# On denotational completeness extended abstract

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**Abstract**

The founding idea of linear logic is the duality between $A$ and $A^\perp$, with values in $\perp$. This idea is at work in the original denotational semantics of linear logic, coherent spaces, but also in the phase semantics of linear logic, where the « bilinear form » which induces the duality is nothing but the product in a monoid $\mathbb{M}$, $\perp$ being an arbitrary subset $\mathbb{B}$ of $\mathbb{M}$. The rather crude phase semantics has the advantage of being complete, and against all predictions, this kind of semantics had some applications. Coherent semantics is not complete for an obvious reason, namely that the coherent space $\Bbbk$ interpreting $\perp$ is too small (one point), hence the duality between $A$ and $A^\perp$ expressed by the cut-rule cannot be informative enough. But $\Bbbk$ is indeed the simplest case of a Par-monoid, i.e. the dual of a comonoid, and it is tempting to replace $\Bbbk$ with any commutative Par-monoid $\mathbb{P}$. Now we can replace coherent spaces with « free $\mathbb{P}$-modules over $\mathbb{P}$ », linear maps with « $\mathbb{P}$-linear maps », with the essential result that all usual constructions remain unchanged : technically speaking cliques are replaced with $\mathbb{P}$-cliques and that's it. The essential intuition behind $\mathbb{P}$ is that it accounts for arbitrary contexts : instead of dealing with $\Gamma, A$, one deals with $A$, but a clique of $\Gamma, A$ can be seen as a $\mathbb{P}$-clique in $A$. In particular all logical rules are now defined only on the main formulas of rules, as operations on $\mathbb{P}$-cliques. The duality between $A$ and $A^\perp$ yields a $\mathbb{P}$-clique in $\Bbbk$, i.e. a clique in $\mathbb{P}$ ; strangely enough, one must keep the phase layer, i.e. a monoid $\mathbb{M}$ (useful in the degenerated case), and the result of the duality is a $\mathbb{MP}$-clique. We specify an arbitrary set $\mathbb{B}$ of such cliques as the interpretation of $\perp$. Soundness and completeness are then easily established for closed $\Pi^1$-formulas, i.e. second-order propositional formulas without existential quantifiers. We must however find the equivalent of $1 \in \mathcal{F}$ (which is the condition for being a « provable fact ») : a $\mathbb{MP}$-clique is *essential* when it does not make use of $\mathbb{M}$ and $\mathbb{P}$, i.e. when it is induced by a clique in $A^\bullet$. We can now state the theorem :

*Let $A$ be a closed $\Pi^1$ formula, and let $a$ be a clique in the (usual) coherent interpretation $A^\bullet$ of $A$, which is the interpretation of a proof of $A$ ; then $a$ (as an essential clique), belongs to the « denotational fact » $A^\circ$ interpreting $A$ for all $\mathbb{M}$, $\mathbb{P}$ and $\mathbb{B}$. Conversely any essential clique with this property comes from a proof of $A$.*

### .1 Classical completeness

The traditional semantical idea is to interpret formulas by some kind of models, thus yielding

- *Soundness* results : a provable formula is true in any model.
- *Completeness* results : a formula true in any model is provable.

This result holds of course for classical logic ; it can be extended to other logical systems : for instance intuitionistic logic is sound and complete w.r.t. Kripke models (and also w.r.t. topological models).

### .2 Phase semantics

A similar result holds for linear logic which is sound and complete w.r.t. phase semantics, see in particular [1], 2.1. Phase semantics is based on a commutative monoid $\mathbb{M}$, together with a distinguished subset $\mathbb{B}$ of $\mathbb{M}$ ; a formula will receive « truth values » in $\mathbb{M}$, i.e. a formula $A$ will be interpreted as a subset $A^*$ of $\mathbb{M}$. The crucial notion is the duality between $A^*$ and $A^{\perp *}$ : given $m \in A^*$, $m' \in A^{\perp *}$, then $mm'$ should be a « truth value » for $\perp$, i.e. $mm' \in \mathbb{B} \quad (= \perp^*)$. One therefore defines *orthogonality* between objects of $\mathbb{M}$ by means of $m \perp m' \iff mm' \in \mathbb{B}$. In this respect the product behaves like a bilinear form ($\langle m, m' \rangle = mm'$) with value in $\mathbb{B}$, which plays the role of the « scalars ». The symmetry of linear logic (involutivity of negation) forces one to interpret any formula by a *fact*, i.e. by a subset $X$ of $\mathbb{M}$ equal to its biorthogonal ; among all facts, the set $\mathbb{B}$ interprets the constant $\perp$ and the « bilinear » form can be seen as the canonical map from $X, X^\perp$ to $\perp$. The interpretation of connectives consists in building new facts from existing ones, and in particular the Tensor product $F \otimes G$ is defined as $(F.G)^{\perp\perp}$ ; the product is therefore used twice, both for the multiplicative conjunction and the duality.

