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A priori estimates and existence of positive solutions for strongly nonlinear problems

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Abstract

In this paper we study the existence of positive solutions for a nonlinear Dirichlet problem involving the m-Laplacian. The nonlinearity considered depends on the first derivatives; in such case, variational methods cannot be applied. So, we make use of topological methods to prove the existence of solutions. We combine a blow-up argument and a Liouville-type theorem to obtain a priori estimates. Some Harnack-type inequalities which are needed in our reasonings are also proved.

Keywords: m-Laplacian; A priori estimates; Blow-up; Harnack inequalities

1. Introduction

In this work we are concerned with the existence of positive solutions for the problem:

$$\begin{cases} \Delta_m u + f(x, u, \nabla u) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\Delta_m u = \mathrm{div}(|\nabla u|^{m-2}\nabla u)$ stands for the usual m-Laplacian, $1 < m < N$, and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous function. Observe that problem (1) does not have, in general, a variational structure.

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The main assumption on the function f is the following, which will be referred throughout the paper as [F]:

$$[\mathrm{F}] \quad \left[\begin{array}{c} u^p - M|\eta|^\alpha \leqslant f(x,u,\eta) \leqslant c_0 u^p + M|\eta|^\alpha, \\ \forall\ (x,u,\eta) \in \Omega \times \mathbb{R} \times \mathbb{R}^N, \text{ where } c_0 \geqslant 1,\ \ M>0, \\ p \in (m-1, m_*-1) \text{ and } \alpha \in \left(m-1, \frac{mp}{p+1}\right). \end{array} \right.$$

Here we denote $m_* = \frac{m(N-1)}{N-m}$. Note that under [F], $u \equiv 0$ is always a solution for (1).

Similar problems have been very studied in the literature, especially when $m=2$, see for instance the classical papers [5] (where the conditions assumed on f are stronger than [F], even if we restrict ourselves to the case $m=2$) and [8] (where f does not depend on the derivatives and some other technical conditions are imposed; in exchange, f is allowed to exhibit any subcritical growth). In [24], a very similar problem is treated in the framework of uniformly elliptic operators (for the case of systems of equations, see [9]). Azizieh and Clément have recently studied the m-Laplacian case, but under some additional conditions; f does not depend on x nor ∇u, $1<m\leqslant 2$, and Ω is a convex domain. The main feature of this work is to remove all these conditions. We also prove some Harnack-type inequalities which may be useful in the study of similar problems.

Another related paper is [16], where problem (1) is also considered, but under different hypotheses on the nonlinearity f.

In order to prove the existence of positive solutions for (1), we will use a degree argument which was first used by Krasnoselskii [13] (see also [8]). The main ingredients of our arguments are some a priori estimates on the pairs (u,λ) solving the problem:

$$\begin{cases} \Delta_m u + f(x,u,\nabla u) + \lambda = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{cases} \tag{2}$$

with $\lambda \geqslant 0$ and $u \in C^1(\bar{\Omega})$.

First, we prove that (2) has no solution at all when $\lambda > \lambda_0$ for a certain λ_0 positive. We use an argument involving the Picone identity for the m-Laplacian (see [1]).

Most part of the paper is devoted to obtaining a priori estimates (in the L^∞ sense) on the weak solutions of (2) when $\lambda \in [0,\lambda_0]$. These a priori estimates will be accomplished by using a blow-up technique, together with Liouville-type theorems (that is, nonexistence results of positive solutions for the so-called "limit problems"). This kind of argument was first used in [12], where the authors made use of the nonexistence of positive solutions for the two problems (see [11,12] respectively).

$$\Delta u + u^p = 0 \quad \text{in } \mathbb{R}^N, \tag{3}$$

$$\begin{cases} \Delta u + u^p = 0 & \text{in } H_+, \\ u(x) = 0 & \text{in } \partial H_+, \end{cases} \tag{4}$$

where $1<p<\frac{N+2}{N-2}$ and $H_+=\{x=(x_1,\dots,x_N)\in\mathbb{R}^N\colon\ x_1>0\}$ is a halfspace in $\mathbb{R}^N$.

Mitidieri and Pohozaev proved in [15] that the problem:

$$\Delta_m u + u^p \leqslant 0 \tag{5}$$

has no positive solution in $\mathbb{R}^N$ when $p\in(m-1,m_*-1)$. However, as far as we know, there is no Liouville-type result for the generic m-Laplacian in the halfspace. This is the main difficulty we have to face.

Suppose, reasoning by contradiction, that there exists a divergent sequence (in the L^∞ norm) of solutions u_n of (2). Take x_n a point at which u_n attain their maxima. As shown in [12], the blow-up method provides a solution in a halfspace when the points x_n approach sufficiently fast (in comparison with the L^∞ norm of u_n) to the boundary of Ω.

In [4], the authors assume that Ω is convex and that $1<m\leqslant 2$; in that case, they use the moving plane method (as developed in [6] for the m-Laplacian) to find out that the sequence x_n cannot approach to the boundary. Then they use the nonexistence result [15].

We use here the same blow-up technique but centered on a certain fixed point $y_0\in\Omega$, instead of x_n. First of all, we have to verify that $u_n(y_0)\to\infty$ when $n\to\infty$. In order to do that, we will compare the values of u_n in different points in Ω through some Harnack-type inequalities. One of these inequalities is due to Trudinger [22], the other being proved in this work, Section 2, by using ideas from Serrin and Zou (see [19]).

Using this procedure, the corresponding problem will be defined in $\mathbb{R}^N$, and we obtain a contradiction with the Liouville result [15]. To the best of the author's knowledge, this variant of the blow-up technique is entirely new in the study of this type of problems. In our opinion, these arguments may also be used in other frameworks.

At the end of the paper we use the sub–super solutions method in order to establish a Ambrosetti–Prodi-type result for problem (2), that is, to study the existence of at least two solutions, depending on the parameter λ.

2. Harnack inequalities

In this section we state the Harnack inequalities given by Theorems 2.3 and 2.4, which will be needed in next section. The first one is a convenient extension of the Theorem 4.1 in [19], whereas the second one is due to Trudinger [22].

We first state and prove some lemmas which will be useful to prove Theorem 2.3. Here and throughout all the paper, C stands for a positive constant which may vary from one expression to another, but is always independent of u.

