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The Schur complement of strictly doubly diagonally dominant matrices and its application ☆

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ABSTRACT

It is known that the Schur complements of doubly diagonally dominant matrices are doubly diagonally dominant. In this paper, we obtain an estimate for the doubly diagonally dominant degree on the Schur complement of strictly doubly diagonally dominant matrices. Then, as an application we obtain that the eigenvalues of the Schur complements are located in the Brauer Ovals of Cassini of the original matrices under certain conditions. As another application, we obtain an upper bound for the infinity norm on the inverse on the Schur complement of strictly doubly diagonally dominant matrices. Further, based on the derived results, we give a kind of iteration called the Schur-based iteration, which can solve large scale linear systems though reducing the order by the Schur complement and can compute out the results faster.

1. Definition and introduction

To begin with, we first recall some notations and definitions. Let $\mathbb{C}^{m\times n}$ ($\mathbb{R}^{m\times n}$) denote the set of all $m \times n$ complex (real) matrices, $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ ($n \geqslant 2$), $N = \{1, 2, \dots, n\}$ and $\alpha \subseteq N$. We write $|A| = (|a_{ij}|)$ and

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$$P_i(A) = \sum_{j\in N, j\neq i} |a_{ij}|, \quad i = 1, 2, \ldots, n.$$

The infinity norm of A is defined as

$$\|A\|_\infty = \max_{1\leqslant i\leqslant n} \{P_i(A) + |a_{ii}|\}.$$

The comparison matrix of A, which is denoted by $\mu(A) = (t_{ij})$, is defined to be

$$t_{ij} = \begin{cases} |a_{ij}|, & \textit{if } i = j; \\ -|a_{ij}|, & \textit{if } i \neq j. \end{cases}$$

Recall that A is a (row) diagonally dominant matrix (D_n) if for all $i \in N$,

$$|a_{ii}| \geqslant P_i(A). \tag{1.1}$$

A is further said to be a strictly diagonally dominant matrix (SD_n) if all the strict inequalities in (1.1) hold. A is a doubly diagonally dominant matrix (DD_n) if for all $i, j \in N$, $i \neq j$,

$$|a_{ii}||a_{jj}| \geqslant P_i(A)P_j(A). \tag{1.2}$$

A is said to be a strictly doubly diagonally dominant matrix (SDD_n) if all the inequalities in (1.2) hold. We call $|a_{ii}| - P_i(A)$ $(i \in N)$ the D_n degree of A with respect to the ith row and $|a_{ii}||a_{jj}| - P_i(A)P_j(A)$ $(i, j \in N,\ i \neq j)$ the DD_n degree of A with respect to the ith row and the jth row. For simplicity, we call the D_n degree and the DD_n degree of A.

A is called an M-matrix if it can be written in the form of $A = mI - P$, where P is a nonnegative matrix, $m > \rho(P)$ and $\rho(P)$ is the spectral radius of P. A matrix A is an H-matrix if $\mu(A)$ is an M-matrix. We denote by $\mathbb{H}_n$ and $\mathbb{M}_n$ the sets of $n \times n$ H-matrices and M-matrices respectively.

It is well known that SD_n and SDD_n are nonsingular(by the *Gerschgorin Circle Theorem* and the *Brauer Ovals of Cassini Theorem*). In addition, if $A \in SDD_n$, there exists at most one index $i_0 \in N$ such that

$$|a_{i_0 i_0}| \leqslant P_{i_0}(A). \tag{1.3}$$

For nonempty index sets $\alpha, \beta \subseteq N$, we denote by $|\alpha|$ the cardinality of α and write $A(\alpha, \beta)$ to mean the submatrix of $A \in \mathbb{C}^{n\times n}$ lying in the rows indexed by α and the columns indexed by β. $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. Supposing $\alpha = \{i_1, i_2, \ldots, i_k\} \subset N$, $\alpha' = N - \alpha = \{j_1, j_2, \ldots, j_l\}$ and the elements of α and α' are both conventionally arranged in increasing order, we denote

$$A_s = A(\alpha \cup \{j_s\}), \quad \text{for } 1 \leqslant s \leqslant l.$$

Further, if $A(\alpha)$ is nonsingular, we define the Schur complement of A with respect to $A(\alpha)$, which is denoted by $A/A(\alpha)$ or simply A/α, to be

$$A(\alpha') - A(\alpha', \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha'). \tag{1.4}$$

The theory of Schur complement plays an important role in many fields such as control theory and computational mathematics. A lot of work have been done on it. It is known that the Schur complements of positive semidefinite matrices are positive semidefinite; similar properties hold for M-matrix, H-matrix, and the inverse of M-matrix (see, e.g. [4]). Carlson and Markham showed that the Schur complements of SDD_n are SDD_n (see [5]). Li and Tsatsomeros [16] and Ikramov [17] independently proved that the Schur complements of DD_n are DD_n. Liu and Huang obtained that the Schur complements of generalized DD_n are generalized DD_n (see [6]). First it was Smith [14] and then other researchers followed and obtained some upper and lower bounds for eigenvalues, singular values and determinants of Schur complement (see, e.g. [7-15]). These properties have been repeatedly used for the convergence of iterations in numerical analysis and for deriving matrix inequalities in matrix analysis (see, e.g. [1, p. 508], [2, p. 58] or [9]).

Meanwhile, investigating the distribution for the eigenvalues of the Schur complement is of great significance as well as estimating the upper bound for the infinity norm of the inverse of the Schur complement. As is known to all, for a non-homogeneous system of linear equation $Ax = b$, the upper bound for $\|A^{-1}\|_\infty$ plays an important role in some iterations for the original system (see, e.g. [20,21]). Furthermore, for $Ax = b$, after reducing order, the system matrix of the derived smaller system is the Schur complement of the original system matrix (see [12,13]). Therefore, the upper bound for the infinity norm of the Schur complement of A is also useful in the iteration.

However, when dealing with practical problems such as the α-stability constraint, which plays an important role in control theory (see, e.g. [18,19]), we just need to know the strip distribution for the real part of the eigenvalues instead of the location for the eigenvalues. So sometimes it is useful and simple to estimate the strip distribution for the real part of the eigenvalues.

By the *Gerschgorin Circles Theorem* and the *Brauer Ovals of Cassini Theorem*, if we know the D_n degree or the DD_n degree of a matrix, we can obtain bounds for the eigenvalues of the matrix correspondingly. Liu and Zhang got some results showing that the D_n degree for the Schur complement of D_n is greater than that of the original matrix (see [8]). Unfortunately, there is no definite relationship between the DD_n degree of the Schur complements on SDD_n and that of the original matrices. This is illustrated by the following example.

