneous coordinate ring $k[G/Q]$ has a vector space basis given by the standard monomials. In particular, the embedding $G/Q \hookrightarrow \mathbb{P}(V(\lambda))$ is projectively normal.*

Proof. Consider the polynomial algebra $S = k[x_\pi \mid \pi \in B(\lambda)]$. We can write any monomial in S as an ordered product: $x_{\pi_1}^{n_1} \cdots x_{\pi_t}^{n_t}$, where $\pi_1 \succ^r \cdots \succ^r \pi_t$.

We define a monomial order $\succ^r$ on the monomials in S as follows: We say $x_\pi \succeq^r x_{\pi'}$ if $\pi \succeq^r \pi'$. If m_1, m_2 are two ordered monomials, then we say $m_1 \succ^r m_2$ if either the degree of m_1 is strictly greater than the degree of m_2, or, if the degree of the two coincides, then we say $m_1 \succ^r m_2$ if m_1 comes before m_2 in the reverse lexicographic ordering with respect to $\succeq^r$.

Let $m = x_{\pi_1} \cdots x_{\pi_s}$ be an ordered monomial. By abuse of notation we call m a standard monomial in S if the sequence $(\pi_1, \ldots, \pi_s)$ is standard. Denote $E \subset S$ the ideal generated by the type of quadratic relations as in Theorem 13, i.e., E is the ideal generated by the relations

$$x_{\pi_1} x_{\pi_2} - \sum a_{\eta_1,\eta_2} x_{\eta_1} x_{\eta_2},$$

where the set of generators runs over all ordered monomials $x_{\pi_1} x_{\pi_2}$ which are not standard, and the coefficients a_{η_1,η_2} are as in Theorem 13.

For $f \in S$, let $in(f)$ be the initial term, i.e., the greatest monomial of f with respect to the chosen monomial ordering $\succ^r$.

Let $m = x_{\mu_1} \cdots x_{\mu_s} \in S$ be an ordered monomial which is not standard, say $x_{\mu_j} x_{\mu_{j+1}}$ is not standard. By the relations above, we have, $x_{\mu_j} x_{\mu_{j+1}} - \sum a_{\eta_1,\eta_2} x_{\eta_1} x_{\eta_2}$ is an element of E, and $\mu_j \wedge \mu_{j+1} \succ^r (\eta_1, \eta_2)$ for all standard monomials with nonzero coefficient a_{η_1,η_2}. One checks easily that this implies $x_{\mu_j} x_{\mu_{j+1}} \succ^r x_{\eta_1} x_{\eta_2}$ for all standard monomials with nonzero coefficient a_{η_1,η_2}.

Set $m_1 = x_{\mu_1} \cdots x_{\mu_{j-1}}$ and $m_2 = x_{\mu_{j+2}} \cdots x_{\mu_s}$, then $m_1 x_{\mu_j} x_{\mu_{j+1}} m_2 - \sum a_{\eta_1,\eta_2} m_1 x_{\eta_1} \times x_{\eta_2} m_2$ is an element of E with initial term m. So any non-standard monomial $m \in S$ occurs as the initial term of an element of E.

It follows by Macaulay's Theorem [6, Theorem 15.3], that the images of the standard monomials form a generating set for the vector space S/E. The canonical epimorphism $S \to k[G/Q]$ defined by $x_\pi \mapsto p_\pi$ factors through S/E (Theorem 13). Since $k[G/Q]_m = H^0(G/Q, \mathcal{L}_{m\lambda}) \simeq V(m\lambda)^*$ for $m \gg 0$, it follows by the character formula (Theorem 3) that the number of standard monomials is equal to the dimension of $k[G/Q]_m$, so they form in fact a basis. It follows that the standard monomials in $(S/E)_m$ are linearly independent for $m \gg 0$.

Note that $\pi = (id)$ is an L–S path of shape λ. For all $\ell > 0$ and any standard monomial $x_{\pi_1} \cdots x_{\pi_t}$, the monomial $x_{\pi_1} \cdots x_{\pi_t} x_{id}^{\ell}$ of degree $t + \ell$ is again standard. So any linear dependence relation between standard monomials of low degree can be made into a linear dependence relation between standard monomials of high degree by multiplying them with a power of x_{id}. As a consequence we see that the standard monomials are linearly independent for all $m \geqslant 0$, and the map $S/E \to k[G/Q]$ is in fact an isomorphism. $\square$

6. Ideals and coordinate rings

We analyze now the homogeneous coordinate ring and the defining ideal of a union of Richardson varieties. Throughout this section we assume $\lambda \in \Lambda_Q^{++}$ and we consider the embedding

$$X_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}} \subset G/Q \hookrightarrow \mathbb{P}\big(V(\lambda)\big).$$

In the following theorem we recall some results from [5,26,27], and apply them to standard monomial theory. Actually, these results could also be obtained in the course of proving the second part of the theorem below with plain standard monomial theory methods (the most important tool would be the quadratic relations, Theorem 13), but the geometric arguments used by Deodhar and Richardson and the algebraic Frobenius splitting methods are much more direct and much shorter, so it seems more appropriate in this case to just quote the results.

Theorem 16. *A Richardson variety X_τ^κ is non-empty if and only if $\tau \geqslant \kappa$, and, in this case, X_τ^κ is an irreducible variety of dimension $\ell(\tau) - \ell(\kappa)$.*

The vanishing ideal $I_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}} \subset k[G/Q]$ for a union of Richardson varieties $X_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}$ has a basis given by the standard monomials which are not standard on $X_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}$, and the homogeneous coordinate ring $k[X_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}]$ has as basis the restrictions of the monomials standard on $X_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}$. Further, the scheme theoretic intersection of two such unions is reduced and is again a union of Richardson varieties.

Proof. The dimension formula and the irreducibility for Richardson varieties follow from the description of $Be_\tau \cap B^- e_\kappa$ by Deodhar [5], who showed that the intersection of the latter is isomorphic to a product of k's and k^*'s, and the proof by Richardson [27], that the closure of such an intersection is a Richardson variety; the reducedness of intersections and unions of Richardson varieties follow from the existence of a compatible Frobenius splitting [26].

It remains to describe the ideals $I_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}$. We consider first Schubert varieties. By definition, the linear span of the cone $\widehat{X}_\tau$ over $X_\tau \subset \mathbb{P}(V(\lambda))$ is the Demazure submodule $V_\tau(\lambda)$. If a standard monomial $p_{\pi_1} \cdots p_{\pi_m}$ is not standard on X_τ, then p_{π_1} is not standard on X_τ and hence vanishes on $V_\tau(\lambda)$ (Theorem 8). It follows that the standard monomials which are not standard on X_τ lie in I_τ.

Since $\mathcal{L}_\lambda$ is ample one knows that $V(m\lambda)^* \simeq k[G/Q]_m$ for $m \gg 0$, and the latter has a basis given by standard monomials of degree m. The monomials which are not standard on X_τ vanish on X_τ and hence on $V_\tau(m\lambda)$, the linear span of the cone $\widehat{X}_\tau \hookrightarrow V(m\lambda)$. Theorem 8 (applied to the weight $m\lambda$) together with Remark 12 implies that $\dim V_\tau(m\lambda)^*$ is equal to the number of standard monomials on X_τ of length m, so the restrictions form a basis. This proves the linear independence for $m \gg 0$. Now one can use the same arguments as in the proof of Proposition 15 to show that the standard monomials of shape $m\lambda$ on X_τ are linearly independent for all $m \geqslant 0$. As a consequence, the standard monomials which are not standard on X_τ form a basis for I_τ.

The proof for an opposite Schubert variety is the same, and the proof for Richardson varieties follows from the reducedness of the scheme theoretic intersection of a Schubert variety and an opposite Schubert variety. The basis given by the standard monomials is compatible with intersections of the ideals, i.e., if $B_{\tau_i}^{\kappa_i}$ is the standard monomial basis for $I_{\tau_i}^{\kappa_i}$, then $B_{\tau_1}^{\kappa_1} \cap B_{\tau_2}^{\kappa_2}$ is a basis for $I_{\tau_1}^{\kappa_1} \cap I_{\tau_2}^{\kappa_2}$. This implies immediately for a union $X_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}$ of Richardson varieties that the standard monomials $p_{\boldsymbol{\pi}}$, not standard on $X_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}$, form a basis for $I_{\boldsymbol{\kappa}}^{\boldsymbol{\sigma}}$. $\square$

7. Filtrations of ideals and ideal sheaves

The standard monomial bases of the coordinate rings and defining ideals suggest certain vector space decompositions: For example, let X_τ^σ be a Richardson variety and suppose π is an L–S path of shape $\lambda \in \Lambda_Q^{++}$ standard on X_τ^σ. The standard monomials of degree m on X_τ^σ starting with p_π are all of the form $p_\pi p_{\boldsymbol{\pi}'}$, where $p_{\boldsymbol{\pi}'}$ is a standard monomial of degree $(m-1)$ such that

$$e(\pi) \geqslant i(\boldsymbol{\pi}') \geqslant e(\boldsymbol{\pi}') \geqslant \sigma.$$

As a vector space, the space of standard monomials of degree m on X_τ^σ starting with p_π can be identified with the space of standard monomials of degree $(m-1)$, standard on $X_{e(\pi)}^\sigma$. The aim of this section is to formulate this vector space decomposition more precisely in terms of filtrations and associated graded modules and sheaves.