Lemma 2.1. *Let u be a positive weak C^1 solution of the inequality*:

$$-\Delta_m u \geqslant u^p - M|\nabla u|^\alpha \tag{6}$$

in a domain $\Omega\subset\mathbb{R}^N$, *where* $p>m-1$ *and* $m-1\leqslant\alpha<\frac{mp}{p+1}$. *Take* $\gamma\in(0,p)$ *and* $\mu\in\left(0,\frac{mp}{p+1}\right)$. *Let* $R_0>0$ *be fixed, and* $0<R<R_0$. *Denote by* B_R *a ball of radius* R *such that* B_{2R} *is included in* Ω.

Then, there exists a positive constant $C=C(N,m,p,\gamma,\mu,R_0)$ *such that*

$$\int_{B_R} u^{\gamma}\leqslant CR^{N-m\gamma/(p+1-m)}, \tag{7}$$

$$\int_{B_R} |\nabla u|^{\mu}\leqslant CR^{N-(p+1)\mu/(p+1-m)}. \tag{8}$$

Remark. Lemma 2.1 is a generalization of Lemma 2.4 of [19] (that lemma treated the inequality $-\Delta_m u\geqslant u^p$ instead of (6); see also [15]). We just sketch the proof, since it basically uses the same ideas. The only difference is that now we have a new term to be estimated.

Proof. We can suppose that the ball B_R is centered at zero. We first focus on proving (7).

Let ξ be a radially symmetric C^2 cut-off function on $B_2(0)$, that is:

1. $\xi(x)=1$ for $|x|\leqslant 1$.
2. ξ has compact support in $B_2(0)$ and $0\leqslant\xi\leqslant 1$.
3. $|\nabla\xi|\leqslant 2$.

Let $d=p-\gamma>0$. We take $\phi=[\xi(x/R)]^k u^{-d}$ as a test function for inequality (6) (we will fix k later), and obtain:

$$d\int_\Omega \xi^k u^{\gamma-p-1}|\nabla u|^m+\int_\Omega \xi^k u^{\gamma}\leqslant\int_\Omega u^{-d}|\nabla u|^{m-1}|\nabla\xi^k|+M\int_\Omega \xi^k u^{-d}|\nabla u|^{\alpha}$$

Now we give some estimates by using the Young inequality in the form

$$ab\leqslant\varepsilon a^q+\varepsilon^{\frac{1}{1-q}}b^{q/(q-1)}\quad\forall q>1,\ \varepsilon>0.$$

Observe that $|\nabla\xi^k|=k\xi^{k-1}|\nabla\xi|\leqslant\xi^k 2k/R\xi$. Using the above Young's inequality with an appropriate ε, we have

$$\int_\Omega u^{-d}\,|\nabla u|^{m-1}|\nabla\xi^k|\leqslant\frac{d}{2}\xi^k u^{\gamma-p-1}|\nabla u|^m+CR^{-m}\int_\Omega \xi^{k-m}u^{\gamma-r},$$

where we denote $r=p+1-m$. Therefore, we have

$$\frac{d}{2}\int_\Omega \xi^k u^{\gamma-p-1}\,|\nabla u|^m+\int_\Omega \xi^k u^{\gamma}\leqslant CR^{-m}\int_\Omega \xi^{k-m}u^{\gamma-r}+M\int_\Omega \xi^k u^{-d}|\nabla u|^{\alpha}. \tag{9}$$

Inequality (9) is very similar to expression (2.8) in [19]; the only difference is that now a second term on the right appears.

We want now to estimate the two right terms of the previous expression.

Step 1: If $\gamma = r$, the arguments in step 2 together with (9) give (7). Moreover, if $\gamma < r$, we apply Hölder inequality to obtain

$$\int_{B_R} u^\gamma \leqslant C R^{N(1-\gamma)/r} \left(\int_{B_R} u^r \right)^{\gamma/r}.$$

So, we obtain the desired estimate, thanks to the case $\gamma = r$.

Let us focus on the case $\gamma > r$. Choose $k = m\gamma/r$, and apply the Young inequality to obtain

$$R^{-m} \int_\Omega \xi^{k-m} u^{\gamma-r} \leqslant \frac{1}{3} \int_\Omega \xi^k u^\gamma + C R^{N-m\gamma/r}.$$

Step 2: We are interested now in the second right term of (9). Using again the Young inequality, we obtain

$$u^{-d} |\nabla u|^\alpha \leqslant \frac{d}{4} u^{\gamma-p-1} |\nabla u|^m + C u^\varphi,$$

where $\varphi = \left(-d - \alpha \frac{\gamma-p-1}{m}\right) \frac{m}{m-\alpha}$. Thus, we have

$$M \int_\Omega \xi^k u^{-d} |\nabla u|^\alpha \leqslant \frac{d}{4} \int_\Omega \xi^k u^{\gamma-p-1} |\nabla u|^m + C \int_\Omega \xi^k u^\varphi.$$

By using the inequality $\alpha < \frac{mp}{p+1}$, one can easily conclude that $\varphi < \gamma$. Then we can deduce from the Young inequality that

$$\int_\Omega \xi^k u^\varphi \leqslant \frac{1}{3} \int_\Omega \xi^k u^\gamma + C R^N.$$

Summing up the results of the two preliminary steps, and taking into account (9), we have

$$\frac{d}{4} \int_\Omega \xi^k u^{\gamma-p-1} |\nabla u|^m + \frac{1}{3} \int_\Omega \xi^k u^\gamma \leqslant C R^{N-m\gamma/r} \tag{10}$$

(recall that $R \leqslant R_0$). The proof of (7) is complete.

To prove (8), first note that $\mu < m$. Use Hölder inequality to get

$$\int_{B_R} |\nabla u|^\mu \leqslant \left(\int_{B_R} u^{\gamma-p-1} |\nabla u|^m \right)^{\mu/m} \left(\int_{B_R} u^{\bar\gamma} \right)^{1-\mu/m},$$

where $\bar\gamma = (p+1-\gamma)\mu/(m-\mu)$. Since $\mu < \frac{mp}{p+1}$, we can choose γ close enough to $p-1$ such that $\bar\gamma < p$. Combining (7) and (10) (now referred to the first term on the left) we deduce the desired inequality. $\square$

In next lemma we state a result of Harnack type due to Serrin (see Theorem 5 in [18]). The formulation we give is exactly that of Lemma 4.2 in [19].

Lemma 2.2. *Let u be a nonnegative weak solution in a domain Ω of*:

$$|\Delta_m u| \leqslant c(x)|\nabla u|^{m-1} + d(x)u^{m-1} + f(x),$$

where $c \in L^{q'}(\Omega)$, $d,\ f \in L^q(\Omega)$, $q' > N$ *and* $q \in (N/m, N/(m-1))$.