Example 1. Let

$$A = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Take $\alpha_1 = \{3\}, \alpha_2 = \{2\}$. Then $A \in SDD_3$,

$$A/\alpha_1 = \begin{pmatrix} 2.5 & -1.5 \\ -1.5 & 2.5 \end{pmatrix}, \quad A/\alpha_2 = \begin{pmatrix} \frac{8}{3} & -\frac{4}{3} \\ -\frac{4}{3} & \frac{5}{3} \end{pmatrix}.$$

By computation,

$$\begin{aligned} &(A/\alpha_1)_{11}(A/\alpha_1)_{22} - (A/\alpha_1)_{12}(A/\alpha_1)_{21} = 6.25 - 2.25 = 4 \\ &< |a_{11}||a_{22}| - (|a_{12}| + |a_{13}|)(|a_{21}| + |a_{23}|) = 9 - 4 = 5; \\ &(A/\alpha_2)_{11}(A/\alpha_2)_{22} - (A/\alpha_2)_{12}(A/\alpha_2)_{21} = \frac{40}{9} - \frac{16}{9} = \frac{8}{3} \\ &> |a_{11}||a_{33}| - (|a_{12}| + |a_{13}|)(|a_{31}| + |a_{32}|) = 6 - 4 = 2. \end{aligned}$$

Thus we see that the DD_n degree of the Schur complement on SDD_n is not necessarily greater or smaller than that of the original matrices.

In this paper, we first obtain an estimate for the DD_n degree on the Schur complement of SDD_n. Then, as an application we obtain that the eigenvalues of the Schur complements are located in the Brauer Ovals of Cassini of the original matrices under certain conditions. As another application, for SDD_n, we obtain an upper bound for $\|A^{-1}\|_\infty$ and extend the result to $\|(A/\alpha)^{-1}\|_\infty$. Further, based on the derived results, we give a kind of iteration called the Schur-based iteration, which can solve large scale linear systems though reducing the order by the Schur complement and can compute out the results faster.

2. The DD_n degree of A/α

Li and Tsatsomeros obtained that the Schur complement of DD_n is DD_n (see [16]). Here we give an estimate for the DD_n degree on the Schur complement of SDD_n, which extends their results. For this, we recall a few results.

Lemma 2.1 (See [3, p. 117, 131]). *If A is an H-matrix, then*

$$[\mu(A)]^{-1} \geqslant |A^{-1}|.$$

Lemma 2.2 (See [3, p. 114]). *If $A \in SD_n$ or SDD_n, then $\mu(A) \in \mathbb{M}_n$, i.e., $A \in \mathbb{H}_n$.*

Lemma 2.3 (See [16, Theorem 2.1]). *If $A \in SD_n$ or SDD_n and α is a proper subset of N, then A/α is in $SD_{|\alpha'|}$ or $SDD_{|\alpha'|}$, where $|\alpha'|$ is the cardinality of α'.*

Lemma 2.4 (See [8, Theorem 2]). *Let $A \in SDD_n$ and let i_0 satisfy (1.3). Then for any index set $\alpha = \{i_1, i_2, \ldots, i_k\}$ containing i_0, $\alpha' = N - \alpha = \{j_1, j_2, \ldots, j_l\}$ and $A/\alpha = (a'_{st})$,*

$$|a'_{tt}| - P_t(A/\alpha) \geqslant |a_{j_tj_t}| - \frac{P_{i_0}(A)}{|a_{i_0i_0}|}P_{j_t}(A) > 0, \quad \text{for } t = 1, 2, \ldots, l. \tag{2.1}$$

Lemma 2.5 (See [24, p. 135]). *If $A \in \mathbb{M}_n$, then* $\det(A) > 0$.

Now we are ready to give the main result of this section.

Theorem 2.1. *Let $A \in SDD_n$, $\alpha = \{i_1, i_2, \ldots, i_k\} \subset N$, $\alpha' = N - \alpha = \{j_1, j_2, \ldots, j_l\}$. Denote $A/\alpha = (a'_{st})$. Then*

$$\begin{aligned}&(i)|a'_{tt}||a'_{ss}| - P_t(A/\alpha)P_s(A/\alpha)\\ &\geqslant \left[|a_{j_tj_t}| - \max_{v\in N-\{j_t\}} \frac{P_v(A)}{|a_{vv}|}P_{j_t}(A)\right]\left[|a_{j_sj_s}| - \max_{i_w\in\alpha} \frac{P_{i_w}(A)}{|a_{i_wi_w}|}P_{j_s}(A)\right] \equiv \omega_{st},\end{aligned} \tag{2.2}$$

$$\begin{aligned}&(ii)|a'_{tt}||a'_{ss}| + P_t(A/\alpha)P_s(A/\alpha)\\ &\leqslant \left[|a_{j_tj_t}| + \max_{v\in N-\{j_t\}} \frac{P_v(A)}{|a_{vv}|}P_{j_t}(A)\right]\left[|a_{j_sj_s}| + \max_{i_w\in\alpha} \frac{P_{i_w}(A)}{|a_{i_wi_w}|}P_{j_s}(A)\right] \equiv \widetilde{\omega}_{st}.\end{aligned} \tag{2.3}$$

Proof. (i) *(a)* If there exists $i_0 \in \alpha \subset N$ satisfying (1.3), then for each $j_t \in \alpha'$,

$$\max_{i_w\in\alpha} \frac{P_{i_w}(A)}{|a_{i_wi_w}|} = \max_{v\in N-\{j_t\}} \frac{P_v(A)}{|a_{vv}|} = \frac{P_{i_0}(A)}{|a_{i_0i_0}|}.$$

From Lemma 2.4 we know that for $t = 1, 2, \ldots, l$,

$$|a'_{tt}| - P_t(A/\alpha) \geqslant |a_{j_tj_t}| - \frac{P_{i_0}(A)}{|a_{i_0i_0}|}P_{j_t}(A) > 0.$$

Therefore, for each $t, s = 1, 2, \ldots, l$, $t \neq s$,

$$\left[|a'_{tt}| - P_t(A/\alpha)\right]\left[|a'_{ss}| - P_s(A/\alpha)\right] > 0.$$

By Lemma 2.3, A/α is in $SDD_{|\alpha'|}$, i.e., for each $t, s = 1, 2, \ldots, l$, $t \neq s$,

$$|a'_{tt}||a'_{ss}| - P_t(A/\alpha)P_s(A/\alpha) > 0.$$

Hence, for each $t, s = 1, 2, \ldots, l$, $t \neq s$,

$$\begin{aligned}|a'_{tt}||a'_{ss}| - P_t(A/\alpha)P_s(A/\alpha) &\geqslant \left[|a'_{tt}| - P_t(A/\alpha)\right]\left[|a'_{ss}| - P_s(A/\alpha)\right]\\ &\geqslant \left[|a_{j_tj_t}| - \max_{v\in N-\{j_t\}} \frac{P_v(A)}{|a_{vv}|}P_{j_t}(A)\right]\left[|a_{j_sj_s}| - \max_{i_w\in\alpha} \frac{P_{i_w}(A)}{|a_{i_wi_w}|}P_{j_s}(A)\right].\end{aligned}$$

Thus we get (2.2).