*Soundness*, i.e. the fact that $1 \in A^*$ for all provable $A$, is proved without problems ; however this is a non-trivial result, since nothing in the notion of commutative monoid $\mathbb{M}$ and arbitrary subset $\mathbb{B}$ makes any reference to the peculiar laws of linear logic. *Completeness* is proved in a more *ad hoc* way : it consists in exhibiting, among all possible pairs, $\mathbb{M}$ the commutative monoid of contexts (i.e. multisets of formulas [1]) and among all possible $\mathbb{B}$ the set of all provable contexts. As usual completeness is slightly frustrating, the only positive point about it being that this particular choice of phase model is particularly inconspicuous in the theorem : « $A$ is provable iff $1 \in A^*$ for any phase model $(M, \mathbb{B})$ » ; this stresses the fact that, if completeness is a desirable result, soundness should not be contrived.

By the way observe that completeness is by nature limited to a specific kind of formulas : first-order formulas, and more generally second-order formulas in which the positive second-order universal (resp. existential) occur only pos-

1. On page 24 of [1] a footnote is missing after « (i.e. multisets of formulas », namely « We ignore the multiplicities of formulas $?A$, so that $I$ is the set of contexts $?\Gamma$. »

itively (resp. negatively) : such formulas are called $\Pi^1$ (their negations are called $\Sigma^1$). Completeness fails for non-$\Pi^1$-formulas [2], hence its denotational extension, which implies usual completeness will be limited to $\Pi^1$-formulas.

### .3 *Categorical completeness*

However these extensions are not completely satisfactory, since we are dealing with constructive systems, for which the notion of proof is central : soundness and completeness only refer to the weaker notion of provability, i.e. w.r.t. models which can distinguish between two formulas, but not between two proofs of the same formula.
However there is a *semantics of proofs* whose general mathematical expression is categorical semantics : a proof of an implication $A \Rightarrow B$ is a morphism from the interpretation $A^*$ of $A$ to $B^*$. Categorical models of intuitionistic and linear logic associate different interpretations to distinct proofs of the same formula ; but to which extent are they complete ? In other terms, given a morphism from $A^*$ to $B^*$ is it the interpretation of a proof of the implication $A \Rightarrow B$ ? Up to now there is no satisfactory solution. Of course it is possible to give the abstract definition of an intuitionistic category (e.g. a CCC, i.e. a Closed Cartesian Category) and to prove some forms of completeness w.r.t. such categories, but it is easy to argue that a CCC is nothing but another presentation of intuitionistic logic, so what ? For the same reason one should reject, as contrived, any linear categorical completeness based upon « linear categories », i.e. upon the categorical axiomatization of linear logic.

### .4 *Denotational completeness*

We shall therefore limit ourselves to concrete categories, and we shall definitely work with *coherent spaces*, the original semantics of linear logic [3]. A general exposition of coherent semantics can be found in [1], see 2.2., from which we borrow the terminology and notations.
Starting with an assignment of coherent spaces to atomic formulas, one can associate a coherent space $A^*$ to any formula $A$, and a clique $\pi^* \sqsubset A^*$ to any proof $\pi$ of $A$. This is obviously the starting point for a soundness theorem, expressing that the rules of linear logic can be interpreted as operations on cliques of coherent spaces. But there is no obvious completeness counterpart, i.e. a result that would basically say that every clique in $A^*$ is of the form $\pi^*$ for some proof $\pi$ of $A$ :

- The empty set is always a clique in $A^*$, whereas the interpretation of a proof is usually nonempty.

2. This is one of the possible readings of Gödel's incompleteness, since the Gödel sentence $G$ can be written $\forall x(\mathbb{N}(x) \Rightarrow F(x))$, with $F(x)$ a first-order and $\mathbb{N}(x)$ (which expresses that $x$ is an integer) a second-order $\Pi^1$-formula : $G$, which is $\Sigma^1$ is true in any model (in fact : true) without being provable.
3. Intuitionistic logic can be seen as a subsystem of linear logic, hence what we are doing applies also to intuitionistic logic.

- All constructions are usually infinite, but recursive in the parameters ; hence non-recursive cliques are not the interpretation of any proof.

In order to fix this failure, one must modify something in the interpretation, e.g. replace coherent spaces with something else, require some additional properties of the cliques etc. But this is a non-trivial endeavor ; in particular most modifications will accept the following extra principles :

- The *mix*-rule, namely the principle $A \otimes B \multimap A \parr B$.
- The identification between the two multiplicative neutrals $1$ and $\perp$, both interpreted by a space $\mathbb{k}$ with one point.
- The identification between the two additive neutrals $0$ and $\top$, both interpreted by an empty space.