Then, for every R such that $B_{2R} \subset \Omega$, *there exists C depending on*

$$N,\ m,\ q,\ q',\ R^{1-N/q'}\,||c||_{L^{q'}},\ R^{m-N/q}\,||d||_{L^q}$$

such that

$$\sup_{B_R} u \leqslant C\left(\inf_{B_R} u + R^{m-N/q}||f||_{L^q}\right).$$

Now, we derive our Harnack inequality from the previous lemmas:

Theorem 2.3. *Suppose that u defined in a domain Ω is a positive weak solution of the inequality*:

$$u^p - M|\nabla u|^\alpha \leqslant \Delta_m u \leqslant c_0 u^p + M|\nabla u|^\alpha + \lambda$$

with $m < p+1 < m_*$, $m-1 \leqslant \alpha \leqslant mp/(p+1)$ *and* $\lambda > 0$. *Let* $R \leqslant R_0$ *such that* $B_{2R} \subset \Omega$. *Then, there exists* $C = C(N, m, p, \alpha, R_0, M)$ *such that*

$$\sup_{B_R} u < C\left(\inf_{B_R} u + R^m \lambda\right).$$

Remark. Theorem 2.3 is a version of Theorem 4.1(b) in [19] (in that theorem, only the case $\alpha = m-1$ is considered).

Proof. We use Lemma 2.2 with $f = \lambda$, $c = M|\nabla u|^{\alpha-m+1}$ and $d = c_0 u^{p+1-m}$. Then, it suffices to prove that

$$R^{m-N/q}||f||_{L^q} < C\lambda R^m, \quad R^{1-N/q'}\,||c||_{L^{q'}} < C, \quad R^{m-N/q}\,||d||_{L^q} < C$$

for certain $C > 0$. To do that, we will use Lemma 2.1.

The first inequality is proved straightforward:

$$R^{m-N/q}||f||_{L^q(B_{2R})} = R^{m-N/q} C R^{N/q} \lambda = C\lambda R^m.$$

We now treat the second one. As explained in [19], we can assume at first that $B_{4R} \subset \Omega$: after that it is easy to prove the same results (increasing C) when $B_{2R} \subset \Omega$.

As in Lemma 2.1, we write $r = p + 1 - m$.

$$||c||_{L^{q'}(B_{2R})} = M\left[\int_{B_{2R}} |\nabla u|^{\mu}\right]^{\frac{1}{q'}} \leqslant CR^{\frac{N-(p+1)\mu/r}{q'}}.$$

In the above expression we have written $\mu = q'(\alpha - m + 1)$ (for some $q' > N$ not yet fixed) and we have used Lemma 2.1. To do that, we need to verify that $\mu < mp/(p+1)$. It suffices to show that $N(\alpha - m + 1) < \frac{mp}{p+1}$ and to choose $q' > N$ close enough to N. Since $\alpha \leqslant \frac{mp}{p+1}$, we just need to verify

$$N\left(\frac{mp}{p+1} - m + 1\right) < \frac{mp}{p+1}.$$

This last inequality is equivalent to $p < m_* - 1$, and therefore $\mu < \frac{mp}{p+1}$ for an appropriate $q' > N$.

Hence we can write

$$R^{1-N/q'}||c||_{L^{q'}(B_{2R})} \leqslant CR^{1-\frac{(p+1)\mu}{rq'}}.$$

For that expression to be bounded (when $R \leqslant R_0$) we just need to show that $1 - \frac{(p+1)\mu}{rq'} \geqslant 0$, but this holds by using the inequality $\alpha \leqslant mp/(p+1)$ (recall that $\mu = q'(\alpha - m + 1)$).

Now we prove $R^{m-N/q}||d||_{L^q} < C$. Assuming again that $B_{4R} \subset \Omega$, we obtain:

$$||d||_{L^q(B_{2R})} = c_0\left[\int_{B_{2R}} u^{\gamma}\right]^{1/q} \leqslant CR^{(N-mq)/q}$$

provided $\gamma < p$, where we denote $\gamma = (p+1-m)q < p$. As before, taking $q > N/m$ as close as necessary to N/m, it suffices to show that $(p+1-m)N/m < p$. But this last inequality is equivalent to $p < m_* - 1 \Leftrightarrow (p+1-m)N/m < p$.

The proof is complete. □

In Section 3 we will also make use of the following weak Harnack inequality, due to Trudinger [22].

Theorem 2.4. *Let $u \geqslant 0$ be a weak solution of the inequality $\Delta_m u \leqslant 0$ in Ω. Take $\gamma \in [1, m_* - 1)$ and $R > 0$ such that $B_{2R} \subset \Omega$. Then there exists $C = C(N, m, \gamma)$ (independent of R) such that*

$$\inf_{B_R} u \geqslant CR^{-n/\gamma}||u||_{L^{\gamma}(B_{2R})}.$$

3. A priori estimates

As we mentioned before, in this section we are interested in obtaining a priori estimates using a blow-up procedure. Actually, we are interested in the problem:

$$\begin{cases} \Delta_m u + f(x,u,\nabla u) + \lambda = 0, & x\in\Omega, \\ u(x) = 0, & x\in\partial\Omega. \end{cases} \tag{11}$$

As we shall see in next section, the parameter λ will be used to make an appropriate homotopy. Hence we are interested in studying the a priori bounds for a solution of (11). Since we are going to use a degree argument as in [8], we will also need the nonexistence of solutions of (11) for λ large. To do that, we will use the following lemma.

Lemma 3.1. *Let u be a positive solution of the problem*:

$$\begin{cases} -\Delta_m u = h(x)\geqslant 0, & x\in\Omega, \\ u(x) = 0, & x\in\partial\Omega. \end{cases}$$

Then

$$\int_\Omega h(x)\frac{\phi_1^m}{u^{m-1}}\leqslant\lambda_1\int_\Omega \phi_1^m,$$

where λ_1 is the first eigenvalue of the m-Laplacian with Dirichlet boundary conditions, and ϕ_1 is the associated eigenfunction.

Moreover, the equality holds if and only if $u(x) = a\phi_1(x)$ for some $a>0$.

This inequality is an easy consequence of the Picone identity for the m-Laplacian, see Theorem 1.1 in [1] (see also [3], Lemma 24). Observe that $\frac{\phi_1^m}{u^{m-1}}$ belongs to $W_0^{1,p}(\Omega)$ since u is positive in Ω and has nonzero outward derivative on the boundary because of the Hopf lemma (see [23]).