(b) If there does not exist any $i_0 \in \alpha \subset N$ satisfying (1.3), then by Lemmas 2.1 and 2.2, for each $t, s = 1, 2, \ldots, l, t \neq s$,

$$
\begin{aligned}
&|a'_{ss}||a'_{tt}| - P_s(A/\alpha)P_t(A/\alpha)\\
&= \left| a_{j_sj_s} - (a_{j_si_1}, \ldots, a_{j_si_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1j_s} \\ \vdots \\ a_{i_kj_s} \end{pmatrix} \right| \left| a_{j_tj_t} - (a_{j_ti_1}, \ldots, a_{j_ti_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t} \end{pmatrix} \right|\\
&\quad - \left[\sum_{\substack{u=1\\u\neq s}}^{l} \left| a_{j_sj_u} - (a_{j_si_1}, \ldots, a_{j_si_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1j_u} \\ \vdots \\ a_{i_kj_u} \end{pmatrix} \right| \right]\\
&\quad \times \left[\sum_{\substack{v=1\\v\neq t}}^{l} \left| a_{j_tj_v} - (a_{j_ti_1}, \ldots, a_{j_ti_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1j_v} \\ \vdots \\ a_{i_kj_v} \end{pmatrix} \right| \right]\\
&\geqslant \left[|a_{j_sj_s}| - (|a_{j_si_1}|, \ldots, |a_{j_si_k}|)\{\mu[A(\alpha)]\}^{-1} \begin{pmatrix} |a_{i_1j_s}| \\ \vdots \\ |a_{i_kj_s}| \end{pmatrix} \right]\\
&\quad \times \left[|a_{j_tj_t}| - (|a_{j_ti_1}|, \ldots, |a_{j_ti_k}|)\{\mu[A(\alpha)]\}^{-1} \begin{pmatrix} |a_{i_1j_t}| \\ \vdots \\ |a_{i_kj_t}| \end{pmatrix} \right]\\
&\quad - \left\{ \sum_{\substack{u=1\\u\neq s}}^{l} \left[|a_{j_sj_u}| + (|a_{j_si_1}|, \ldots, |a_{j_si_k}|)\{\mu[A(\alpha)]\}^{-1} \begin{pmatrix} |a_{i_1j_u}| \\ \vdots \\ |a_{i_kj_u}| \end{pmatrix} \right] \right\}\\
&\quad \times \left\{ \sum_{\substack{v=1\\v\neq t}}^{l} \left[|a_{j_tj_v}| + (|a_{j_ti_1}|, \ldots, |a_{j_ti_k}|)\{\mu[A(\alpha)]\}^{-1} \begin{pmatrix} |a_{i_1j_v}| \\ \vdots \\ |a_{i_kj_v}| \end{pmatrix} \right] \right\}\\
&\overset{def.}{=} \overline{B}\\
&= \left[|a_{j_sj_s}| - (|a_{j_si_1}|, \ldots, |a_{j_si_k}|)\{\mu[A(\alpha)]\}^{-1} \begin{pmatrix} |a_{i_1j_s}| \\ \vdots \\ |a_{i_kj_s}| \end{pmatrix} \right]
\end{aligned}
$$

$$
\begin{aligned}
&\times \Bigg[|a_{j_tj_t}| - \max_{v\in N-\{j_t\}} \frac{P_v(A)}{|a_{vv}|} P_{j_t}(A) + \max_{v\in N-\{j_t\}} \frac{P_v(A)}{|a_{vv}|} P_{j_t}(A) \\
&-(|a_{j_ti_1}|,\ldots,|a_{j_ti_k}|)\{\mu[A(\alpha)]\}^{-1}\begin{pmatrix}|a_{i_1j_t}|\\ \vdots \\ |a_{i_kj_t}|\end{pmatrix}\Bigg] \\
&-\left\{\sum_{\substack{u=1\\u\neq s}}^{l}\left[|a_{j_sj_u}|+(|a_{j_si_1}|,\ldots,|a_{j_si_k}|)\{\mu[A(\alpha)]\}^{-1}\begin{pmatrix}|a_{i_1j_u}|\\ \vdots \\ |a_{i_kj_u}|\end{pmatrix}\right]\right\} \\
&\times\left\{\sum_{\substack{v=1\\v\neq t}}^{l}\left[|a_{j_tj_v}|+(|a_{j_ti_1}|,\ldots,|a_{j_ti_k}|)\{\mu[A(\alpha)]\}^{-1}\begin{pmatrix}|a_{i_1j_v}|\\ \vdots \\ |a_{i_kj_v}|\end{pmatrix}\right]\right\} \\
&=\left[|a_{j_sj_s}|-(|a_{j_si_1}|,\ldots,|a_{j_si_k}|)\{\mu[A(\alpha)]\}^{-1}\begin{pmatrix}|a_{i_1j_s}|\\ \vdots \\ |a_{i_kj_s}|\end{pmatrix}\right]\left[|a_{j_tj_t}|-\max_{v\in N-\{j_t\}}\frac{P_v(A)}{|a_{vv}|}P_{j_t}(A)\right] \\
&+\left[|a_{j_sj_s}|-(|a_{j_si_1}|,\ldots,|a_{j_si_k}|)\{\mu[A(\alpha)]\}^{-1}\begin{pmatrix}|a_{i_1j_s}|\\ \vdots \\ |a_{i_kj_s}|\end{pmatrix}\right] \\
&\times\left[\max_{v\in N-\{j_t\}}\frac{P_v(A)}{|a_{vv}|}P_{j_t}(A)-(|a_{j_ti_1}|,\ldots,|a_{j_ti_k}|)\{\mu[A(\alpha)]\}^{-1}\begin{pmatrix}|a_{i_1j_t}|\\ \vdots \\ |a_{i_kj_t}|\end{pmatrix}\right] \\
&-\left\{\sum_{\substack{u=1\\u\neq s}}^{l}\left[|a_{j_sj_u}|+(|a_{j_si_1}|,\ldots,|a_{j_si_k}|)\{\mu[A(\alpha)]\}^{-1}\begin{pmatrix}|a_{i_1j_u}|\\ \vdots \\ |a_{i_kj_u}|\end{pmatrix}\right]\right\} \\
&\times\left\{\sum_{\substack{v=1\\v\neq t}}^{l}\left[|a_{j_tj_v}|+(|a_{j_ti_1}|,\ldots,|a_{j_ti_k}|)\{\mu[A(\alpha)]\}^{-1}\begin{pmatrix}|a_{i_1j_v}|\\ \vdots \\ |a_{i_kj_v}|\end{pmatrix}\right]\right\}. \qquad (2.4)
\end{aligned}
$$