### .5 *The denotational duality*

The only reasonable idea is to build a duality between $X$ and $X^\perp$ : there is a canonical bilinear map from $X \otimes X^\perp$ into $\mathbb{k}$, where $\mathbb{k}$ is the unit coherent space interpreting the constant $\perp$ ; concretely, if $a \sqsubset X$ and $b \sqsubset X^\perp$, then the clique $\langle a, b\rangle$ (which has at most one point) is the singleton $\mathbb{k}$ when $a \cap b \neq \emptyset$, and $\emptyset$ otherwise. The idea would be to select a set $\mathbb{B}$ of cliques in $\mathbb{k}$, and to define $a \perp b \iff \langle a, b\rangle \in \mathbb{B}$ when $a \sqsubset X$ and $b \sqsubset X^\perp$ ; a formula would therefore be interpreted by a *denotational fact*, i.e. a set of cliques in $X$ equal to its biorthogonal.
The idea is not too bad, but it eventually fails for want of suitable $\mathbb{B}$ (only four possible choices). For instance, if $\mathbb{B}$ is empty, a denotational fact will either be empty or consist of all cliques in $X$ ; on the other hand, if $\mathbb{B}$ is non-empty, we must accept the elements of $\mathbb{B}$ as the interpretation of proofs of $\perp$, and more generally that both $A$ and $A^\perp$ might have proofs, which goes against completeness… unless we admit that cliques of $A^*$ which are accepted will eventually be refused when completeness is at stake. So among the elements of a denotational fact it is necessary to distinguish between two classes of citizens, the higher kind, *essential* cliques being the subclass to which completeness applies… but there is no immediate way to make such a distinction.

### .6 *Expanding the category*

The solution comes from a close examination of the completeness argument w.r.t. phase semantics : one introduces the monoid of provable contexts. But since we are replacing « provability » with « proofs », one should instead consider the set $\mathbb{B}$ of proofs of arbitrary contexts $\Gamma$ (or rather their denotational interpretation). Indeed one can build a gigantic coherent space $\mathbb{P}$, a kind of « infinite $\parr$ » of all coherent spaces $A^*$. As to its structure, $\mathbb{P}$ is a kind of monoid, exactly a Par-monoid, i.e. it is equipped with a « Par-multiplication » : $\mathbb{P} \parr \mathbb{P} \multimap \mathbb{P}$ and a « Par-neutral » : $\mathbb{k} \multimap \mathbb{P}$ [4]. $\mathbb{B}$ can be seen as a set of cliques in $\mathbb{P}$. Now the basic idea is to replace plain proofs of $A$ (seen

4. The typical Par-monoids are spaces $?X$.

as cliques in $A^*$) with proofs of $\Gamma, A$ (seen as cliques in $\mathbb{P} \parr A^*$). The duality between $X$ and $X^{*\perp}$ becomes a duality between $\mathbb{P} \parr X$ and $\mathbb{P} \parr X^\perp$ : given a clique $a \sqsubset \mathbb{P} \parr X$, a clique $b \sqsubset \mathbb{P} \parr X^\perp$, the interpretation of the cut-rule yields a clique $c \sqsubset \mathbb{P} \parr \mathbb{P}$, which can be mapped by « Par-multiplication » to a clique $\langle a, b\rangle \sqsubset \mathbb{P}$. The basic orthogonality is therefore $\langle a, b\rangle \in \mathbb{B}$, and one can elaborate the semantics on this basis (i.e. a denotational fact is set of cliques in $\mathbb{P} \parr X$ equal to its biorthogonal, etc.). Since $\mathbb{B}$ is far from being empty, a denotational fact will hardly be empty, hence not all inhabitants should compete when completeness is at stake ; but if we restrict to inhabitants that are induced by a clique in $X$ by means of the « Par-identity », then we obtain completeness ; indeed those cliques correspond to proofs with empty contexts. But remember that completeness should not be achieved at the price of a contrived soundness ; fortunately, we can forget our particular $\mathbb{P}$ and $\mathbb{B}$ and observe that the interpretation works without any hypothesis on them, just as soundness w.r.t. phase semantics works for arbitrary $\mathbb{B}$. It remains to give a status to our use of a Par-monoid and the answer is extremely simple : all usual notions of linearity are replaced with $\mathbb{P}$-linearity, the familiar case being nothing more than the case $\mathbb{P} = \mathbb{k}$. This is clearly analogous to the replacement of commutative groups with $R$-modules, the ground case being the case $R = \mathbb{Z}$.

The fact that certain proofs have empty interpretations forces one to slightly complicate this very simple pattern : an additional commutative monoid $\mathbb{M}$ (which only matters in the case of empty cliques) must also be introduced. This copes with the degenerated cases of coherent semantics, i.e. empty cliques, in which the denotational information is absent, which forces one to deal with « truth values ». The modification induced by the auxiliary monoid to coherent semantics is modest, almost invisible, and our redaction tries to forget about it ; but it is a natural modification, involving a notion of $\mathbb{M}$-linearity with very satisfactory properties.

## .7 *What has been achieved ?*

This is always a delicate question, when we speak about completeness. For instance the first reaction of Yves Lafont in September 86 to phase semantics was something like « abstract nonsense », whereas later developments (including recent works by Lafont) suggest a less severe judgement. For the same reason, one should not be too harsh against the use of abstract monoids and abstract Par-monoids : eventually some application of this abstract nonsense will be found. Moreover, conceptually speaking, the individuation of a structure of « module » over a monoid and/or a Par-monoid induces an additional dimension in denotational semantics, which was obviously missing.

## BIBLIOGRAPHY

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