Proposition 3.2. *There exists $\lambda_0>0$ such that problem* (11) *has no positive solutions for any $\lambda\geqslant\lambda_0$.*

Proof. Suppose that u is a positive solution for (11). Previous lemma yields that

$$\int_\Omega [f(x,u,\nabla u)+\lambda]\frac{\phi_1^m}{u^{m-1}}\leqslant\lambda_1\int_\Omega \phi_1^m.$$

Define

$$l = \min\left\{\frac{\lambda+t^p}{t^{m-1}}:\ t\geqslant 0\right\}.$$

One can even give an explicit expression of $l = l(\lambda)$, and it is easy to show that $\lim_{\lambda\to\infty} l(\lambda) = \infty$. Using previous definition and property [F], it follows:

$$\begin{aligned}\int_\Omega [f(x,u,\nabla u)+\lambda]\frac{\phi_1^m}{u^{m-1}} &\geqslant \int_\Omega [|u|^p - M|\nabla u|^\alpha + \lambda]\frac{\phi_1^m}{u^{m-1}}\\ &\geqslant -\int_\Omega [M|\nabla u|^\alpha]\frac{\phi_1^m}{u^{m-1}} + l\int_\Omega \phi_1^m\\ &\Rightarrow -\int_\Omega [M|\nabla u|^\alpha]\frac{\phi_1^m}{u^{m-1}} \leqslant (\lambda_1 - l)\int_\Omega \phi_1^m. \qquad (12)\end{aligned}$$

We claim that the left integral expression is bounded below. If so, (12) provides a bound for l, and therefore for λ (recall that $\lim_{\lambda\to\infty} l(\lambda) = \infty$).

In order to prove the claim, we use the same arguments developed in the proof of Lemma 2.1. We use now as a test function the quotient $\psi = \frac{\phi_1^m}{u^{m-1}}$, which belongs to $W_0^{1,m}(\Omega)$ as we explained above. We will argue as in the proof of Lemma 2.1, with $\xi = \phi_1$, $R = 1$, $k = m$, $\gamma = r = p - m + 1$ and $d = m - 1$. By multiplying by the test function ψ, we obtain

$$\begin{aligned}&(m-1)\int_\Omega \phi_1^m u^{-m}|\nabla u|^m + \int_\Omega \phi_1^m u^{p-m+1}\\ &\quad\leqslant \int_\Omega m\phi_1^{m-1}u^{-(m-1)}|\nabla u|^{m-1}|\nabla\phi_1| + M\int_\Omega \phi_1^m u^{1-m}|\nabla u|^\alpha.\end{aligned}$$

Note that $\nabla\phi_1$ is bounded in $\bar{\Omega}$. So, we can estimate the first right expression by using an trivial version of the Young inequality ($x < \varepsilon x^a + C$, with $a > 1$, C depending on ε) we obtain, analogously to (9):

$$\frac{m-1}{2}\int_\Omega \phi_1^m u^{-m}|\nabla u|^m + \int_\Omega \phi_1^m u^{p-m+1} \leqslant C + M\int_\Omega \phi_1^m u^{1-m}|\nabla u|^\alpha. \qquad (13)$$

We can argue exactly as in step 2 (Lemma 2.1), to conclude that

$$\int_\Omega \phi_1^m u^{1-m}|\nabla u|^\alpha \leqslant \frac{m-1}{4}\int_\Omega \phi_1^m u^{-m}|\nabla u|^m + \frac{1}{2}\int_\Omega \phi_1^m u^{p-m+1} + C. \qquad (14)$$

Using again the Young inequality, and combining the previous inequality with (13) (analogously to (10)), we get

$$\frac{m-1}{4}\int_\Omega \phi_1^m u^{-m}|\nabla u|^m + \frac{1}{2}\int_\Omega \phi_1^m u^{p-m+1} \leqslant C.$$

Thus, (14) provides us with the desired bound, and the claim is proved. □

We now give a priori estimates on the solutions u of (11), when λ is bounded.

Proposition 3.3. *Assume that* [F] *holds, and that* $\lambda < \lambda_0$ *for some* λ_0 *fixed. Then, there exists* $C > 0$ *such that* $||u|| < C$ *for any* C^1 *solution* u *of* (11), *where* $|| \cdot ||$ *denotes the uniform norm.*

Proof. Suppose, by contradiction, that there exist $\lambda_n < \lambda_0$, $u_n > 0$ such that u_n is solution of (11) with λ substituted by λ_n, and that $||u_n|| \to \infty$.

We will use a blow-up argument to get a contradiction. Usually, the blow-up technique is always used around the points x_n in which u_n attain their maxima. However, in so doing it may happen that $d(x_n, \partial\Omega) \to 0$; as we mentioned in the introduction, in that case we may get into trouble.

In this proposition we make the blow-up procedure around a fixed point y_0 in Ω. To do that, we first need to assure that $u_n(y_0) \to \infty$, which is the main difficulty of this proof. After that, we will use the Harnack inequalities to prove some uniform estimates, which are needed to show the convergence of the method.

Let x_n be a point in Ω such that $u_n(x_n) = ||u_n|| = S_n$. Denote $\delta_n = d(x_n, \partial\Omega)$. In order to prove that $u_n(y_0) \to \infty$ for some $y_0 \in \Omega$, we proceed in several steps:

Step 1: *There exists* $c > 0$ *such that* $c < \delta_n S_n^{(p+1-m)/m}$. This will be accomplished by a blow-up argument around the maxima x_n. As we shall see, we will not need to pass to the limit, but to use some regularity results. Here (and through the rest of the paper), we use c to denote positive constants, which may vary from one expression to another, but are always independent of n. Define

$$w_n(x) = S_n^{-1} u_n(y),$$

where $y = M_n x + x_n$ and $M_n > 0$ will be defined later. The functions w_n are well defined at least in $B(0, \delta_n M_n^{-1})$, and $w_n(0) = ||w_n|| = 1$.