We have two types of boundaries of a Richardson variety X_τ^σ, the *positive* and the *opposite* or *negative* boundary:

$$\partial^+ X_\tau^\sigma = \bigcup_{\sigma \leqslant \kappa < \tau} X_\kappa^\sigma \quad \text{and} \quad \partial^- X_\tau^\sigma = \bigcup_{\sigma < \kappa \leqslant \tau} X_\tau^\kappa.$$

Correspondingly, let $\partial^+ I_\tau^\sigma \subset k[X_\tau^\sigma]$ be the defining ideal of $\partial^+ X_\tau^\sigma$ and let $\partial^- I_\tau^\sigma \subset k[X_\tau^\sigma]$ be the defining ideal of $\partial^- X_\tau^\sigma$.

The standard monomials (or rather the images), standard on X_τ^σ, form a basis of $k[X_\tau^\sigma]$. By Theorem 16, $\partial^+ I_\tau^\sigma$ has a basis given by the standard monomials $p_{\boldsymbol{\pi}}$ on X_τ^σ such that $i(\boldsymbol{\pi}) = \tau$, and $\partial^- I_\tau^\sigma$ has a basis given by the standard monomials $p_{\boldsymbol{\pi}}$ on X_τ^σ such that $e(\boldsymbol{\pi}) = \sigma$.

The partial orders on L–S paths introduced in Section 2 are not only strongly related to the B–module structure (see Section 3) but also to the ideal structure of the homogeneous coordinate ring of Richardson varieties.

Denote by S_τ^+ *the set of L–S paths of shape* λ *such that* $i(\pi)=\tau$. Note that this set is positive saturated (see Section 3) with a unique maximal element: the path (τ).

Theorem 17. *Let* X_τ^σ *be a Richardson variety and let* $\partial^+ X_\tau^\sigma$ *be its positive boundary. Consider a sequence of subsets* $S_j \subset S_\tau^+$ *such that* $|S_j| = j$, $(\tau)\in S_j$ *for* $j>0$, *and* S_j *is positive saturated for all* j:

$$S_0 = \emptyset \subset S_1 \subset \cdots \subset S_{N-1} \subset S_N = S_\tau^+.$$

Set $(\partial^+ I_\kappa^\sigma)_j = \sum_{\pi\in S_j} k[X_\kappa^\tau]p_\pi$, *then the filtration by* T*-stable ideals*

$$0 \subset \left(\partial^+ I_\kappa^\sigma\right)_1 \subset \cdots \subset \left(\partial^+ I_\kappa^\sigma\right)_N = \partial^+ I_\kappa^\sigma$$

is such that the subquotients are as $k[X_\kappa^\sigma]$*-modules and* T*-modules isomorphic to*:

$$\left(\partial^+ I_\kappa^\sigma\right)_j / \left(\partial^+ I_\kappa^\sigma\right)_{j-1} \simeq k\left[X_{e(\pi)}^\sigma\right](-1)\otimes \chi_{-\pi(1)}, \quad \{\pi\} = S_j - S_{j-1},$$

where $k[X_{e(\pi)}^\sigma]$ *denotes the homogeneous coordinate ring of* $X_{e(\pi)}^\sigma \subset X_\kappa^\sigma$. *The isomorphism is induced by the graded morphism*

$$\left(\partial^+ I_\kappa^\sigma\right)_j \to k\left[X_{e(\pi)}^\sigma\right](-1), \quad \left(p_\pi f_\pi + \sum_{\eta\in S_{j-1}} p_\eta f_\eta\right) \mapsto f_\pi|_{X_{e(\pi)}^\sigma}.$$

Remark 18. (i) There is a corresponding obvious version of the theorem for the ideal $\partial^- I_\kappa^\sigma$.

(ii) If $\sigma = id$, i.e., $X_\kappa^\sigma = X_\kappa$ is a Schubert variety, then the filtration can in addition be chosen to be B-equivariant.

(iii) If κ is the class of the longest word in W/W_λ, i.e., $X_\kappa^\sigma = X^\sigma$ is an opposite Schubert variety, then the filtration for the ideal $\partial^- I_\kappa^\sigma$ can in addition be chosen to be B^--equivariant.

Proof of Theorem 17. Consider the vector space $J_j \subset k[X_\kappa^\sigma]$ spanned by all standard monomials starting with a p_π, $\pi\in S_j$, then $J_j \subset (\partial^+ I_\kappa^\sigma)_j$. Since S_j is positive saturated, Corollary 14 implies in fact that J_j is an ideal and hence, by the definition, $J_j = (\partial^+ I_\kappa^\sigma)_j$. So the latter has the standard monomials on X_κ^σ starting with a p_π, $\pi\in S_j$, as a basis.

This shows that the subquotient $(\partial^+ I_\kappa^\sigma)_j/(\partial^+ I_\kappa^\sigma)_{j-1}$ has as basis the images of the standard monomials on X_κ^σ starting with p_π, $\{\pi\} = S_j - S_{j-1}$.

Let $f = \sum_{\eta\in S_j} p_\eta f_\eta$ be an element of $(\partial^+ I_\kappa^\sigma)_j$. Let $\pi\in S_j$ be the unique element such that $\pi\notin S_{j-1}$. Write $f_\pi = \sum a_{\boldsymbol{\eta}} p_{\boldsymbol{\eta}} \in k[X_\kappa^\sigma]$ as a linear combination of standard monomials. Let $f_{\pi,1} = \sum a_{\boldsymbol{\eta}} p_{\boldsymbol{\eta}}$ be the sum of those summands in the expression above such that $p_\pi p_{\boldsymbol{\eta}}$ is standard, and set $f_{\pi,2} = f_\pi - f_{\pi,1}$.

The quadratic relations (Corollary 14) imply that $p_\pi f_{\pi,2} \in (\partial^+ I_\kappa^\sigma)_{j-1}$, and Theorem 16 shows that the restriction of f_π to $X_{e(\pi)}^\sigma$ coincides with the corresponding restriction of $f_{\pi,1}$. It follows that the map

$$\left(\partial^+ I_\kappa^\sigma\right)_j \to k\left[X_{e(\pi)}^\sigma\right](-1), \quad \left(p_\pi f_\pi + \sum_{\eta \in S_{j-1}} p_\eta f_\eta\right) \mapsto f_\pi|_{X_{e(\pi)}^\sigma}$$

is a well defined graded $k[X_\kappa^\sigma]$-module homomorphism. One checks easily that the map is surjective, has $(\partial^+ I_\kappa^\sigma)_{j-1}$ as kernel and is T-equivariant up to a twist by the character corresponding to the weight of p_π. But this is the same as to say that that the homomorphism induces an isomorphism between the T- and $k[X_\kappa^\sigma]$-modules $(\partial^+ I_\kappa^\sigma)_j/(\partial^+ I_\kappa^\sigma)_{j-1}$ and $k[X_{e(\pi)}^\sigma](-1) \otimes \chi_{-\pi(1)}$. $\square$

We consider the corresponding sheaves. Let $\mathcal{O}_{X_\kappa^\sigma}$ be the structure sheaf of $X_\kappa^\sigma \subset G/Q$ and let $\partial^+ \mathcal{I}_\kappa^\sigma$ be the sheaf of ideals of $\partial^+ X_\kappa^\sigma \subset X_\kappa^\sigma$, we get an exact sequence

$$0 \to \partial^+ \mathcal{I}_\kappa^\sigma \to \mathcal{O}_{X_\kappa^\sigma} \to \mathcal{O}_{\partial^+ X_\kappa^\sigma} \to 0.$$

Since $\mathcal{L}_\lambda$ is ample, the filtration of $\partial^+ I_\kappa^\sigma$ (Theorem 17) provides a filtration of $\partial^+ \mathcal{I}_\kappa^\sigma$ as sheaf of $\mathcal{O}_{X_\kappa^\sigma}$-module and as a T-sheaf (note that this generalizes Theorem 5.3 in [21]):

Corollary 19 (Pieri–Chevalley type formula). *The filtration in Theorem* 17 *of the vanishing ideal $\partial^+ I_\kappa^\sigma$ induces a filtration of $\partial^+ \mathcal{I}_\kappa^\sigma$ such that the associated graded sheaf is, as T-equivariant sheaf of $\mathcal{O}_{X_\kappa^\sigma}$-modules, the direct sum*

$$\operatorname{grad} \partial^+ \mathcal{I}_\kappa^\sigma = \bigoplus \mathcal{O}_{X_{e(\pi)}^\sigma}(-1) \otimes \chi_{-\pi(1)}$$

where the sum runs over all L–S paths π of shape λ, standard on X_κ^σ and such that $i(\pi) = \kappa$.

8. Vanishing theorems and standard monomials

As before, we assume $\lambda \in \Lambda_Q^{++}$. The vanishing of the higher cohomology groups proved below can also be proved using an appropriate compatible Frobenius splitting, see [2, Lemma 5]. But since assuming the vanishing would not shorten the proof very much and, on the other hand, the proof shows very nicely how the different ingredients, the filtration and the combinatorial character formula, converge now to give the desired SMT for the cohomology groups, we decided to keep the plain SMT-theoretic proof here since it also may serve as a model for other types of varieties.

Theorem 20. *Let X_κ^σ be a union of Richardson varieties.*

(i) *$H^i(X_\kappa^\sigma, \mathcal{L}_{m\lambda}) = 0$ for all $i \geqslant 1$ and all $m \geqslant 1$, and for an irreducible variety one has in addition $H^i(X_\kappa^\sigma, \mathcal{O}_{X_\kappa^\sigma}) = 0$ for all $i \geqslant 1$.*

(ii) *The standard monomials, standard on X_κ^σ of degree m, form a basis for $H^0(X_\kappa^\sigma, \mathcal{L}_{m\lambda})$ for all $m \geqslant 1$.*

Proof. The proof is by induction on the maximal dimension of the irreducible components and on the number of irreducible components of maximal dimension and on m. The theorem holds if $\dim X_\kappa^\sigma = 0$.