On the other hand,

$$|a_{j_sj_s}| - (|a_{j_si_1}|, \ldots, |a_{j_si_k}|)\{\mu[A(\alpha)]\}^{-1} \begin{pmatrix} |a_{i_1j_s}| \\ \vdots \\ |a_{i_kj_s}| \end{pmatrix}$$

$$= |a_{j_sj_s}| - \max_{i_w \in \alpha} \frac{P_{i_w}(A)}{|a_{i_wi_w}|} \sum_{t=1}^{k} |a_{j_si_t}|$$

$$+ \max_{i_w \in \alpha} \frac{P_{i_w}(A)}{|a_{i_wi_w}|} \sum_{t=1}^{k} |a_{j_si_t}| - (|a_{j_si_1}|, \ldots, |a_{j_si_k}|)\{\mu[A(\alpha)]\}^{-1} \begin{pmatrix} |a_{i_1j_s}| \\ \vdots \\ |a_{i_kj_s}| \end{pmatrix}$$

$$= |a_{j_sj_s}| - \max_{i_w \in \alpha} \frac{P_{i_w}(A)}{|a_{i_wi_w}|} \sum_{t=1}^{k} |a_{j_si_t}|$$

$$+ \frac{1}{\det \mu[A(\alpha)]} \cdot \det \begin{pmatrix} \max_{i_w \in \alpha} \frac{P_{i_w}(A)}{|a_{i_wi_w}|} \sum_{t=1}^{k} |a_{j_si_t}| & -|a_{j_si_1}| \;\ldots\; -|a_{j_si_k}| \\ -|a_{i_1j_s}| & \\ \vdots & \mu[A(\alpha)] \\ -|a_{i_kj_s}| & \end{pmatrix}$$

$$\stackrel{def.}{=} |a_{j_sj_s}| - \max_{i_w \in \alpha} \frac{P_{i_w}(A)}{|a_{i_wi_w}|} \sum_{t=1}^{k} |a_{j_si_t}| + \frac{1}{\det \mu[A(\alpha)]} \cdot \det B_1. \tag{2.5}$$

In B_1, for any $r = 1, 2, \ldots, k$, we have

$$|a_{i_ri_r}| \max_{i_w \in \alpha} \frac{P_{i_w}(A)}{|a_{i_wi_w}|} \sum_{t=1}^{k} |a_{j_si_t}| \geqslant |a_{i_ri_r}| \frac{P_{i_r}(A)}{|a_{i_ri_r}|} \sum_{t=1}^{k} |a_{j_si_t}|$$

$$= P_{i_r}(A) \sum_{t=1}^{k} |a_{j_si_t}| = \left(\sum_{u=1}^{l} |a_{i_rj_u}| + \sum_{\substack{v=1 \\ v \neq r}}^{k} |a_{i_ri_v}| \right) \sum_{t=1}^{k} |a_{j_si_t}|$$

$$\geqslant \left(|a_{i_rj_s}| + \sum_{\substack{v=1 \\ v \neq r}}^{k} |a_{i_ri_v}| \right) \sum_{t=1}^{k} |a_{j_si_t}|; \tag{2.6}$$

and for any $q, r = 1, 2, \ldots, k$, we have

$$|a_{i_qi_q}||a_{i_ri_r}| > P_{i_q}(A)P_{i_r}(A)$$

$$= \left(\sum_{u=1}^{l} |a_{i_qj_u}| + \sum_{\substack{v=1 \\ v \neq q}}^{k} |a_{i_qi_v}| \right) \left(\sum_{u=1}^{l} |a_{i_rj_u}| + \sum_{\substack{v=1 \\ v \neq r}}^{k} |a_{i_ri_v}| \right)$$

$$\geqslant \left(|a_{i_qj_s}| + \sum_{\substack{v=1 \\ v \neq q}}^{k} |a_{i_qi_v}| \right) (|a_{i_rj_s}| + \sum_{\substack{v=1 \\ v \neq r}}^{k} |a_{i_ri_v}|). \tag{2.7}$$

Therefore $B_1 \in SDD_{|\alpha|+1}$, by Lemma 2.2, then $\mu(B_1) \in \mathbb{M}_n$. Obviously, $B_1 = \mu(B_1)$. Hence, $B_1 \in \mathbb{M}_n$. By Lemma 2.5, $\det B_1 > 0$. Thus from (2.5) we have

$$
\begin{aligned}
&|a_{j_sj_s}| - (|a_{j_si_1}|, \ldots, |a_{j_si_k}|)\{\mu[A(\alpha)]\}^{-1}\begin{pmatrix} |a_{i_1j_s}| \\ \vdots \\ |a_{i_kj_s}| \end{pmatrix} \\
&\geqslant |a_{j_sj_s}| - \max_{i_w\in\alpha}\frac{P_{i_w}(A)}{|a_{i_wi_w}|}\sum_{t=1}^{k}|a_{j_si_t}| \\
&\geqslant |a_{j_sj_s}| - \max_{i_w\in\alpha}\frac{P_{i_w}(A)}{|a_{i_wi_w}|}P_{j_s}(A) > 0. \qquad (2.8)
\end{aligned}
$$