Easy computations show that

$$\nabla w_n(x) = S_n^{-1} M_n \nabla u_n(y),$$

$$\Delta_m w_n(x) = S_n^{1-m} M_n^m \Delta_m u_n(y).$$

Note that the above equality must be understood in the weak sense. Then, we have

$$\begin{aligned} -\Delta_m w_n(x) &= S_n^{1-m} M_n^m (f(y, u_n(y), \nabla u_n(y)) + \lambda_n) \\ &= S_n^{1-m} M_n^m (f(M_n x + x_n, S_n w_n(x), S_n M_n^{-1} \nabla w_n(x)) + \lambda_n) \\ &:= \theta_n(x, w_n, \nabla w_n). \end{aligned}$$

We choose $M_n = S_n^{\overline{\frac{m-1-p}{m}}}$. Note that since $p > m-1$, $M_n \to 0$. Using condition [F], we obtain

$$|\theta_n(x, w, \zeta)| \leqslant c_0 |w|^p + M S_n^{-p} S_n^{\alpha \overline{\frac{p+1}{m}}} |\zeta|^\alpha + \lambda_n S_n^{-p}. \tag{15}$$

Observe that the term $MS_n^{-p}S_n^{\alpha\frac{p+1}{m}}$ tends to zero as n tends to infinity. Moreover, since $\lambda_n S_n^{-p}$ also converges to 0,

$$|\theta_n(x,w,\zeta)| \leqslant c_0|w|^p + |\zeta|^\alpha + 1 \tag{16}$$

for n large enough. In fact, we can deduce from [F] that

$$\theta_n(x,w,\zeta) - w^p \geqslant -MS_n^{-p}S_n^{\alpha\frac{p+1}{m}}|\zeta|^\alpha + \lambda_n S_n^{-p} \to 0. \tag{17}$$

Now we use a $C^{1,\tau}$ regularity result up to the boundary due to Lieberman [14], to conclude from (16) that $||\nabla w_n|| \leqslant C$ for certain $C>0$ independent of n. We now argue as in [12]. Let $y_n \in \partial\Omega$ such that $d(x_n,y_n)=\delta_n$; then, by using the mean value theorem,

$$1 = w_n(0) - w_n(M_n^{-1}(y_n - x_n)) \leqslant ||\nabla w_n|| M_n^{-1}\delta_n \leqslant CM_n^{-1}\delta_n.$$

The proof of step 1 is concluded.

Remark. Observe that if $M_n^{-1}\delta_n$ is unbounded, passing to the limit (and taking into account (17)), we would obtain a weak (C^1) positive solution for the problem

$$\Delta_m u + u^p \leqslant 0 \quad \text{in } \mathbb{R}^N, \tag{18}$$

contradicting the Liouville result of [15]. However, we cannot study in such way the case $M_n^{-1}\delta_n$ bounded. If we did, we would obtain a positive solution in a halfspace, as in [12]; since we do not know if such solutions may exist or not, we do not arrive to a contradiction. That is the reason why we develop a blow-up argument around a fixed point $y_0 \in \Omega$.

Step 2: *There exists* $\gamma \in (0, m_* - 1)$ *such that*

$$\int_{B(x_n,\delta_n/2)} |u_n|^\gamma \to \infty.$$

First, we use Theorem 2.3 in that ball:

$$S_n = \max_{B(x_n,\delta_n/2)} u_n \leqslant C\left[\min_{B(x_n,\delta_n/2)} u_n + \lambda_n \frac{\delta_n^m}{2^m}\right].$$

Since λ_n and δ_n are bounded, we have that $\min_{B(x_n,\delta_n/2)} \geqslant cS_n$ for certain $c>0$. Using that inequality and step 1, we obtain

$$\int_{B(x_n,\delta_n/2)} |u_n|^\gamma \geqslant cS_n^\gamma \delta_n^N \geqslant cS_n^\gamma S_n^{(m-1-p)N/m}.$$

It is easy to show now that

$$\frac{(p+1-m)N}{m} < m_* - 1 \;\Leftrightarrow\; p < m_* - 1.$$

Thus we can choose $(p+1-m)N/m < \gamma < m_* - 1$, and the claim is proved.

Step 3: *There exists* $y_0 \in \Omega$ *such that* $u_n(y_0) \to \infty$. We now need to use the smoothness of the boundary of Ω (in fact, C^2 regularity suffices). So, we can find $\varepsilon > 0$, $y_n \in \Omega$ such that:

- $d(y_n, \partial\Omega) = 2\varepsilon$ for all $n \in \mathbb{N}$.
- $B(x_n, \delta_n/2) \subset B(y_n, 2\varepsilon)$ for all $n \in \mathbb{N}$.

The fact that ε can be chosen independent of n is due to the compactness and regularity of $\partial\Omega$. We now use Theorem 2.4 and step 2 to conclude

$$\min_{B(y_n,\varepsilon)} u_n \geqslant c \left(\int_{B(y_n,2\varepsilon)} |u_n|^\gamma \right)^{1/\gamma} \to +\infty.$$

Taking a subsequence if necessary, we can assume that $y_n \to y_0 \in \Omega$. For n large, we have that $y_0 \in B(y_n, \varepsilon)$, and hence $u_n(y_0) \to +\infty$.

Remark. Observe that, in particular, $u_n(y) \to +\infty$ for any $y \in B(y_0, \varepsilon/2)$. In fact, it is easy to prove, by using Harnack inequality (Theorem 2.3), that $u_n(y) \to +\infty$ for any $y \in \Omega$.

Once we have found $y_0 \in \Omega$ such that $u_n(y_0) \to \infty$, we make use of the blow-up technique around y_0. Let $\bar{S}_n = u_n(y_0)$, and define

$$w_n(x) = (\bar{S}_n)^{-1} u_n(y),$$

where $y = \bar{M}_n x + y_0$ and $\bar{M}_n = \bar{S}_n^{\frac{m-1-p}{m}}$.

Our intention is now to pass to the limit when $n \to \infty$. To do that, we first need an uniform bound on the sequence w_n. Observe that, when using blow-up around the maxima of u_n, we immediately get $||w_n|| = 1$. This type of bound is needed to obtain uniform $C^{1,\tau}$ estimates, which are essential in the proof of the convergence of the sequence.

Take $\delta = d(y_0, \partial\Omega) > 0$. We apply then the Harnack inequality given in Theorem 2.3, to obtain

$$\max_{B(x_0,\delta/2)} u_n < C \left[\min_{B(x_0,\delta/2)} u_n + \lambda_n (\delta/2)^m \right].$$

It follows that $\max_{B(0,\bar{M}_n^{-1}\delta/2)} w_n < C$, and clearly $\bar{M}_n^{-1}\delta/2 \to +\infty$.

Define then

$$z_n(x) = \begin{cases} w_n(x), & |x| < \bar{M}_n^{-1}\delta/2, \\ 0, & \text{otherwise.} \end{cases}$$

We can now apply the usual convergence argument (see [12], for instance) to the sequence z_n taking into account that $z_n \leqslant C$. Fixed a ball $B(0,R) \subset \mathbb{R}^n$, we can take n large enough so that $2R < \delta \bar{M}_n^{-1}\delta/2$. We use the definition of θ_n given in the step 1 of this proof, but with S_n, M_n substituted by $\bar{S}_n$, $\bar{M}_n$. Note that the expressions (15), (34) and (15) are also valid here (when S_n is replaced with $\bar{S}_n$).