Assume that the theorem holds for all unions of Richardson varieties of dimension smaller than n. Suppose now $\dim X_\kappa^\sigma = n$. Consider the exact sequence

$$0 \to \partial^+\mathcal{I}_\kappa^\sigma \otimes \mathcal{L}_\lambda \to \mathcal{O}_{X_\kappa^\sigma} \otimes \mathcal{L}_\lambda \to \mathcal{O}_{\partial^+ X_\kappa^\sigma} \otimes \mathcal{L}_\lambda \to 0.$$

Since $\partial^+ X_\kappa^\sigma$ is a union of Richardson varieties of smaller dimension, the vanishing theorem holds and hence $H^i(\partial^+ X_\kappa^\sigma, \mathcal{L}_\lambda) = 0$ for $i \geqslant 1$. Moreover, by induction, the global sections on $\partial^+ X_\kappa^\sigma$ can be lifted to global sections on G/Q, so the restriction map $H^0(X_\kappa^\sigma, \mathcal{L}_\lambda) \to H^0(\partial^+ X_\kappa^\sigma, \mathcal{L}_\lambda)$ is surjective.

By Corollary 19, the sheaf $\partial^+\mathcal{I}_\kappa^\sigma \otimes \mathcal{L}_\lambda$ admits a filtration such that the associated graded is just a direct sum of structure sheafs of Richardson varieties of the form $\mathcal{O}_{X_{\kappa'}^\sigma}$, where $\kappa \geqslant \kappa'$. By induction, for $\kappa' < \kappa$ the vanishing for the higher cohomology groups holds. The associated graded has only one subquotient isomorphic to $\mathcal{O}_{X_\kappa^\sigma}$, the contribution coming from the path (κ). So the long exact sequence in cohomology splits into isomorphisms $H^i(X_\kappa^\sigma, \mathcal{O}_{X_\kappa^\sigma}) \simeq H^i(X_\kappa^\sigma, \mathcal{L}_\lambda)$ for $i \geqslant 1$ and a short exact sequence

$$0 \to H^0\big(X_\kappa^\sigma, \partial^+\mathcal{I}_\kappa^\sigma \otimes \mathcal{L}_\lambda\big) \to H^0\big(X_\kappa^\sigma, \mathcal{L}_\lambda\big) \to H^0\big(\partial^+ X_\kappa^\sigma, \mathcal{L}_\lambda\big) \to 0,$$

so $\dim H^0(X_\kappa^\sigma, \mathcal{L}_\lambda) = \dim H^0(\partial^+ X_\kappa^\sigma, \mathcal{L}_\lambda) + \dim H^0(X_\kappa^\sigma, \partial^+\mathcal{I}_\kappa^\sigma \otimes \mathcal{L}_\lambda)$. Consider the right hand side of the equation. By induction, the first term is equal to the number of L–S paths of shape λ standard on $\partial^+ X_\kappa^\sigma$, and the second is by the filtration equal to the number of L–S paths of shape λ standard on X_κ^σ and such that $i(\pi) = \kappa$. This together is the number of L–S paths of shape λ standard on X_κ^σ. Since $k[X_\kappa^\sigma]_1 \hookrightarrow H^0(X_\kappa^\sigma, \mathcal{L}_\lambda)$, this proves that the path vectors form a basis of $H^0(X_\kappa^\sigma, \mathcal{L}_\lambda)$.

For $m > 1$ one gets an exact sequence:

$$0 \to \partial^+\mathcal{I}_\kappa^\sigma \otimes \mathcal{L}_{m\lambda} \to \mathcal{O}_{X_\kappa^\sigma} \otimes \mathcal{L}_{m\lambda} \to \mathcal{O}_{\partial^+ X_\kappa^\sigma} \otimes \mathcal{L}_{m\lambda} \to 0.$$

The same arguments as above (induction on the dimension or on m and the filtration of $\partial^+\mathcal{I}_\kappa^\sigma \otimes \mathcal{L}_{m\lambda}$) show that the associated long exact sequence in cohomology splits into isomorphisms $H^i(X_\kappa^\sigma, \mathcal{L}_{(m-1)\lambda}) \simeq H^i(X_\kappa^\sigma, \mathcal{L}_{m\lambda})$ for $i \geqslant 1$ and a short exact sequence for the global sections. The same counting argument as above proves that the standard monomials form a basis of $H^0(X_\kappa^\sigma, \mathcal{L}_{m\lambda})$. Further, for $i \geqslant 1$ we have isomorphisms:

$$H^i\big(X_\kappa^\sigma, \mathcal{O}_{X_\kappa^\sigma}\big) \simeq H^i\big(X_\kappa^\sigma, \mathcal{L}_\lambda\big) \simeq H^i\big(X_\kappa^\sigma, \mathcal{L}_{m\lambda}\big), \quad \forall m \geqslant 1.$$

Since $\mathcal{L}_\lambda$ is ample, it follows that $H^i(X_\kappa^\sigma, \mathcal{L}_{m\lambda}) = 0$ for all $i \geqslant 1, m \geqslant 0$.

Suppose now X_{κ}^{σ} is a union of Richardson varieties. Let X_{κ}^{σ} be an irreducible component of maximal dimension and denote $X_{\kappa'}^{\sigma'}$ the union of the remaining irreducible components. We have an exact sequence:

$$0 \to \mathcal{O}_{X_{\kappa}^{\sigma}} \to \mathcal{O}_{X_{\kappa}^{\sigma}} \oplus \mathcal{O}_{X_{\kappa'}^{\sigma'}} \to \mathcal{O}_{X_{\kappa'}^{\sigma'} \cap X_{\kappa}^{\sigma}} \to 0.$$

If we tensor the sequence with $\mathcal{L}_{m\lambda}$, $m \geqslant 1$, then we know by induction on the dimension, respectively the number of irreducible components of maximal dimension that the higher cohomology groups of the second and the last term vanish, and further, for the global sections the last map is surjective. It follows that the higher cohomology groups $H^i(X_{\kappa}^{\sigma}, \mathcal{L}_{m\lambda})$ vanish.

It remains to count the dimensions of the spaces of global sections. Since one has standard monomial theory for the last two terms by induction on the dimension, respectively the number of irreducible components of maximal dimension, one checks easily that in fact $\dim H^0(X_{\kappa}^{\sigma}, \mathcal{L}_{m\lambda})$ is equal to the number of standard monomials on X_{κ}^{σ} of degree m. The same argument as above proves hence that the standard monomials form a basis of $\dim H^0(X_{\kappa}^{\sigma}, \mathcal{L}_{m\lambda})$. □

9. The non-regular case and pointed unions

Fix a dominant weight $\lambda \in \Lambda_Q^+$. Note that in the general case not all results carry over. We have the following two examples (cf. [2, remark following Lemma 5]).

Consider $G = SL_2$, so $G/B = \mathbb{P}^1$. Let $X_{\kappa}^{\sigma} = \{0, \infty\}$ be the union of the two T-fixed points. Then $\dim H^0(G/B, \mathcal{O}_{G/B}) = 1$ but $\dim H^0(X_{\kappa}^{\sigma}, \mathcal{O}_{X_{\kappa}^{\sigma}}) = 2$, so the restriction map for global sections is not surjective.

In $\mathbb{P}^1 \times \mathbb{P}^1$, let X_{κ}^{σ} be the union of the four $\mathbb{P}^1$'s: $\{0\} \times \mathbb{P}^1$, $\{\infty\} \times \mathbb{P}^1$, $\mathbb{P}^1 \times \{0\}$ and $\mathbb{P}^1 \times \{\infty\}$, then $H^1(X_{\kappa}^{\sigma}, \mathcal{O}_{X_{\kappa}^{\sigma}}) \neq 0$.

But we get a SMT (standard monomial theory) and vanishing theorems also in the non-regular case for a special class of unions of Richardson varieties. In the inductive procedure used to construct standard monomial theory, we need a class of varieties which includes (i) all Richardson varieties, (ii) their boundaries $\partial^+ X_{\tau}^{\kappa}$ and $\partial^- X_{\tau}^{\kappa}$, and (iii) if Y is in this class and $Y = X \cup Y'$ is such that X is irreducible and Y' is a union of irreducible components of Y, then X, $Y' \cap X$ and Y' are also in this class.

The example above shows: we can expect that the global restriction map is surjective for all varieties in this class only if it is not possible to construct a non-trivial union of points by using the operations in (ii) and (iii). This leads to the following definition:

Definition 21. *A union of Richardson varieties Y is called pointed* if there exists a $\kappa \in W/W_Q$ such that $Y = \bigcup_j X_{\kappa}^{\sigma_j}$, or if there exists a $\sigma \in W/W_Q$ such that $Y = \bigcup_j X_{\kappa_j}^{\sigma}$.

Simple examples for such pointed unions are arbitrary unions of Schubert varieties or arbitrary unions of opposite Schubert varieties. It is easy to check that the class of pointed unions of Richardson varieties has the three properties above.

A first step to construct SMT is the generalization of Corollary 19, but first the definition of the boundary has to be adjusted. For $\tau \in W/W_Q$, set $\overline{\tau} \equiv \tau \bmod W_\lambda$, and denote $p_{\overline{\tau}}$ the extremal weight vector in $H^0(G/Q, \mathcal{L}_\lambda)$ corresponding to the L–S path $(\overline{\tau})$ of type λ.