So

$$
\overline{B} = \det\begin{bmatrix}
|a_{j_sj_s}| & -\sum\limits_{\substack{u=1\\u\neq s}}^{l}|a_{j_sj_u}| & -|a_{j_si_1}| \cdots -|a_{j_si_k}| & 0 \cdots 0 \\
-\sum\limits_{\substack{v=1\\v\neq t}}^{l}|a_{j_tj_v}| & \max\limits_{v\in N-\{j_t\}}\frac{P_v(A)}{|a_{vv}|}P_{j_t}(A) & 0 \ldots 0 & -|a_{j_ti_1}| \cdots -|a_{j_ti_k}| \\
-|a_{i_1j_s}| & -\sum\limits_{\substack{u=1\\u\neq s}}^{l}|a_{i_1j_u}| & & \\
\vdots & \vdots & \mu[A(\alpha)] & 0 \\
-|a_{i_kj_s}| & -\sum\limits_{\substack{u=1\\u\neq s}}^{l}|a_{i_kj_u}| & & \\
-\sum\limits_{\substack{v=1\\v\neq t}}^{l}|a_{i_1j_v}| & -|a_{i_1j_t}| & & \\
\vdots & \vdots & 0 & \mu[A(\alpha)] \\
-\sum\limits_{\substack{v=1\\v\neq t}}^{l}|a_{i_kj_v}| & -|a_{i_kj_t}| & &
\end{bmatrix}
$$

$$
\times\frac{1}{\det\begin{pmatrix}\mu[A(\alpha)] & 0 \\ 0 & \mu[A(\alpha)]\end{pmatrix}} \stackrel{def.}{=} \frac{1}{\{\det\mu[A(\alpha)]\}^2}\det\hat{B}. \qquad (2.9)
$$

It is easy to show that $\hat{B} \in SDD_{2|\alpha|+2}$ by the fact $A \in SDD_n$ and the assumption *(b)*. By Lemma 2.2, then $\mu(\hat{B}) \in \mathbb{M}_n$. Obviously, $\hat{B} = \mu(\hat{B})$. Hence, $\hat{B} \in \mathbb{M}_n$. By Lemma 2.3, thus $\det\hat{B} > 0$.

Therefore, from (2.4), (2.8) and (2.9), we obtain (2.2).

With a similar method we can prove (ii). □

3. Distribution of the eigenvalues

By the famous *Gerschgorin Circles Theorem* or *Brauer Ovals of Cassini Theorem*, the eigenvalues of Schur complement could be estimated after calculating out the Schur complement. In this section, as application we present some locations for eigenvalues of the Schur complements of SDD_n by the elements of the original matrix.

Lemma 3.1 (Brauer Ovals of Cassini Theorem)**.** *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. Then the eigenvalues of A are in the union of the following sets:*

$$U_{ij} = \{z \in \mathbb{C} \mid |z - a_{ii}||z - a_{jj}| \leqslant P_i(A)P_j(A)\}, \quad i, j = 1, 2, \ldots, n,\ i \neq j.$$

Theorem 3.1. *Let $A \in SDD_n$, $\alpha = \{i_1, i_2, \ldots, i_k\} \subset N$, $\alpha' = N - \alpha = \{j_1, j_2, \ldots, j_l\}$. Set $A/\alpha = (a'_{st})$. Then for every eigenvalue λ of A/α, there exist $1 \leqslant s, t \leqslant l$, $s \neq t$ such that*

$$\text{(i)} \left|\lambda - \frac{\det A_t}{\det A(\alpha)}\right|\left|\lambda - \frac{\det A_s}{\det A(\alpha)}\right| \leqslant 2\left[|a_{j_tj_t}| \max_{i_w\in\alpha} \frac{P_{i_w}(A)}{|a_{i_wi_w}|}P_{j_s}(A) + |a_{j_sj_s}| \max_{v\in N-\{j_t\}} \frac{P_v(A)}{|a_{vv}|}P_{j_t}(A)\right]; \tag{3.1}$$

$$\text{(ii)} \left|\lambda - \frac{\det A_t}{\det A(\alpha)}\right|\left|\lambda - \frac{\det A_s}{\det A(\alpha)}\right| \leqslant \left[|a_{j_tj_t}| + \max_{v\in N-\{j_t\}} \frac{P_v(A)}{|a_{vv}|}P_{j_t}(A)\right]\left[|a_{j_sj_s}| + \max_{i_w\in\alpha} \frac{P_{i_w}(A)}{|a_{i_wi_w}|}P_{j_s}(A)\right]. \tag{3.2}$$

Proof. By the Brauer Ovals of Cassini Theorem, there exist $1 \leqslant s, t \leqslant l$, $s \neq t$ such that

$$|\lambda - a'_{tt}||\lambda - a'_{ss}| \leqslant P_t(A/\alpha)P_s(A/\alpha). \tag{3.3}$$

(i) From (2.2) we have

$$\begin{aligned}
&P_t(A/\alpha)P_s(A/\alpha)\\
&\leqslant |a'_{tt}||a'_{ss}| - \left[|a_{j_tj_t}| - \max_{v\in N-\{j_t\}} \frac{P_v(A)}{|a_{vv}|}P_{j_t}(A)\right]\left[|a_{j_sj_s}| - \max_{i_w\in\alpha} \frac{P_{i_w}(A)}{|a_{i_wi_w}|}P_{j_s}(A)\right]\\
&= \left|a_{j_sj_s} - (a_{j_si_1}, \ldots, a_{j_si_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_s}\\ \vdots \\ a_{i_kj_s}\end{pmatrix}\right|\left|a_{j_tj_t} - (a_{j_ti_1}, \ldots, a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_t}\\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right|\\
&\quad - \left[|a_{j_tj_t}| - \max_{v\in N-\{j_t\}} \frac{P_v(A)}{|a_{vv}|}P_{j_t}(A)\right]\left[|a_{j_sj_s}| - \max_{i_w\in\alpha} \frac{P_{i_w}(A)}{|a_{i_wi_w}|}P_{j_s}(A)\right]\\
&\leqslant \left[|a_{j_sj_s}| + (|a_{j_si_1}|, \ldots, |a_{j_si_k}|)\{\mu[A(\alpha)]\}^{-1}\begin{pmatrix} |a_{i_1j_s}|\\ \vdots \\ |a_{i_kj_s}|\end{pmatrix}\right]\\
&\quad \left[|a_{j_tj_t}| + (|a_{j_ti_1}|, \ldots, |a_{j_ti_k}|)\{\mu[A(\alpha)]\}^{-1}\begin{pmatrix} |a_{i_1j_t}|\\ \vdots \\ |a_{i_kj_t}|\end{pmatrix}\right]
\end{aligned}$$