Hence, we can use the regularity result [10,21] (which is possible thanks to (16)) to conclude that $||z_n||_{C^{1,\tau}} < C$ in $B(0,R)$ for certain C independent of n. Therefore, z_n converges in the C^1 norm (up to a subsequence) to a certain function z_0. Observe that $z_0(0) = 1$. Applying (17) with $|x| < \bar{M}_n^{-1} d/2$, we obtain that:

$$\theta_n(x, z_n, \nabla z_n) - z_n^p \geqslant -\bar{M}\bar{S}_n^{-p}\bar{S}_n^{\alpha\frac{p+1}{m}} C^\alpha + \lambda_n \bar{S}_n^{-p} \to 0 \quad (n \to \infty).$$

Thus, z_0 is a nonnegative (weak) solution of the problem $\Delta_m z_0 + z_0^p \leqslant 0$ in $B(0,R)$. The strong maximum principle of Vázquez [23] yields that w_0 is actually positive in that ball.

Since R is arbitrary, using a diagonal procedure, we can take a subsequence (still denoted by z_n) such that z_n converge to z_0 in compact sets of $\mathbb{R}^n$ (in the norm C^1), where z_0 is now defined in all $\mathbb{R}^n$. Then, we obtain that z_0 is a positive solution of problem (18), which is a contradiction with [15]. □

4. Existence results and final remarks

In this section we prove the existence of positive solutions for (1). In order to do that, we use a version of a theorem of Krasnoselskii [13] (see also [8]) about the existence of fixed points on compact operators defined in a cone. As we shall see, Propositions 3.2 and 3.3 will be needed.

Theorem 4.1. *Let $\mathscr{C}$ be a cone in a Banach space and $K : \mathscr{C} \to \mathscr{C}$ a compact operator such that $K(0) = 0$. Assume that there exists $r > 0$, verifying*:

(a) *$u \neq tK(u)$ for all $||u|| = r$, $t \in [0,1]$.*

Assume also that there exist a compact homotopy $H : [0,1] \times \mathscr{C} \to \mathscr{C}$, and $R > r$ such that:

(b1) *$K(u) = H(0,u)$ for all $u \in \mathscr{C}$.*
(b2) *$H(t,u) \neq u$ for any $||u|| = R$, $t \in [0,1]$.*
(b3) *$H(1,u) \neq u$ for any $||u|| \leqslant R$.*

Let $D = \{u \in \mathscr{C} : r < ||u|| < R\}$. Then, K has a fixed point in D.

Proof. We only sketch the proof, since these type of results are well-known (see [8,13]). Denote by $i_{\mathscr{C}}$ the topological index in the cone $\mathscr{C}$, and $B_s = B(0,s) \cap \mathscr{C}$ for any $s>0$ (for a definition and the properties of the topological index in cones, see [7]).

First of all, hypothesis (b3) implies that $i_{\mathscr{C}}(H(1,\cdot), B_R) = 0$. By using the homotopy H (and (b1)–(b2)), we deduce that $i_{\mathscr{C}}(K, B_R)$ is also zero. However, $i_{\mathscr{C}}(K, B_r) = 1$ because of condition (a). Then, the excision property of the index yields that $i_{\mathscr{C}}(K, D) = -1$, and hence K must have a fixed point in D. □

Let us state some notations. We consider $C(\bar{\Omega})$ as a Banach space equipped with the uniform norm $||\cdot||$, and $C^{1,\tau}(\bar{\Omega})$ with the usual Hölder norm $||\cdot||_{C^{1,\tau}}$.

For each function $v \in C(\bar{\Omega})$, we denote by $T(v) \in C^{1,\tau}(\Omega)$ the unique weak solution of the problem:

$$\Delta_m T(v) + v = 0$$

with zero Dirichlet conditions in the boundary of Ω. It is well known that $T: C(\bar{\Omega}) \to C^{1,\tau}(\bar{\Omega})$ is a continuous operator, and maps bounded sets into bounded sets (see Lemma 1.1 in [4], for instance).

Define $N: C^{1,\tau}(\bar{\Omega}) \to C(\bar{\Omega})$, $N(u) = f(x, u, \nabla u)$. From the continuity of f and the compactness of the inclusion $C^{1,\tau}(\bar{\Omega}) \hookrightarrow C^1(\bar{\Omega})$, we deduce that N is compact.

Define also $K = T \circ N : C^{1,\tau}(\bar{\Omega}) \to C^{1,\tau}(\bar{\Omega})$, which is also compact.

We denote by $\mathscr{C} \subset C^{1,\tau}(\bar{\Omega})$ the subset of nonnegative functions. Clearly, $\mathscr{C}$ is a cone. Moreover, K maps $\mathscr{C}$ into $\mathscr{C}$ because of the maximum principle (in fact, if $u \in \mathscr{C} - \{0\}$, then $K(u)$ is strictly positive). We are interested in finding nontrivial fixed points of K in $\mathscr{C}$.

We finally state the main result of this paper:

Theorem 4.2. *Consider the problem*:

$$\begin{cases} \Delta_m u + f(x, u, \nabla u) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{19}$$

where Ω is a bounded C^2 domain in $\mathbb{R}^N$, $1<m<N$ and f is a nonnegative continuous function verifying [F]. *Then*, (19) *has at least one positive solution*.

Proof. We use Theorem 4.1. First, let us verify condition (a):

Take u in $\mathscr{C} - \{0\}$ such that $u = tK(u)$ for certain $t \in [0,1]$. We rewrite this equality as

$$\Delta_m u + t^{m-1} f(x, u, \nabla u) = 0 \tag{20}$$

with zero Dirichlet boundary conditions. We multiply by u and integrate to obtain

$$\begin{aligned}\int_\Omega |\nabla u|^m = t^{m-1}\int_\Omega f(x,u,\nabla u) \leqslant c_0 \int_\Omega u^{p+1} + M\int_\Omega |\nabla u|^\alpha u \\ \leqslant C\left(\int_\Omega |\nabla u|^m\right)^{(p+1)/m} + \left(\int_\Omega |\nabla u|^m\right)^{\alpha/m}\cdot\left(\int_\Omega u^{m/(m-\alpha)}\right)^{(m-\alpha)/m} \\ \leqslant C\left[\left(\int_\Omega |\nabla u|^m\right)^{(p+1)/m} + \left(\int_\Omega |\nabla u|^m\right)^{(\alpha+1)/m}\right].\end{aligned}$$

In the previous computations we have used Hölder and Poincaré inequalities, as well as condition [F]. Now, since $p+1>m$, $\alpha+1>m$, we deduce that there exists $c>0$ such that $\int_\Omega |\nabla u|^m > c$. Hence we can choose $r>0$ small enough such that $||u||_{C^{1,\tau}} \leqslant r \Rightarrow \int_\Omega |\nabla u|^m < c$; condition (a) is then proved.