Proposition 22. *Let $P_\lambda \supset Q$ be the parabolic subgroup associated to the weight λ, and for $\tau \geqslant \kappa \in W/W_Q$ consider the projection*:

$$\begin{tikzcd} \phi : X_\tau^\kappa \arrow[rr] \arrow[dr] & & G/P_\lambda \\ & G/Q \arrow[ur] & \end{tikzcd}$$

Either $X_\tau^\kappa \subset \phi^{-1}(e_{\overline{\tau}})$, in which case the line bundle $\mathcal{L}_\lambda|_{X_\tau^\kappa}$ is trivial, or the preimage of the set of zeros of $p_{\overline{\tau}}$ below, called the boundary of X_τ^κ with respect to λ (*or just the λ-boundary*), *is not empty*:

$$\partial_\lambda^+ X_\tau^\kappa = \{ y \in X_\tau^\kappa \mid p_{\overline{\tau}}(y) = 0 \} \neq \emptyset.$$

Remark 23. We use the notation $\partial_\lambda^+ X_\tau^\kappa$ in the following for the corresponding variety with its induced reduced structure.

Proof. Let $Z = \phi(X_\tau^\kappa)$ be the image of the T-equivariant map ϕ. If $Z = e_{\overline{\tau}}$, then $X_\tau^\kappa \subset \phi^{-1}(e_{\overline{\tau}})$ and the restriction of $\mathcal{L}_\lambda$ to the fibre is constant. Otherwise $\dim Z \geqslant 1$ and hence Z admits at least two T-fixed points, say $e_{\overline{\tau}}$ and $e_{\overline{\sigma}}$ for some $\overline{\sigma} \neq \overline{\tau}$. Let $y \in X_\tau^\kappa$ be such that $\phi(y) = e_{\overline{\sigma}}$, then $p_{\overline{\tau}}(y) = p_{\overline{\tau}}(e_{\overline{\sigma}}) = 0$ and hence $y \in \partial_\lambda^+ X_\tau^\kappa$, so the latter is not empty. □

Remark 24. Let $\overline{\tau}$ be as above and let $\partial^+ X_{\overline{\tau}} = \bigcup X_{\overline{\sigma}_i} \subset G/P_\lambda$ be the boundary of $X_{\overline{\tau}}$. Let $X_{\sigma_i} \subset G/Q$ be the preimage of $X_{\overline{\sigma}_i}$, then

$$\partial_\lambda^+ X_\tau^\kappa = \bigcup (X_\tau^\kappa \cap X_{\sigma_i}),$$

and this description as union of intersections holds scheme theoretically. In particular, the λ-boundary is either empty or a pointed union of Richardson varieties. In view of Proposition 22, we shall henceforth consider only those X_τ^κ for which the λ-boundary is non-empty.

It remains to define *standardness on X_τ^κ* for the non-regular case: If $\pi = (\sigma_0, \ldots, \sigma_p, a_1, \ldots, a_p)$ is an L–S path of shape λ, then π is called *standard on X_τ^κ* if there exist *lifts* $\tilde{\sigma}_0, \ldots, \tilde{\sigma}_p \in W/W_Q$, i.e., $\tilde{\sigma}_i \equiv \sigma_i \bmod W_\lambda$ for all $i = 0, \ldots, r$, such that

$$\tau \geqslant \tilde{\sigma}_0 \geqslant \tilde{\sigma}_1 \geqslant \cdots \geqslant \tilde{\sigma}_p \geqslant \kappa.$$

Such a sequence $(\tilde{\sigma}_0, \ldots, \tilde{\sigma}_p)$ is called a *defining chain for* π (on X_τ^κ).

For the λ-boundary $\partial_\lambda^+ X_\tau^\kappa$ of the Richardson variety X_τ^κ, let $\mathcal{I}_{\partial_\lambda^+ X_\tau^\kappa}$ be the corresponding sheaf of ideals.

Theorem 25. *The twisted sheaf* $\mathcal{I}_{\partial_\lambda^+ X_\tau}(\lambda) = \mathcal{I}_{\partial_\lambda^+ X_\tau^\kappa} \otimes \mathcal{L}_\lambda$ *admits a filtration, such that the associated graded is isomorphic* (*as* $\mathcal{O}_{X_\kappa^\sigma}$*-sheaf and as* T*-sheaf*) *to a direct sum of structure sheaves, twisted by a character*:

$$\operatorname{grad} \mathcal{I}_{\partial_\lambda^+ X_\tau^\kappa}(\lambda) \simeq \bigoplus_{\pi \in S} \mathcal{O}_{X^\kappa_{e(\pi)}} \otimes \chi_{-\pi(1)},$$

where S *is the set of all L–S paths* π *of shape* λ, *standard on* X_τ^κ *and such that* $i(\pi) \equiv \tau \bmod W_\lambda$.

The rest of this section is devoted to the proof of Theorem 25. For the rest of this section we fix a regular dominant weight $\rho \in \Lambda_Q^{++}$. Let $\pi \in B(\lambda)$ be as above and let $\eta_1, \ldots, \eta_s, \eta'_1, \ldots, \eta'_q \in B(\rho)$.

Definition 26. A sequence $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_s, \pi, \eta'_1, \ldots, \eta'_q)$ of shape $(s\rho, \lambda, q\rho)$, is called *standard* if there exists a defining chain $(\tilde\sigma_0, \ldots, \tilde\sigma_p)$ for π such that

$$e(\eta_1) \geqslant i(\eta_2) \geqslant \cdots \geqslant e(\eta_s) \geqslant \tilde\sigma_0 \geqslant \cdots \geqslant \tilde\sigma_p \geqslant i(\eta'_1) \geqslant e(\eta'_1) \geqslant \cdots \geqslant i(\eta'_q).$$

The sequence $\boldsymbol{\eta}$ is called *standard on* X_τ^κ if in addition $\tau \geqslant i(\eta_1)$ (respectively $\tau \geqslant \tilde\sigma_0$ if $s = 0$) and $e(\eta'_q) \geqslant \kappa$ (respectively $\tilde\sigma_p \geqslant \kappa$ if $q = 0$). The sequence $(\tilde\sigma_0, \ldots, \tilde\sigma_p)$ is then called a *defining chain* for $\boldsymbol{\eta}$ on X_τ^κ.

The monomial $p_{\boldsymbol{\eta}} = p_{\eta_1} \cdots p_{\eta_s} p_\pi p_{\eta'_1} \cdots p_{\eta'_q}$ is called *standard* (on X_τ^κ) if the sequence $\boldsymbol{\eta}$ is so. The monomial is called *standard on a union of Richardson varieties* if it is standard on at least one irreducible component. We have the following global basis given by standard monomials:

Theorem 27 [20]. *The standard monomials of shape* $(s\rho, \lambda, q\rho)$ *form a basis for* $H^0(G/Q, \mathcal{L}_{\lambda+(s+q)\rho})$ *for all* $s, q \geqslant 0$.

The notion of the final and initial direction has to be adapted to standard sequences. Let $\boldsymbol{\eta}$ be a standard sequence on X_τ^κ. The set of all defining chains for $\boldsymbol{\eta}$ on X_τ^κ is partially ordered: We say $(\tilde\sigma_0, \ldots, \tilde\sigma_p) \geqslant (\tilde\sigma'_0, \ldots, \tilde\sigma'_p)$ if and only if $\tilde\sigma_0 \geqslant \tilde\sigma'_0, \ldots, \tilde\sigma_p \geqslant \tilde\sigma'_p$. One knows by Deodhar's Lemma that for this partial order there exists a unique minimal and a unique maximal defining chain (see for example [15,16,19]).

Definition 28. If $s \geqslant 1$, then set $i(\boldsymbol{\eta}) = i(\eta_1)$ as before. If $s = 0$, then set $i(\boldsymbol{\eta}) = \tilde\sigma_0$ for the unique minimal defining chain $(\tilde\sigma_0, \ldots, \tilde\sigma_p)$.

If $q \geqslant 1$, then set $e(\boldsymbol{\eta}) = e(\eta'_q)$. If $q = 0$, then set $e(\boldsymbol{\eta}) = \tilde\sigma_p$ for the unique maximal defining chain $(\tilde\sigma_0, \ldots, \tilde\sigma_p)$.

Remark 29. If $q \geqslant 1$ or $s \geqslant 1$, then $i(\boldsymbol{\eta})$ and $e(\boldsymbol{\eta})$ are independent of the Richardson variety X_τ^κ. If $q = s = 0$, this is not the case: For $G = SL_3$, $Q = B$, $\lambda = \omega_1$, $\eta = (id)$ and $X_\tau^\kappa = X_{id}^{s_1 s_2}$, we have $i(\eta) = (id)$, but for $X_\tau^\kappa = X_{s_1}^{s_1 s_2}$, we have $i(\eta) = (s_1)$.

Remark 30. The path model theory provides for all $s, q \geqslant 0$, $q + s > 0$, a natural weight preserving bijection between the set of all L–S paths π of shape $\lambda + (s+q)\rho$ such that $i(\pi) = \tau$ and $e(\pi) = \kappa$ and the set of standard sequences $\boldsymbol{\eta}$ of shape $(s\rho, \lambda, q\rho)$ such that $i(\boldsymbol{\eta}) = \tau$ and $e(\boldsymbol{\eta}) = \kappa$, see [19].