$$
\begin{aligned}
&-\left[|a_{j_tj_t}|-\max_{v\in N-\{j_t\}}\frac{P_v(A)}{|a_{vv}|}P_{j_t}(A)\right]\left[|a_{j_sj_s}|-\max_{i_w\in\alpha}\frac{P_{i_w}(A)}{|a_{i_wi_w}|}P_{j_s}(A)\right]\\
&\leqslant\left[|a_{j_tj_t}|+\max_{v\in N-\{j_t\}}\frac{P_v(A)}{|a_{vv}|}P_{j_t}(A)\right]\left[|a_{j_sj_s}|+\max_{i_w\in\alpha}\frac{P_{i_w}(A)}{|a_{i_wi_w}|}P_{j_s}(A)\right]\\
&-\left[|a_{j_tj_t}|-\max_{v\in N-\{j_t\}}\frac{P_v(A)}{|a_{vv}|}P_{j_t}(A)\right]\left[|a_{j_sj_s}|-\max_{i_w\in\alpha}\frac{P_{i_w}(A)}{|a_{i_wi_w}|}P_{j_s}(A)\right]\\
&=2\left[|a_{j_tj_t}|\max_{i_w\in\alpha}\frac{P_{i_w}(A)}{|a_{i_wi_w}|}P_{j_s}(A)+|a_{j_sj_s}|\max_{v\in N-\{j_t\}}\frac{P_v(A)}{|a_{vv}|}P_{j_t}(A)\right]. \qquad (3.4)
\end{aligned}
$$

Additionally, for any $s \in N - \alpha$,

$$
\begin{aligned}
|\lambda - a'_{ss}| &= \left|\lambda - a_{j_sj_s} + (a_{j_si_1}, \ldots, a_{j_si_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_s} \\ \vdots \\ a_{i_kj_s}\end{pmatrix}\right| \\
&= |\lambda - \det(A_s/A(\alpha))| = \left|\lambda - \frac{\det A_s}{\det A(\alpha)}\right|.
\end{aligned}
$$

Similarly,

$$
|\lambda - a'_{tt}| = \left|\lambda - \frac{\det A_t}{\det A(\alpha)}\right|.
$$

Hence, from (3.3) and (3.4), we get (3.1).
(ii) Similarly, we can obtain (3.2) from (2.3) and (3.3). □

4. Upper bound for $\|(A/\alpha)^{-1}\|_\infty$

Varah (see [20]) obtained the following upper bound for the infinity norm of an H-matrix A

$$
\|A^{-1}\|_\infty \leqslant \max_{i\in N}\frac{1}{|a_{ii}| - P_i(A)}. \qquad (4.1)
$$

In this section, we give a new upper bound for $\|A^{-1}\|_\infty$ which improves (4.1) and extend the result to $\|(A/\alpha)^{-1}\|_\infty$ by employing Theorem 2.1.

Theorem 4.1. *Let $A = (a_{ij}) \in SDD_n$, $M = (m_{ij}) \in \mathbb{C}^{n\times n}$. Then*

$$
\|A^{-1}M\|_\infty \leqslant \max_{i,j\in N, i\neq j}\frac{|a_{jj}|\sum_{t=1}^n |m_{it}| + P_i(A)\sum_{t=1}^n |m_{jt}|}{|a_{ii}||a_{jj}| - P_i(A)P_j(A)}. \qquad (4.2)
$$

Particularly,

$$
\|A^{-1}\|_\infty \leqslant \max_{i,j\in N, i\neq j}\frac{|a_{jj}| + P_i(A)}{|a_{ii}||a_{jj}| - P_i(A)P_j(A)}. \qquad (4.3)
$$

Proof. Since $A \in SDD_n$, from Lemma 2.1 and Lemma 2.2 we have $[\mu(A)]^{-1} \geqslant |A^{-1}|$. Denote

$$(x_1, \ldots, x_n)^\top = |A^{-1}M|\,(1, \ldots, 1)^\top,$$
$$\left(x'_1, \ldots, x'_n\right)^\top = [\mu(A)]^{-1}|M|\,(1, \ldots, 1)^\top,$$
$$x'_{i_0} = \max_{1\leqslant i\leqslant n} x'_i, \quad x'_{j_0} = \max_{1\leqslant i\leqslant n, i\neq i_0} x'_i.$$

Then we have

$$\begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix} = [\mu(A)]^{-1}|M| \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \geqslant |A^{-1}||M| \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \geqslant |A^{-1}M| \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

and

$$[\mu(A)] \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix} = |M| \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} |a_{11}|x'_1 - \sum\limits_{t\neq 1} |a_{1t}|x'_t \\ \vdots \\ |a_{nn}|x'_n - \sum\limits_{t\neq n} |a_{nt}|x'_t \end{pmatrix} = \begin{pmatrix} \sum\limits_{t=1}^{n} |m_{1t}| \\ \vdots \\ \sum\limits_{t=1}^{n} |m_{nt}| \end{pmatrix}. \tag{4.4}$$

Considering the i_0th row and the j_0th row of (4.4)

$$\sum_{t=1}^{n} |m_{i_0t}| = |a_{i_0i_0}|x'_{i_0} - \sum_{t\neq i_0} |a_{i_0t}|x'_t \geqslant |a_{i_0i_0}|x'_{i_0} - \sum_{t\neq i_0} |a_{i_0t}|x'_{j_0},$$
$$\sum_{t=1}^{n} |m_{j_0t}| = |a_{j_0j_0}|x'_{j_0} - \sum_{t\neq j_0} |a_{j_0t}|x'_t \geqslant |a_{j_0j_0}|x'_{j_0} - \sum_{t\neq j_0} |a_{j_0t}|x'_{i_0},$$

we can easily get

$$|a_{j_0j_0}| \sum_{t=1}^{n} |m_{i_0t}| + \sum_{t\neq i_0} |a_{i_0t}| \sum_{t=1}^{n} |m_{j_0t}| \geqslant x'_{i_0} \left(|a_{i_0i_0}||a_{j_0j_0}| - \sum_{t\neq i_0} |a_{i_0t}| \sum_{t\neq j_0} |a_{j_0t}| \right).$$

So

$$\begin{aligned} \|A^{-1}M\|_\infty = \max_{1\leqslant i\leqslant n} x_i &\leqslant x'_{i_0} \\ &\leqslant \frac{|a_{j_0j_0}| \sum_{t=1}^{n} |m_{i_0t}| + \sum_{t\neq i_0} |a_{i_0t}| \sum_{t=1}^{n} |m_{j_0t}|}{|a_{i_0i_0}||a_{j_0j_0}| - \sum_{t\neq i_0} |a_{i_0t}| \sum_{t\neq j_0} |a_{j_0t}|} \\ &\leqslant \max_{i,j\in N, i\neq j} \frac{|a_{jj}| \sum_{t=1}^{n} |m_{it}| + \sum_{t\neq i} |a_{it}| \sum_{t=1}^{n} |m_{jt}|}{|a_{ii}||a_{jj}| - \sum_{t\neq i} |a_{it}| \sum_{t\neq j} |a_{jt}|}. \end{aligned}$$

Thus we obtain (4.2).