We define $H:[0,1]\times\mathscr{C}\to\mathscr{C}$ as $H(t,u) = T[N(u)+t\lambda_0]$ where λ_0 is given by Proposition 3.2. Obviously, the image of H is contained in $\mathscr{C}$, again because of the maximum principle and $C^{1,\tau}$ estimates.

Condition (b1) of Theorem 4.1 is clearly verified. To prove (b2), observe that the equality $H(t,u)=u$ is equivalent to

$$\Delta_m u + f(x,u,\nabla u) + t\lambda_0 = 0 \tag{21}$$

with zero Dirichlet boundary conditions in Ω. In Proposition 3.3 we proved that the solutions of such problem are a priori bounded in their uniform norm. Making use again of the $C^{1,\tau}$ estimates [14], we deduce that we can find $R>0$ such that all solutions of (21) verify the inequality $||u||_{C^{1,\tau}} < R$. Condition (b2) is then verified.

Finally, Proposition 3.2 clearly implies that (b3) holds (in fact, for any $u\in\mathscr{C}$). The proof is complete. □

Remark. We can also use some continuation ideas in the spirit of [17] to prove the existence of a continuum of solutions. Specifically, define

$$\mathscr{S} = \{(\lambda,u)\in\mathbb{R}\times\mathscr{C}\colon\ u \text{ is solution of } (11)\}.$$

Taking condition (a) into account, we can argue exactly as in the Appendix of [4] to prove the existence of a connected set $\mathscr{A}\subset\mathscr{S}$ joining the trivial solution $(0,0)$ and $(0,u)$ where u is a positive solution for (1).

In the next theorem we make use of the sub-super solutions method to prove an Ambrosetti–Prodi type result (see [2]), under a certain additional condition on f.

Theorem 4.3. *Under the same hypotheses of Theorem* 4.2, *consider the boundary value problem*:

$$\begin{cases}\Delta_m u + f(x,u,\nabla u) + \lambda = 0, & x\in\Omega,\\ u(x) = 0, & x\in\partial\Omega,\end{cases} \tag{22}$$

where $\lambda \geqslant 0$. *As a consequence of Theorem* 4.2, *there exists* $\lambda^* > 0$ *such that* (22) *has a solution if and only if* $\lambda \leqslant \lambda^*$.

Assume also that f satisfies that, for each $R > 0$, *there exists* $\mu > 0$ *such that*:

$$f(x, u, \eta) + \mu u^{m-1} \text{ is increasing in } u, \ 0 \leqslant u \leqslant R$$

for any $x \in \Omega$, $\eta \in \mathbb{R}^N$. *Then, for any* $\lambda_0 \in [0, \lambda^*)$, *there exist at least two different solutions for* (22).

Proof. Take u^* a solution of problem (22) for $\lambda = \lambda^*$, $R = ||u^*||$, and fix $0 < \lambda_0 < \lambda^*$. Define $g(x, u, \eta) = \lambda_0 + f(x, u, \eta) + \mu u^{m-1}$ which is increasing in u, if $0 \leqslant u \leqslant R$. Clearly, the functions 0 and u^* are sub and super solutions (respectively) for the problem

$$\begin{cases} -\Delta_m u + \mu u^{m-1} = g(x, u, \nabla u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{23}$$

Define $Q_\mu : \mathscr{C} \to \mathscr{C}$ given by $Q_\mu(u) = w$, where w is the unique solution of the problem:

$$\begin{cases} -\Delta_m w + \mu w^{m-1} = g(x, u, \nabla u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

In order to use the sub–super solution method, we need to prove some monotonicity properties on Q_μ. In fact, we claim that if $u, v \in \mathscr{C}$, $u \leqslant v$, $||v|| \leqslant R$, then $Q_\mu(u) \leqslant Q_\mu(v)$.

Reasoning by contradiction, assume that there exist u, $v \in \mathscr{C}$, $||v|| \leqslant R$, $u \leqslant v$, such that the functions $w = Q_\mu(u)$, $z = Q_\mu(v)$ verify that $l = \max\{w - z\} > 0$. Take $\bar{R} > 0$ such that $|\nabla u(x)| < \bar{R}$, $|\nabla v(x)| < \bar{R}$ for all $x \in \bar{\Omega}$.

We choose $l/2 < k < l$ arbitrarily close to l; we shall use the test function $\phi = (w - z - k)^+$. Clearly, ϕ is zero outside the compact set $A_k = \{x \in \Omega: w(x) - z(x) \geqslant k\}$.

Take an arbitrary $\varepsilon > 0$, and choose an appropriate $\delta > 0$ such that $g(x, u, \eta) \leqslant g(x, v, \xi) + \varepsilon$ for all $(x, u, v, \eta, \xi) \in K$:

$$K = \{(x, u, v, \eta, \xi) \in A_{l/2} \times [0, R]^2 \times B_{\bar{R}}^2 \colon u \leqslant v, \ |\eta - \xi| \leqslant \delta\},$$

where $B_{\bar{R}}$ stands for the ball in $\mathbb{R}^N$ centered at zero and with radius $\bar{R}$.

Note that $\nabla z = \nabla w$ in A_l, and

$$A_l = \bigcap_{n=1}^{\infty} A_{l-1/n}.$$

So, we can choose $k = k(\delta)$ close enough to l such that $|\nabla z(x) - \nabla w(x)| < \delta$ for every $x \in A_k$.

We now use the test function ϕ in the weak equalities:

$$-\Delta_m w + \mu w^{m-1} = g(x, u, \nabla u), \tag{24}$$

$$-\Delta_m z + \mu z^{m-1} = g(x, v, \nabla v). \tag{25}$$

Integrating in Ω and subtracting, we obtain

$$\int_{A_k} [|\nabla w|^{m-2}\nabla w - |\nabla z|^{m-2}\nabla z] \cdot (\nabla w - \nabla z) + \mu \int_{A_k} [w^{m-1} - z^{m-1}]\phi$$
$$= \int_{A_k} [g(x, u, \nabla u) - g(x, v, \nabla v)]\phi.$$

The first summand in the right term is positive (in fact, more precise estimates are known, see for instance [20]). So, we can write:

$$\mu \int_{A_k} [w^{m-1} - z^{m-1}]\phi \leqslant \int_{A_k} [g(x, u, \nabla u) - g(x, v, \nabla v)]\phi \leqslant \varepsilon \int_{A_k} \phi.$$

Note that in A_k, $w - z \geqslant k > l/2$; so, it suffices to take ε small enough so that $\mu(w^{m-1} - z^{m-1}) > \varepsilon$, to get a contradiction. The claim is proved.