Proposition 31. *The standard monomials of shape* $(0, \lambda, 0)$ *are linearly independent on a pointed union of Richardson varieties* Y*. Further, for all* $s + q \geqslant 1$*, the standard monomials on* Y *of shape* $(s\rho, \lambda, q\rho)$ *form a basis for* $H^0(Y, \mathcal{L}_{\lambda+(s+q)\rho})$ *and the corresponding higher cohomology groups vanish.*

Proof. Theorem 20 implies for $q + s \geqslant 1$ that the higher cohomology groups vanish, and, by Remark 30, the number of standard monomials, standard on Y, is equal to the dimension of $H^0(Y, \mathcal{L}_{\lambda+(s+q)\rho})$. So to prove the proposition in this case, it is sufficient to show that the standard monomials span the space of global sections. We consider only the case $q \geqslant 1$, the case $s \geqslant 1$ can be proved similarly.

The proof for a union of (opposite) Schubert varieties X_τ has been given in [20]. Suppose $Y = \bigcup_j X^\kappa_{\tau_j}$ is a pointed union of Richardson varieties. Denote by Z the union of Schubert varieties $Z = \bigcup_j X_{\tau_j}$. The restriction map

$$H^0(Z, \mathcal{L}_{\lambda+(q+s)\rho}) \to H^0(Y, \mathcal{L}_{\lambda+(q+s)\rho}), \tag{4}$$

is surjective by Theorem 20. The standard monomials standard on Y are also standard on Z. But a standard monomial $p_{\boldsymbol{\eta}}$, standard on Z, is not standard on Y if and only if $e(\boldsymbol{\eta}) \ngeqslant \kappa$. But in this case the restriction of $p_{\boldsymbol{\eta}}|_Y$ vanishes ($q > 0$!), so, by the surjectivity of (4), the standard monomials, standard on Y, span the space of global sections.

If $Y = \bigcup_j X^{\kappa_j}_\tau$, then let Z be the Schubert variety $Z = X_\tau$. As above, the restriction map on the global sections is surjective, and the same arguments show that a standard monomial, standard on Z but not on Y, vanishes identically on Y, which finishes the proof also in this case.

It remains to consider the case $q = s = 0$. Suppose first $Y = \bigcup_j X^\kappa_{\tau_j}$ is a pointed union of Richardson varieties and $p_{\pi_1}, \ldots, p_{\pi_t} \in H^0(G/Q, \mathcal{L}_\lambda)$ are standard on Y, but $(\sum_{i=1}^t a_i p_{\pi_i})|_Y = 0$. The sequence (κ) is an L–S path of shape ρ, let p_κ be the corresponding path vector. By the definition of standardness, the monomials $p_{\pi_i} p_\kappa$ of shape $(0, \lambda, \rho)$ are standard on Y.

It follows that the sum $\sum_{i=1}^t a_i p_{\pi_i} p_\kappa = 0$ would be a linear dependence relation of standard monomials of shape $(0, \lambda, \rho)$, in contradiction to what has been proved above. For $Y = \bigcup_j X^{\kappa_j}_\tau$ the arguments are similar, instead of p_κ one takes the path vector $p_\tau \in H^0(G/Q, \mathcal{L}_\rho)$ and deduces the linear independence of the $p_{\pi_1}, \ldots, p_{\pi_t}$ by the linear independence of the $p_\tau p_{\pi_1}, \ldots, p_\tau p_{\pi_t}$. $\square$

We come now to the proof of Theorem 25.

Proof. The exact sequence $0 \to \mathcal{I}_{\partial^+_\lambda X^\kappa_\tau} \to \mathcal{O}_{X^\kappa_\tau} \to \mathcal{O}_{\partial^+_\lambda X^\kappa_\tau} \to 0$ induces an exact sequence

$$0 \to \mathcal{I}_{\partial^+_\lambda X^\kappa_\tau}(\lambda) \to \mathcal{O}_{X^\kappa_\tau} \otimes \mathcal{L}_\lambda \to \mathcal{O}_{\partial^+_\lambda X^\kappa_\tau} \otimes \mathcal{L}_\lambda \to 0.$$

Since ρ is regular, one gets for $m \gg 0$ an induced exact sequence

$$\begin{aligned} 0 \to\ & H^0\big(X_\tau^\kappa, \mathcal{I}_{\partial_\lambda^+ X_\tau^\kappa}(\lambda) \otimes \mathcal{L}_{m\rho}\big) \\ &\to H^0\big(X_\tau^\kappa, \mathcal{L}_{\lambda+m\rho}\big) \to H^0\big(\partial_\lambda^+ X_\tau^\kappa, \mathcal{L}_{\lambda+m\rho}\big) \to 0. \end{aligned} \tag{5}$$

Denote $S_{\bar\tau}^+$ *the set of L–S paths of shape* λ*, standard on* X_τ^κ *and such that* $\pi = (\bar\tau, \dots)$. This set is positive saturated, i.e., if π, π' are elements of this set and $\pi' \geqslant \eta \geqslant \pi$, then η is an element of this set too (see Section 3). This set has a unique maximal element: the path $(\bar\tau)$. By deleting step by step one element, it is possible to get a sequence of subsets

$$S_0 = \emptyset \subset S_1 \subset \cdots \subset S_{N-1} \subset S_N = S_{\bar\tau}^+$$

such that $(\bar\tau) \in S_j$ for all $j > 0$ and the S_j are positive saturated for all j. Fix $M \gg 0$ so that (5) holds for all $m \geqslant M$ and consider the R-module

$$I_{\bar\tau}^\kappa(\lambda) := \bigoplus_{m \geqslant M} H^0\big(X_\tau^\kappa, \mathcal{L}_{m\rho} \otimes \mathcal{I}_{\partial_\lambda^+ X_\tau^\kappa}(\lambda)\big).$$

The path vectors p_π, $\pi \in S_{\bar\tau}^+$, are sections in $H^0(X_\tau^\kappa, \mathcal{L}_\lambda)$ which vanish on $\partial_\lambda^+ X_\tau^\kappa$, this follows immediately from the description of the boundary in Remark 24 and the fact that the p_π vanishes on $\partial^+ X_{\bar\tau}$. Set

$$\big(I_{\bar\tau}^\kappa(\lambda)\big)_j = \sum_{\pi \in S_j} \left(\sum_{m \geqslant M} H^0\big(X_\kappa^\tau, \mathcal{L}_{m\rho}\big) p_\pi \right),$$

then $(I_{\bar\tau}^\kappa(\lambda))_j$ is a T-stable R-submodule of $I_{\bar\tau}^\kappa(\lambda)$ and hence this defines a filtration of $I_{\bar\tau}^\kappa(\lambda)$ by T-stable R-submodules

$$0 \subset \big(I_{\bar\tau}^\kappa(\lambda)\big)_1 \subset \cdots \subset \big(I_{\bar\tau}^\kappa(\lambda)\big)_N = I_{\bar\tau}^\kappa(\lambda).$$

The middle term in the short exact sequence (5) has as basis the standard monomials of shape $(0, \lambda, m\rho)$, standard on X_τ^κ, and the last term has as basis the standard monomials of shape $(0, \lambda, m\rho)$, standard on $\partial_\lambda^+ X_\tau^\kappa$ (see Proposition 31). Let $I(\lambda + m\rho)$ be the set of standard monomials of shape $(0, \lambda, m\rho)$, standard on X_τ^κ but not standard on $\partial_\lambda^+ X_\tau^\kappa$. These are the standard monomials of the form $p_{\boldsymbol\eta} = p_\pi p_{\boldsymbol\eta'}$, $\pi \in S_{\bar\tau}^+$ and $p_{\boldsymbol\eta'}$ of shape $m\rho$. These monomials vanish on $\partial_\lambda^+ X_\tau^\kappa$. It follows that the $p_{\boldsymbol\eta}$, $\boldsymbol\eta \in I(\lambda + m\rho)$, form a basis of the first term.

The next step is to prove that the basis is compatible with the filtration. So suppose $p_\pi \in H^0(X_\tau^\kappa, \mathcal{L}_\lambda)$ is such that $\pi \in S_{\bar\tau}^+$ and let p_η be a path vector in $H^0(X_\tau^\kappa, \mathcal{L}_\rho)$ and standard on X_τ^κ. Suppose the monomial $p_\pi p_\eta$ of shape $(0, \lambda, \rho)$ is not standard on X_τ^κ. Since $\rho \in \Lambda^{++}$, this implies automatically that the product is not standard on the Schubert variety X_τ. Now the quadratic relations in [21, Theorem 3.13], show:

$$p_\pi p_\eta = \sum_{\pi', \eta'} b_{\pi', \eta'} p_{\pi'} p_{\eta'}$$

gives a presentation of the product as linear combination of standard monomials, standard on X_τ, where $(\pi', \eta') \geqslant (\pi, \eta) \geqslant^r (\pi', \eta')$. Such a standard monomial $p_\pi p_\eta$ is then not standard on X_τ^κ if and only if $e(\eta') \ngeqslant \kappa$, in which case the monomial vanishes on X_τ^κ. But this implies that we have the same type of relation for non-standard products also on X_τ^κ and hence:

$$p_\pi p_\eta = \sum_{(\pi',\eta') \geqslant (\pi,\eta) \geqslant^r (\pi',\eta')} b_{\pi',\eta} p_{\pi'} p_{\eta'}, \quad \text{all } p_{\pi'} p_{\eta'} \text{ are standard on } X_\tau^\kappa. \tag{6}$$