Particularly, when $M = I = diag(1, 1, \ldots, 1)$, we obtain (4.3). □

Remark. (4.3) improves (4.1), which can be demonstrated by the following computations:

$$\begin{aligned}
&\frac{|a_{jj}|+P_i(A)}{|a_{ii}||a_{jj}|-P_i(A)P_j(A)}\\
&=\frac{|a_{jj}|+P_i(A)}{|a_{ii}||a_{jj}|-|a_{jj}|P_i(A)+|a_{jj}|P_i(A)-P_i(A)P_j(A)}\\
&=\frac{|a_{jj}|+P_i(A)}{|a_{jj}|(|a_{ii}|-P_i(A))+P_i(A)(|a_{jj}|-P_j(A))}\\
&\leqslant\frac{|a_{jj}|+P_i(A)}{\min\limits_{i\in N}\{|a_{ii}|-P_i(A)\}(|a_{jj}|+P_i(A))}\\
&=\frac{1}{\min\limits_{i\in N}\{|a_{ii}|-P_i(A)\}}=\max_{i\in N}\frac{1}{|a_{ii}|-P_i(A)}.
\end{aligned}$$

Theorem 4.2. *Let $A=(a_{ij})\in SD_n$, $\alpha=\{i_1,i_2,\ldots,i_k\}\subset N$, $\alpha'=N-\alpha=\{j_1,j_2,\ldots,j_l\}$ and $M=(m_{ij})\in\mathbb{C}^{l\times l}$. Then*

$$\begin{aligned}
&\|(A/\alpha)^{-1}M\|_\infty\\
&\leqslant\max_{\substack{1\leqslant s,t\leqslant l\\ s\neq t}}\frac{\left(|a_{j_tj_t}|+\max_{i_w\in\alpha}\frac{P_{i_w}(A)}{|a_{i_wi_w}|}P_{j_t}(A)\right)\sum_{u=1}^{l}|m_{su}|+\left(\max_{i_w\in\alpha}\frac{P_{i_w}(A)}{|a_{i_wi_w}|}P_{j_s}(A)+P_{j_s}(A)\right)\sum_{u=1}^{l}|m_{tu}|}{\left[|a_{j_tj_t}|-\max_{v\in N-\{j_t\}}\frac{P_v(A)}{|a_{vv}|}P_{j_t}(A)\right]\left[|a_{j_sj_s}|-\max_{i_w\in\alpha}\frac{P_{i_w}(A)}{|a_{i_wi_w}|}P_{j_s}(A)\right]}.
\end{aligned}\tag{4.5}$$

Particularly,

$$\|(A/\alpha)^{-1}\|_\infty\leqslant\max_{\substack{1\leqslant s,t\leqslant l\\ s\neq t}}\frac{|a_{j_tj_t}|+P_{j_s}(A)+\max_{i_w\in\alpha}\frac{P_{i_w}(A)}{|a_{i_wi_w}|}\left[P_{j_t}(A)+P_{j_s}(A)\right]}{\left[|a_{j_tj_t}|-\max_{v\in N-\{j_t\}}\frac{P_v(A)}{|a_{vv}|}P_{j_t}(A)\right]\left[|a_{j_sj_s}|-\max_{i_w\in\alpha}\frac{P_{i_w}(A)}{|a_{i_wi_w}|}P_{j_s}(A)\right]}.\tag{4.6}$$

Proof. Denote $A/\alpha=(a'_{st})$. From Theorem 4.1 we have

$$\|(A/\alpha)^{-1}M\|_\infty\leqslant\max_{\substack{1\leqslant s,t\leqslant l\\ s\neq t}}\frac{|a'_{tt}|\sum_{u=1}^{l}|m_{su}|+P_s(A/\alpha)\sum_{u=1}^{l}|m_{tu}|}{|a'_{ss}||a'_{tt}|-P_s(A/\alpha)P_t(A/\alpha)}.\tag{4.7}$$

Similar to (2.8),

$$\begin{aligned}
|a'_{tt}|&=\left|a_{j_tj_t}-(a_{j_ti_1},\ldots,a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix}a_{i_1j_t}\\ \vdots\\ a_{i_kj_t}\end{pmatrix}\right|\\
&\leqslant|a_{j_tj_t}|+(|a_{j_ti_1}|,\ldots,|a_{j_ti_k}|)\{\mu[A(\alpha)]\}^{-1}\begin{pmatrix}|a_{i_1j_t}|\\ \vdots\\ |a_{i_kj_t}|\end{pmatrix}\\
&\leqslant|a_{j_tj_t}|+\max_{i_w\in\alpha}\frac{P_{i_w}(A)}{|a_{i_wi_w}|}P_{j_t}(A).
\end{aligned}\tag{4.8}$$

Further, by Theorem 1 of [8],

$$P_s(A/\alpha) \leqslant |a'_{ss}| - |a_{j_sj_s}| + P_{j_s}(A) \leqslant \max_{i_w\in\alpha} \frac{P_{i_w}(A)}{|a_{i_wi_w}|} P_{j_s}(A) + P_{j_s}(A). \tag{4.9}$$

Since $A \in SD_n$, it is obvious that $A \in SDD_n$. Therefore, from Theorem 2.1,

$$\begin{aligned} &|a'_{tt}||a'_{ss}| - P_s(A/\alpha)P_t(A/\alpha) \\ &\geqslant \left[|a_{j_tj_t}| - \max_{v\in N-\{j_t\}} \frac{P_v(A)}{|a_{vv}|} P_{j_t}(A)\right]\left[|a_{j_sj_s}| - \max_{i_w\in\alpha} \frac{P_{i_w}(A)}{|a_{i_wi_w}|} P_{j_s}(A)\right]. \end{aligned} \tag{4.10}$$

From (4.7), (4.8), (4.9) and (4.10) we obtain (4.5).

In particular, taking $M = I = diag(1, 1, \ldots, 1)$, we obtain (4.6). $\square$

5. Numerical examples

In this section, we firstly give a numerical example to estimate the bounds for the eigenvalues on the Schur complements of SDD_n. Then, we present another numerical example to illustrate that the iteration can compute out the results faster in reducing the order of large matrix while the methods in [12,13] could not work.