Note that we would have obtained also a contradiction if, instead of (24), (25), one assumes the inequalities:

$$-\Delta_m w + \mu w^{m-1} \leqslant g(x, u, \nabla u),$$

$$-\Delta_m z + \mu z^{m-1} \geqslant g(x, v, \nabla v).$$

So, we can follow a typical iterating procedure to assure the existence of a solution u_0 for (23), $0 \leqslant u_0 \leqslant u^*$ (actually, $u_0 > 0$ in Ω).

In order to prove the existence of another solution, we argue by contradiction, assuming that u_0 is the unique solution for (23). Since $\lambda^* > \lambda_0$, we can choose $\varepsilon > 0$ small such that the function $u^* + \varepsilon$ is also a super-solution for (23). By hypothesis, there are no solutions in the boundary of the set:

$$\mathscr{U} = \{u \in C\colon\ u(x) < u^*(x) + \varepsilon\ \forall x \in \Omega\}.$$

Then, the index $i_{\mathscr{C}}(Q_\mu, \mathscr{U})$ is well defined and, in fact, $i_{\mathscr{C}}(Q_\mu, \mathscr{U}) = 1$. To prove that, it suffices to use the homotopy:

$$H : [0, 1] \times \mathscr{C} \to \mathscr{C},\ \ H(t, u) = tQ_\mu(u)$$

and to take into account that $Q_\mu(\bar{\mathscr{U}}) \subset \bar{\mathscr{U}}$ (due to the monotonicity of Q_μ).

Since the fixed points of the operators Q_μ do not depend on μ, we can use the homotopy

$$[0, \mu] \times \mathscr{C} \ni (t, u) \mapsto Q_t(u)$$

to show that $i_{\mathscr{C}}(Q_0, \mathscr{U}) = 1$. But in Theorem 4.2 we proved that $i_{\mathscr{C}}(Q_0, B_R) = i_{\mathscr{C}}(K, B_R) = 0$, and it implies the existence of another solution which yields the desired contradiction. □

Open Problem. An interesting open problem is the case with higher exponents, specifically $m_* - 1 \leqslant p < m^* - 1$ where $m^* = \frac{Nm}{N-m}$ is the Sobolev critical exponent for the inclusion $W^{1,m}(\Omega) \hookrightarrow L^p(\Omega)$. In [19] Serrin and Zou proved Liouville type results for the problem:

$$\Delta_m u + u^p = 0 \quad \text{in } \mathbb{R}^N$$

for $p \in (m-1, m^* - 1)$. However, it seems difficult to obtain the a priori estimates in the same way we have done here.

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References

[1] W. Allegretto, Y.X. Huang, A Picone's identity for the p-Laplacian and applications, Nonlinear Anal. 32 (7) (1998) 819–830.
[2] A. Ambrosetti, G. Prodi, On the inversion of some differentiable mappings with singularities between Banach spaces, Ann. Mat Pura Appl. 4 (93) (1972) 231–246.
[3] D. Arcoya, J.L. Gámez, Bifurcation theory and related problems: anti-maximum principle and resonance, Comm. Partial Differential Equations 26 (2001) 1879–1911.
[4] C. Azizieh, P. Clément, A priori estimates and continuation methods for positive solutions of p-laplace equations, J. Diff. Eq. 179 (2002) 213–245.
[5] H. Brezis, R.E.L. Turner, On a class of superlinear elliptic problems, Comm. Partial Differential Equations 2 (1977) 601–614.
[6] L. Damascelli, F. Pascella, Monotonicity and symmetry of solutions of p-Laplace equations, $1 < p < 2$, via the moving plane method, Ann. Scuola. Norm. Pisa, Cl. Sci. 26 (4) (1998) 689–707.
[7] E.N. Dancer, Fixed point index calculations and applications, in: M. Maetzu, A. Vignoli (Eds.), Topological Nonlinear Analysis; Degree, Singularity and Variations, Birkhauser, Boston, 1995, pp. 303–340.
[8] D. de Figueiredo, P.L. Lions, R.D. Nussbaum, A priori estimates and existence of positive solutions of semilinear elliptic equations, J. Math. Pures Appl. 61 (1982) 41–63.
[9] D. de Figueiredo, J. Yang, A priori bounds for positive solutions of a non-variational elliptic system, Comm. Partial Differential Equations 26 (2001) 2305–2321.
[10] E. Di Benedetto, $C^{1,\alpha}$ local regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal. 7 (1983) 827–850.
[11] B. Gidas, J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math. 34 (1981) 525–598.

[12] B. Gidas, J. Spruck, A priori bounds for positive solutions of nonlinear elliptic equations, Comm. Partial Differential Equations 6 (1981) 883–901.
[13] M.A. Krasnoselskii, Fixed point of cone-compressing or cone-extending operators, Soviet. Math. Dokl. 1 (1960) 1285–1288.
[14] G.M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. 12 (1988) 1203–1219.
[15] E. Mitidieri, S.I. Pohozaev, The absence of global positive solutions to quasilinear elliptic inequalities, Dokl. Math. 57 (1998) 250–253;
E. Mitidieri, S.I. Pohozaev, The absence of global positive solutions to quasilinear elliptic inequalities, Dokl. Akad. Nauk. 359 (1998) 456–460.
[16] M. Montenegro, M. Montenegro, Existence and nonexistence of solutions for quasilinear elliptic equations, J. Math. Anal. Appl. 245 (2000) 303–316.
[17] P.H. Rabinowitz, Some aspects of nonlinear eigenvalue problems, Rocky Mountain J. Math. 3 (1973) 161–202.
[18] J. Serrin, Local behavior of solutions of quasilinear equations, Acta Math. 111 (1964) 247–302.
[19] J. Serrin, H. Zou, Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities, Acta Math. 189 (2002) 79–142.
[20] J. Simon, Régularité de la solution d'un problème aux limites non linèaires, Ann. Fac. Sc. Toulouse Math. 3 (6) (1981) 247–274.
[21] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, J. Diff. Eq. 51 (1984) 126–150.
[22] N. Trudinger, On Harnack type inequalities and their applications to quasilinear elliptic equations, Comm. Pure Appl. Math. 20 (1967) 721–747.
[23] J.L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optim. 12 (1984) 191–202.
[24] X.-J. Wang, Y.-B. Deng, Existence of multiple solutions to nonlinear elliptic equations of nondivergence form, J. Math. Anal. Appl. 189 (1995) 617–630.