The sets S_j are positively saturated, and hence (as in the regular case) the subspace spanned by elements of degree m in the jth filtration part:

$$\big(I_{\tau,m}^\kappa(\lambda)\big)_j = H^0\big(X_\tau^\kappa, \mathcal{I}_{\partial_\lambda^+ X_\tau^\kappa}(\lambda) \otimes \mathcal{L}_{m\rho}\big)_j$$

has a basis given by the standard monomials, standard on X_τ^κ, starting with a p_π, $\pi \in S_j$. Let $\{\pi\} = S_j - S_{j-1}$. As in the regular case, the map

$$\big(I_{\tau,m}^\kappa(\lambda)\big)_j \to H^0\big(X_{e(\pi)}^\kappa, \mathcal{L}_{m\rho}\big), \quad \left(p_\pi f_\pi + \sum_{\eta \in S_{j-1}} p_\eta f_\eta\right) \mapsto f_\pi|_{X_{e(\pi)}^\kappa}$$

induces an isomorphism $(I_{\tau,m}^\kappa(\lambda))_j / (I_{\tau,m}^\kappa(\lambda))_{j-1} \to H^0(X_{e(\pi)}^\tau, \mathcal{L}_{m\rho})$. Since this holds for all $m \geqslant M$, this induces a filtration of the sheaf

$$0 \subset \big(\mathcal{I}_{\partial_\lambda^+ X_\tau^\kappa}(\lambda)\big)_1 \subset \big(\mathcal{I}_{\partial_\lambda^+ X_\tau^\kappa}(\lambda)\big)_2 \subset \cdots \subset \big(\mathcal{I}_{\partial_\lambda^+ X_\tau^\kappa}(\lambda)\big)_N = \mathcal{I}_{\partial_\lambda^+ X_\tau^\kappa}(\lambda)$$

such that the associated graded is isomorphic to the direct sum

$$\operatorname{grad} \mathcal{I}_{\partial_\lambda^+ X_\tau^\kappa}(\lambda) = \bigoplus_{\pi \in S} \mathcal{O}_{X_{e(\pi)}^\kappa} \otimes \chi_{-\pi(1)}. \qquad \square$$

10. SMT in the non-regular case

We have the following generalization of Theorem 20 to the non-regular case:

Theorem 32. *Suppose $\lambda \in \Lambda_Q^+$ is a dominant weight and let Y be a pointed union of Richardson varieties.*

(i) $H^i(Y, \mathcal{L}_{m\lambda}) = 0$ *for all* $i \geqslant 1$ *and all* $m \geqslant 1$.
(ii) *The standard monomials, standard on Y of degree m, form a basis for $H^0(Y, \mathcal{L}_{m\lambda})$ for all $m \geqslant 1$.*

Proof. We give only a sketch of the proof for $m = 1$ (respectively $m = 0$ in (i)), the case $m \geqslant 2$ can be proved in the same way as in the regular case. The proof of the appropriate version of Proposition 31 for monomials of type $(s\rho, m\lambda, q\rho)$ is the same.

The proof is (as in the regular case) by induction on the number of components of maximal dimension and on the dimension. The theorem holds obviously in case Y is a point. We assume now that the theorem holds for all pointed unions of Richardson varieties of dimension smaller than n. Suppose X_κ^σ is of dimension n and consider the projection $\phi : X_\kappa^\sigma \to G/P_\lambda$.

If $\partial_\lambda^+ X_\kappa^\sigma = \emptyset$, then $X_\kappa^\sigma \subset p^{-1}(e_{\bar\kappa})$ by Proposition 22, and the restriction of the line bundle to X_κ^σ is trivial. The theorem holds obviously in this case in view of Theorem 20.

If $\partial_\lambda^+ X_\kappa^\sigma \neq \emptyset$, then the inclusion $\mathcal{I}_{\partial_\lambda^+ X_\kappa^\sigma} \hookrightarrow \mathcal{O}_{X_\kappa^\sigma}$, tensored by $\mathcal{L}_\lambda$, induces an exact sequence

$$0 \to \mathcal{I}_{\partial_\lambda^+ X_\kappa^\sigma}(\lambda) \to \mathcal{O}_{X_\kappa^\sigma} \otimes \mathcal{L}_\lambda \to \mathcal{O}_{\partial_\lambda^+ X_\kappa^\sigma} \otimes \mathcal{L}_\lambda \to 0. \tag{7}$$

Since $\partial_\lambda^+ X_\kappa^\sigma$ is a pointed union of Richardson varieties of smaller dimension, the vanishing theorem for higher cohomology groups holds, i.e., for $i \geqslant 1$ we have $H^i(\partial_\lambda^+ X_\kappa^\sigma, \mathcal{L}_\lambda) = 0$. By Theorem 25, the sheaf $\mathcal{I}_{\partial_\lambda^+ X_\kappa^\sigma}(\lambda)$ admits a filtration such that the associated graded sheaf is isomorphic to a direct sum of sheaves of the form $\mathcal{O}_{X_{e(\pi)}^\sigma} \otimes \chi_{-\pi(1)}$. Now Theorem 20(i) states that $H^i(X_\kappa^\sigma, \mathcal{O}_{X_\kappa^\sigma}) = 0$ for all $i \geqslant 1$, we conclude by the long exact sequence associated to the filtration that $H^i(X_\kappa^\sigma, \mathcal{I}_{\partial_\lambda^+ X_\kappa^\sigma}(\lambda)) = 0$ for all $i \geqslant 1$. So the long exact sequence associated to the short exact sequence (7) shows that $H^i(X_\kappa^\sigma, \mathcal{L}_\lambda) = 0$ for all $i \geqslant 1$. Further, the short exact sequence

$$0 \to H^0\big(X_\kappa^\sigma, \mathcal{I}_{\partial_\lambda^+ X_\kappa^\sigma}(\lambda)\big) \to H^0\big(X_\kappa^\sigma, \mathcal{L}_\lambda\big) \to H^0\big(\partial_\lambda^+ X_\kappa^\sigma, \mathcal{L}_\lambda\big) \to 0$$

shows $\dim H^0(X_\kappa^\sigma, \mathcal{L}_\lambda) = \dim H^0(\partial_\lambda^+ X_\kappa^\sigma, \mathcal{L}_\lambda) + \dim H^0(X_\kappa^\sigma, \mathcal{I}_{\partial_\lambda^+ X_\kappa^\sigma}(\lambda))$. Consider the right hand side of the equation. By induction, the first term is equal to the number of L–S paths of shape λ standard on $\partial_\lambda^+ X_\kappa^\sigma$, and the second is by the filtration equal to the number of L–S paths of shape λ standard on X_κ^σ and such that $i(\pi) = \kappa$. This together is the number of L–S paths of shape λ standard on X_κ^σ. Since we have already proved that the restrictions of the corresponding path vectors remain linearly independent (Proposition 31), this proves that the path vectors form a basis of $H^0(X_\kappa^\sigma, \mathcal{L}_\lambda)$.

The proof for a pointed union of Richardson varieties proceeds now as at the end of the proof of Theorem 20. □

Remark 33. The result (i) in Theorem 32 is also proved in [2] (cf. [2, Proposition 1]).

11. Standard monomials and equivariant K-theory

In this section we will show that the notion of a "standard monomial" is directly related to geometry of Richardson varieties, and this can be best expressed in terms of equivariant K-theory.

The Chow ring $\mathrm{Chow}(G/B)$ has a $\mathbb{Z}$-basis consisting of $[X_\tau]$, the class in $\mathrm{Chow}(G/B)$ defined by the cycle represented by the Schubert variety X_τ, $\tau \in W$. Let $\lambda \in \Lambda$ (= the weight lattice), $\mathcal{L}_\lambda$ the associated line bundle. The class $[c_1(\mathcal{L}_\lambda)]$ is defined by a codimension one cycle. We have, the following expression for the product $[X_\tau] \cdot [c_1(\mathcal{L}_\lambda)]$ in the Chow ring:

$$[X_\tau] \cdot [c_1(\mathcal{L}_\lambda)] = \sum a_i [X_{\tau_i}] \tag{1}$$

where the summation runs over all Schubert divisors X_{τ_i} in X_τ; further, one has an explicit expression for a_i due to Chevalley (cf. [3]). When $G = SL(n)$, (1) is just the classical formula of Pieri.

Denote by $K(G/B)$ the Grothendieck ring of (isomorphism classes of) coherent sheaves on G/B. Then one knows that the classes $[\mathcal{O}_{X_\tau}]$ in $K(G/B)$ of the structure sheaves $\mathcal{O}_{X_\tau}$ of the Schubert varieties X_τ form a $\mathbb{Z}$-basis of $K(G/B)$. Let

$$[\mathcal{O}_{X_\tau}] \cdot [\mathcal{L}_\lambda] = \sum_{\kappa \in W} a^\lambda_{\tau,\kappa} [\mathcal{O}_{X_\kappa}]. \tag{2}$$

More generally, let $K_T(G/B)$ be the Grothendieck ring of T-equivariant sheaves on G/B. By [11] one knows again that the classes $[\mathcal{O}_{X_\tau}]_T$ of the structure sheaves of the Schubert varieties form a $\mathbb{Z}[\Lambda]$-basis, where $\mathbb{Z}[\Lambda]$ is the group algebra of the weight lattice Λ. Let

$$[\mathcal{O}_{X_\tau}]_T \cdot [\mathcal{L}_\lambda]_T = \sum_{\kappa \in W} C^\lambda_{\tau,\kappa} [\mathcal{O}_{X_\kappa}]_T \tag{3}$$

where $C^\lambda_{\tau,\kappa}$'s are formal sums of characters of T with integral coefficients. Assume now that $\lambda \in \Lambda^+$. Then one knows that in (1) all $a_i \geqslant 0$ (cf. [3]). The integers $a^\lambda_{\tau,\kappa}$ in (2) were determined by Fulton and Lascoux (cf. [7]) for the case $G = SL(n)$; they provide a formula for $a^\lambda_{\tau,\kappa}$ (using the combinatorics of Grothendieck polynomials). The general case was treated by Mathieu using representation theory (cf. [23]), who shows that $C^\lambda_{\tau,\kappa}$ are effective i.e., formal sums of characters with positive integral coefficients. For an L–S path $\pi = (\tau_1 > \cdots > \tau_r; a_1 < \cdots < a_r)$ we set the initial direction as $i(\pi) = \tau_1$ and the final direction $e(\pi) = \tau_r$.