Example 2. Let

$$A = \begin{pmatrix} 1.3 & 0.2 & 0.3 & 0.4 & 0.5 \\ 0.2 & 2 & 0.4 & 0.5 & 0.1 \\ 0.3 & 0.4 & 2 & 0.1 & 0.2 \\ 0.4 & 0.5 & 0.1 & 3 & 0.3 \\ 0.5 & 0.1 & 0.2 & 0.3 & 3 \end{pmatrix}, \quad \alpha = \{1, 2\}.$$

Then $N - \alpha = \{3, 4, 5\}$, $\det A(\alpha) = 2.56$, $A/A(\alpha) = \begin{pmatrix} 1.8672 & -0.0711 & 0.0805 \\ -0.0711 & 2.7793 & 0.1410 \\ 0.0805 & 0.1410 & 2.8074 \end{pmatrix}$,

$$P_1(A) = 1.4, \quad P_2(A) = 1.2, \quad P_3(A) = 1, \quad P_4(A) = 1.3, \quad P_5(A) = 1.1;$$

$$\det A_3 \equiv \det \begin{pmatrix} 1.3 & 0.2 & 0.3 \\ 0.2 & 2 & 0.4 \\ 0.3 & 0.4 & 2 \end{pmatrix} = 4.78; \quad \det A_4 \equiv \det \begin{pmatrix} 1.3 & 0.2 & 0.4 \\ 0.2 & 2 & 0.5 \\ 0.4 & 0.5 & 3 \end{pmatrix} = 7.115;$$

$$\det A_5 \equiv \det \begin{pmatrix} 1.3 & 0.2 & 0.5 \\ 0.2 & 2 & 0.1 \\ 0.5 & 0.1 & 3 \end{pmatrix} = 7.187; \quad \frac{P_1(A)}{|a_{11}|} = \max_{v\in N} \frac{P_v(A)}{|a_{vv}|} = \max_{i_w\in\alpha} \frac{P_{i_w}(A)}{|a_{i_wi_w}|} = \frac{14}{13}.$$

According to Theorem 3.1, by computation we know that the eigenvalue z of A/α satisfies

$$z \in U_1 = \{z \mid |z-1.87||z-2.78| \leqslant 12.06\} \cup \{|z-1.87||z-2.81| \leqslant 11.20\} \cup \{|z-2.78||z-2.81| \leqslant 15.51\}.$$

According to Theorem 4.2,

$$\|(A/\alpha)^{-1}\|_\infty \leqslant 4.39.$$

Example 3. Consider a system of linear equation $Ax = b$, where

$$A = \begin{pmatrix} M & B \\ C & D \end{pmatrix}, \quad b = \begin{pmatrix} 3 & 3 & \cdots & 3 \end{pmatrix}^T_{1\times 100},$$

$$M = \begin{pmatrix} 70 & -40 & & & \\ -40 & 130 & -40 & & \\ & \ddots & \ddots & \ddots & \\ & & -40 & 130 & -40 \\ & & & -40 & 130 \end{pmatrix}_{50\times 50},$$

$$B = C^T = \begin{pmatrix} 0 & 0 & \cdots & 0 & 40 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ -40 & 0 & \cdots & 0 & 0 \end{pmatrix}_{50\times 50},$$

$$D = \begin{pmatrix} 51\times 130 & -400 & & & & \\ -400 & 52\times 130 & -400 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -400 & 98\times 130 & -400 & \\ & & & -400 & 99\times 130 & -400 \\ & & & & -400 & 1300000 \end{pmatrix}_{50\times 50}.$$

Obviously, the first row of A is not D_n. Hence we could not directly use the methods in [12,13] to reduce the order.

It is evident that A is real symmetric and SDD_n with positive diagonal entries. By Lemma 2.1, A/D are also real symmetric and SDD_n with positive diagonal entries. Thus, A, D and A/D are positive definite (see [22, p. 23]). Consequently, we can convert the original system into the following systems by using the Schur complement

Table 1
Results of Example 3($\varepsilon = 10^{-6}$).

Computer condition : Pentium(R) 4 CPU 3.2 GHz, extended memory 512 M

	CGM	SCGM		CGM	SCGM
x_5	0.060917	0.060299	x_{60}	0.000442	0.000429
x_{10}	0.059857	0.060002	x_{65}	0.000384	0.000392
x_{15}	0.059857	0.060000	x_{70}	0.000333	0.000361
x_{20}	0.059857	0.060000	x_{75}	0.000321	0.000336
x_{25}	0.059857	0.060000	x_{80}	0.000329	0.000313
x_{30}	0.059857	0.060000	x_{85}	0.000318	0.000292
x_{35}	0.059857	0.060000	x_{90}	0.000333	0.000275
x_{40}	0.059857	0.060001	x_{95}	0.000261	0.000260
x_{45}	0.059835	0.059902	x_{100}	−0.000000	−0.000000
x_{50}	0.039804	0.039601	cputime	0.031250 s	0.015625 s
x_{55}	0.000458	0.000472	$\|Mx - b\|_2$	1.436645	0.023226

$$A/Dy = f, \tag{5.1}$$

$$Dz = g - Cy, \tag{5.2}$$

where

$$y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{50} \end{pmatrix}, \quad f = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{50} \end{pmatrix} - BD^{-1} \begin{pmatrix} b_{51} \\ b_{52} \\ \vdots \\ b_{100} \end{pmatrix}, \quad z = \begin{pmatrix} x_{51} \\ x_{52} \\ \vdots \\ x_{100} \end{pmatrix}, \quad g = \begin{pmatrix} b_{51} \\ b_{52} \\ \vdots \\ b_{100} \end{pmatrix}.$$

Then we can first solve (5.1) and then (5.2) by the conjugate gradient method. We call this method the Schur-based conjugate gradient method.

For any given accuracy, applying the Schur-based conjugate gradient method, we could perform approximate calculation to satisfy required accuracy. In this example, if we choose the accuracy as $\varepsilon = 10^{-6}$, the results of computation are given out in Table 1.

As A/D, D and A are all nonsingular, the rank of A is greater than that of A/D and D. On the other hand, we know from Theorem 3.1 that the eigenvalues of A/D and D are more concentrated than those of A. So we predict that the Schur-based conjugate gradient method will compute faster than the ordinary conjugate gradient method (see, e.g. [23, p. 312–317]).

In fact, solving the original system by the conjugate gradient method needs 37 iteration steps and it takes 0.031250 s cputime to compute out x; solving (5.1) and (5.2) by the conjugate gradient method needs 13 and 11 iteration steps respectively and it takes 0.015625 s total cputime to compute out x.

From Table 1 we see that the Schur-based conjugate gradient method (SCGM) is much better than the ordinary conjugate gradient method (CGM).

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