We assume for simplicity that λ is *regular dominant*. Using the path model theory, Pittie and Ram (cf. [24]) gave an explicit determination of $C^\lambda_{\tau,\kappa}$'s:

$$[\mathcal{O}_{X_\tau}]_T \cdot [\mathcal{L}_\lambda]_T = \sum_{i(\pi) \leqslant \tau} [\mathcal{O}_{X_{e(\pi)}}]_T \mathrm{e}^{\pi(1)} \tag{4}$$

where the summation runs over L–S paths π of shape λ.

An effective version of the relation (4) is proved in [21] by constructing a filtration $\mathcal{F} := \{\mathcal{F}^i\}$ for $\mathcal{O}_{X_\tau} \otimes \mathcal{L}_\lambda$ by B-equivariant $\mathcal{O}_{G/B}$-modules such that each subquotient, as a B-equivariant $\mathcal{O}_{G/B}$-sheaf, is isomorphic to $\mathcal{O}_{X_{e(\pi)}} \otimes \chi_{-\pi(1)}$, the structure sheaf $\mathcal{O}_{X_{e(\pi)}}$, twisted by the character $-\pi(1)$ of B, for a suitable π such that $i(\pi) \leqslant \tau$.

Fixing a $\kappa \leqslant \tau$, we obtain from (4) that the coefficient of $[\mathcal{O}_{X_\kappa}]_T$ on the right-hand side of (4) equals

$$\sum_{\{\pi \mid \tau \geqslant i(\pi), e(\pi) = \kappa\}} \mathrm{e}^{\pi(1)}.$$

Using the results of the preceding sections, we give this character a representation theoretic interpretation. By tensoring the exact sequence

$$0 \to \mathcal{I}_{\partial^- X_\tau^\kappa} \to \mathcal{O}_{X_\tau^\kappa} \to \mathcal{O}_{\partial^- X_\tau^\kappa} \to 0$$

by $\mathcal{L}_\lambda$, and writing the cohomology exact sequence, we get the exact sequence

$$0 \to H^0\big(X_\tau^\kappa, \mathcal{I}_{\partial^- X_\tau^\kappa}(\lambda)\big) \to H^0\big(X_\tau^\kappa, \mathcal{L}_\lambda\big) \to H^0\big(\partial^-(X_\tau^\kappa), \mathcal{L}_\lambda\big) \to 0.$$

Hence we obtain $\dim H^0(X_\tau^\kappa, \mathcal{I}_{\partial^- X_\tau^\kappa}(\lambda)) = \#\{p_\pi \mid i(\pi) \leqslant \tau, e(\pi) = \kappa\}$. Since these vectors form a basis for the kernel of the second map, it follows that $\{p_\pi \mid i(\pi) \leqslant \tau, e(\pi) = \kappa\}$ is actually a basis of $H^0(X_\tau^\kappa, \mathcal{I}_{\partial^- X_\tau^\kappa}(\lambda))$. In the non-regular case, we have to work with the λ-boundary $\partial_\lambda^- X_\tau^\kappa$ in the place of $\partial^- X_\tau^\kappa$. Thus we obtain

Theorem 34. *With notations as above, let λ be dominant. We have*

$$\begin{aligned} C_{\tau,\kappa}^\lambda &= \operatorname{Char} H^0\big(X_\tau^\kappa, \mathcal{L}_\lambda \otimes \mathcal{I}_{\partial^- X_\tau^\kappa}\big), \\ a_{\tau,\kappa}^\lambda &= \#\big\{\pi \text{ L–S path, shape } \lambda \mid \tau \geqslant i(\pi),\ e(\pi) = \kappa\big\}. \end{aligned}$$

The above result is also proved in [1] using a flat family with generic fiber $\cong \operatorname{diag}(X_\tau) \subset X_\tau \times X_\tau$, and the special fiber $\cong \bigcup_{x \leqslant \tau} X_x \times X_\tau^x$. We will give a different construction of the flat family using SMT. Again the connection can be most directly established in the language of equivariant K-theory.

Let $X_\tau^\kappa \subset G/Q$ be a Richardson variety. On the one hand, consider the Richardson variety X_τ^κ diagonally embedded in $Z = G/Q \times G/Q$, we write ΔX_τ^κ for this variety. On the other hand, consider the subvariety

$$Y = \bigcup_{\substack{\kappa \in W/W_Q \\ \kappa \leqslant \sigma \leqslant \tau}} X_\sigma^\kappa \times X_\tau^\sigma \subset X_\tau^\kappa \times X_\tau^\kappa \subset Z.$$

We denote the corresponding structure sheaves $\mathcal{O}_{\Delta X_\tau^\kappa}$, respectively $\mathcal{O}_Y$; we are interested in describing their classes in the Grothendieck group $K_T(Z)$ of T-equivariant coherent sheaves of $\mathcal{O}_Z$-modules on Z. It turns out that these are intimately related to the notion of *standard monomials*.

The existence of a filtration and a flat family as in Theorem 35 below is also proved in [2]. The existence of a filtration is proved in [1] for any Cohen–Macaulay subvariety X of a flag variety, in general position with respect to opposite Schubert varieties (note that the assumptions on X hold for Schubert varieties). It is interesting to note that while in this paper the existence of a filtration and a flat family in Theorem 35 is obtained as a consequence of SMT, in [2] it is the starting point towards developing a SMT! It should be pointed out that the construction of the flat family in [2] is much more direct and explicit. The point of the approach used here is: the quadratic straightening relations (Theorem 13, Corollary 14) deform into the most simple form one can imagine: any product of monomials which is non-standard is zero. So the main point in presenting the approach here is to give an explicit link between the Gröbner basis style touch of SMT and the K-theory of flag varieties. The construction here can be viewed as a special case of the more general theory developed by Chrivi of deformations of LS-algebras into discrete LS-algebras, see [4].

Theorem 35. *In $K_T(Z)$, the following equality of classes of T-equivariant coherent sheaves of $\mathcal{O}_Z$-modules holds*:

$$[\mathcal{O}_{\Delta X_\tau^\kappa}]_T = [\mathcal{O}_Y]_T = \sum_{\substack{\sigma \in W/W_Q \\ \kappa \leqslant \sigma \leqslant \tau}} \big[\partial^+ \mathcal{I}_\sigma^\kappa \otimes \mathcal{O}_{X_\tau^\sigma}\big]_T = \sum_{\substack{\sigma \in W/W_Q \\ \kappa \leqslant \sigma \leqslant \tau}} \big[\mathcal{O}_{X_\sigma^\kappa} \otimes \partial^- \mathcal{I}_\tau^\sigma\big]_T.$$

More precisely, there exist T-stable filtrations of $\mathcal{O}_Y$ as sheaf of $\mathcal{O}_Z$-modules, such that the associated graded is the direct sum of the sheaves $\partial^+ \mathcal{I}_\sigma^\kappa \otimes \mathcal{O}_{X_\tau^\sigma}$, respectively the direct sum of the sheaves $\mathcal{O}_{X_\sigma^\kappa} \otimes \partial^- \mathcal{I}_\tau^\sigma$, where $\kappa \leqslant \sigma \leqslant \tau$.

Further, there exists a flat T-equivariant family of subvarieties of Z with basis the affine line, such that the generic fibre is isomorphic to the diagonal Richardson variety ΔX_τ^κ, and the special fibre is Y.

Proof. We use again the correspondence between graded modules over the homogeneous coordinate ring and coherent sheaves. Fix a (Q-regular) dominant weight $\lambda \in \Lambda_Q^{++}$ and let $B_\tau^\kappa(n\lambda)$ be the set of standard sequences of length n of L–S paths of shape λ, standard on X_τ^κ.

Denote $\widehat{R}$ the homogeneous coordinate ring corresponding to the embedding $Z \hookrightarrow \mathbb{P}(H^0(Z, \mathcal{L}_\lambda \otimes \mathcal{L}_\lambda)^*)$, so that

$$\widehat{R} = \bigoplus_{n \geqslant 0} H^0(Z, \mathcal{L}_{n\lambda} \otimes \mathcal{L}_{n\lambda}),$$

and let $\widehat{R}_\tau^\kappa$ be the quotient:

$$\widehat{R}_\tau^\kappa = \bigoplus_{n \geqslant 0} H^0\big(X_\tau^\kappa, \mathcal{L}_{n\lambda}\big) \otimes H^0\big(X_\tau^\kappa, \mathcal{L}_{n\lambda}\big) = \bigoplus_{n \geqslant 0} H^0\big(X_\tau^\kappa \times X_\tau^\kappa, \mathcal{L}_{n\lambda} \otimes \mathcal{L}_{n\lambda}\big).$$

Consider for $n \geqslant 1$ the subspace (this is definitely not an ideal)

$$SM_\tau^\kappa(\lambda)_n = \big\langle p_{\boldsymbol{\pi}_1} \otimes p_{\boldsymbol{\pi}_2} \mid \boldsymbol{\pi}_1, \boldsymbol{\pi}_2 \in B_\tau^\kappa(n\lambda),\ p_{\boldsymbol{\pi}_1} p_{\boldsymbol{\pi}_2} \text{ standard in } H^0(X_\tau^\kappa, \mathcal{L}_{2n\lambda})\big\rangle$$

and set $SM_\tau^\kappa(\lambda)_0 = k$. Consider the graded vector space:

$$SM_\tau^\kappa(\lambda) = \bigoplus_{n \geqslant 0} SM_\tau^\kappa(\lambda)_n.$$

We will present two ways to make this graded vector space into a graded $\widehat{R}$-module.

First note that the subvariety Y is a union of Richardson varieties (for the group $G \times G$ with Borel subgroup $B \times B$). So by Theorem 20, the following restriction map is T-equivariant, surjective, and maps the subspace $SM_\tau^\kappa(\lambda)$ isomorphically onto its image:

$$\widehat{R}_\tau^\kappa \to \widehat{R}_Y = \bigoplus_{n \geqslant 0} H^0(Y, \mathcal{L}_{n\lambda} \otimes \mathcal{L}_{n\lambda}).$$

The kernel is in this case also easy to describe: If the product $p_{\boldsymbol{\pi}_1} p_{\boldsymbol{\pi}_2}$ is not standard, then $e(\boldsymbol{\pi}_1) \not\geqslant i(\boldsymbol{\pi}_2)$ and $e(\boldsymbol{\pi}_2) \not\geqslant i(\boldsymbol{\pi}_1)$; hence either $p_{\boldsymbol{\pi}_1}|_{X_\sigma^\kappa} \equiv 0$ or $p_{\boldsymbol{\pi}_2}|_{X_\tau^\sigma} \equiv 0$. It follows that the restriction of $p_{\boldsymbol{\pi}_1} \otimes p_{\boldsymbol{\pi}_2}$ on all irreducible components $X_\sigma^\kappa \times X_\tau^\sigma$ vanishes. Since the tensor products $p_{\boldsymbol{\pi}_1} \otimes p_{\boldsymbol{\pi}_2}$, $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2 \in B_\tau^\kappa(n\lambda)$, form a basis, it follows that the kernel of the map is spanned by all $p_{\pi_1} \otimes p_{\pi_2}$ such that $\pi_1, \pi_2 \in B_\tau^\kappa(n\lambda)$ and $p_{\boldsymbol{\pi}_1} p_{\boldsymbol{\pi}_2}$ is not standard in $H^0(X_\tau^\kappa, \mathcal{L}_{2n\lambda})$.

Summarizing, by the T-equivariant graded vector space isomorphism of $SM_\tau^\kappa(\lambda) \to \widehat{R}_Y$ we have endowed $SM_\tau^\kappa(\lambda)$ in a T-equivariant way with the structure of an $\widehat{R}$-module.

Next consider the product map

$$\widehat{R}_\tau^\kappa \to \widehat{R}_{\Delta X_\tau^\kappa} = \bigoplus_{n \geqslant 0} H^0\big(\Delta X_\tau^\kappa, \mathcal{L}_{n\lambda} \otimes \mathcal{L}_{n\lambda}\big).$$

Again by Theorem 20, this map induces a T-equivariant isomorphism of graded vector spaces $SM_\tau^\kappa(\lambda) \to \widehat{R}_{\Delta X_\tau^\kappa}$, so this induces a different structure as $\widehat{R}$-module on $SM_\tau^\kappa(\lambda)$.

Consider the $\mathbb{Z}[\Lambda]$-linear map $\chi_\lambda : K_T(Z) \to \mathbb{Z}[\Lambda]$ defined on the class of a T-equivariant coherent sheaf $\mathcal{F}$ as follows:

$$\chi_\lambda\big([\mathcal{F}]\big) = \sum_{i \geqslant 0} (-1)^i \operatorname{Char} H^i\big(Z, \mathcal{F} \otimes_{\mathcal{O}_Z} (\mathcal{L}_\lambda \otimes \mathcal{L}_\lambda)\big).$$

If two elements of $K_T(Z)$ do not coincide, say $\sum a_{\mathcal{F}}[\mathcal{F}] \neq \sum a_{\mathcal{F}'}[\mathcal{F}']$, then it is well known (see for example [21]) that there exists an $n \in \mathbb{N}$ such that

$$\chi_{n\lambda}\left(\sum a_{\mathcal{F}}[\mathcal{F}]\right) \neq \chi_{n\lambda}\left(\sum a_{\mathcal{F}'}\big[\mathcal{F}'\big]\right).$$

Now by the vanishing of the higher cohomology (Theorem 20) and the T-equivariant graded vector space isomorphisms above, we get:

$$\chi_{n\lambda}(\mathcal{O}_Y) = \operatorname{Char} SM_\tau^\kappa(\lambda)_n = \chi_{n\lambda}(\mathcal{O}_{\Delta X_\tau^\kappa})$$

and hence $[\mathcal{O}_Y] = [\mathcal{O}_{\Delta X_\tau^\kappa}]$ in $K_T(Z)$.

Next fix a numeration $\kappa = \sigma_1, \sigma_2, \ldots, \tau = \sigma_q$ of the elements between κ and τ such that $\sigma_j > \sigma_i$ implies $j > i$. Set $Y^j = \bigcup_{1 \leqslant i \leqslant j} X_{\sigma_i}^\kappa \times X_\tau^{\sigma_i}$, so we get a filtration

$$Y^1 = X_\kappa^\kappa \times X_\tau^\kappa \subset Y^2 \subset \cdots \subset Y^q = Y.$$

Set $\widehat{R}_Y(j) = \bigoplus_{n\geqslant 0} H^0(Y^j, \mathcal{L}_{n\lambda} \otimes \mathcal{L}_{n\lambda})$, this ring again has a basis consisting of standard monomials. Set $K^1 = \widehat{R}_Y$, and, for $j = 1, \dots, q$, let $K^{j+1} \subset \widehat{R}_Y$ be the kernel of the restriction map $\widehat{R}_Y \to \widehat{R}_Y(j)$. We get a filtration

$$K^{q+1} = 0 \subset K^q \subset \cdots \subset K^2 \subset K^1 = \widehat{R}_Y.$$

The kernels K^j have again a basis by standard monomials. In fact, recall the basis elements $p_{\boldsymbol{\pi}_1} \otimes p_{\boldsymbol{\pi}_2}$ in $SM_\tau^\kappa(\lambda)$, then K^j is spanned by those such that $i(\boldsymbol{\pi}_1) = \sigma_i$ for some $i \geqslant j$. Consider the map

$$K^j \to H^0\big(X_{\sigma_j}^\kappa \times X_\tau^{\sigma_j}, \mathcal{L}_{n\lambda} \otimes \mathcal{L}_{n\lambda}\big), \quad s \mapsto s|_{X_{\sigma_j}^\kappa \times X_\tau^{\sigma_j}}.$$

It follows immediately that K^{j+1} is the kernel of this map, and, by the proof of Theorem 34, we get an isomorphism of $\widehat{R}$-modules:

$$K^j / K^{j+1} \simeq \bigoplus_{n\geqslant 0} H^0\big(X_{\sigma_j}^\kappa, \mathcal{I}_{\partial^+ X_{\sigma_j}^\kappa}(n\lambda)\big) \otimes H^0(X_\tau^{\sigma_j}, \mathcal{L}_{n\lambda}).$$

Since all these maps are T-equivariant, translated into the language of sheaves of $\mathcal{O}_Z$-modules, this means that the filtration of $\widehat{R}_Y$ induces a T-stable filtration of $\mathcal{O}_Y$ as $\mathcal{O}_Z$-sheaf such that the associated graded is a direct sum of the T-equivariant $\mathcal{O}_Z$-sheaves $\mathcal{I}_{\partial^+ X_{\sigma_j}^\kappa} \otimes \mathcal{O}_{X_\tau^{\sigma_j}}$.

The proof for the second filtration is similar and is left to the reader. To describe the flat family note that the kernel of the product map

$$\widehat{R}_\tau^\kappa \to \widehat{R}_{\Delta X_\tau^\kappa} = \bigoplus_{n\geqslant 0} H^0\big(\Delta X_\tau^\kappa, \mathcal{L}_{n\lambda} \otimes \mathcal{L}_{n\lambda}\big)$$

is the ideal generated by the commutation relations $p_{\pi_1} \otimes p_{\pi_2} - p_{\pi_2} \otimes p_{\pi_1}$ for all $\pi_1, \pi_2 \in B_\tau^\kappa(\lambda)$, and if neither $p_{\pi_1} p_{\pi_2}$ nor $p_{\pi_2} p_{\pi_1}$ is standard in $H^0(X_\tau^\kappa, \mathcal{L}_{2\lambda})$, then we have the additional relations

$$p_{\pi_1} \otimes p_{\pi_2} - \sum a_{\eta_1,\eta_2} p_{\eta_1} \otimes p_{\eta_2}$$

where the $p_{\eta_1} p_{\eta_2}$ are standard and the coefficients $a_{\eta_1,\eta_2} \neq 0$ only if $(\eta_1, \eta_2) \succeq \pi_1 \wedge \pi_2 \succeq^r (\eta_1, \eta_2)$. This follows easily from Theorem 13, using a monomial order as in Proposition 15. Further, these elements form a reduced Gröbner basis for the ideal. The existence of the flat deformation follows now from standard Gröbner basis arguments. $\square$

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