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Hyponormality and subnormality of block Toeplitz operators

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Abstract

In this paper, we are concerned with hyponormality and subnormality of block Toeplitz operators acting on the vector-valued Hardy space $H^2_{\mathbb{C}^n}$ of the unit circle.

First, we establish a tractable and explicit criterion on the hyponormality of block Toeplitz operators having bounded type symbols via the triangularization theorem for compressions of the shift operator.

Second, we consider the gap between hyponormality and subnormality for block Toeplitz operators. This is closely related to Halmos's Problem 5: Is every subnormal Toeplitz operator either normal or analytic? We show that if Φ is a matrix-valued rational function whose co-analytic part has a coprime factorization then every hyponormal Toeplitz operator T_Φ whose square is also hyponormal must be either normal or analytic.

Third, using the subnormal theory of block Toeplitz operators, we give an answer to the following "Toeplitz completion" problem: find the unspecified Toeplitz entries of the partial block Toeplitz matrix

$$A := \begin{bmatrix} U^* & ? \\ ? & U^* \end{bmatrix}$$

so that A becomes subnormal, where U is the unilateral shift on H^2.

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1. Introduction

Toeplitz operators, block Toeplitz operators and (block) Toeplitz determinants (i.e., determinants of sections of (block) Toeplitz operators) arise naturally in several fields of mathematics and in a variety of problems in physics, especially, in quantum mechanics. For example, the spectral theory of Toeplitz operators plays an important role in the study of solvable models in quantum mechanics [25] and in the study of the one-dimensional Heisenberg Hamiltonian of ferromagnetism [24]; the theory of block Toeplitz determinants is used in the study of the classical dimer model [5] and in the study of the vicious walker model [41]; the theory of block Toeplitz operators is also used in the study of Gelfand–Dickey Hierarchies (cf. [9]). On the other hand, the theory of hyponormal and subnormal operators is an extensive and highly developed area, which has made important contributions to a number of problems in functional analysis, operator theory, and mathematical physics (see, for example, [40,49,57] for applications to related mathematical physics problems). Thus, it becomes of central significance to describe in detail hyponormality and subnormality for Toeplitz operators. This paper focuses primarily on hyponormality and subnormality of block Toeplitz operators with rational symbols. For the general theory of subnormal and hyponormal operators, we refer to [10,51].

To describe our results, we first need to review a few essential facts about (block) Toeplitz operators, and for that we will use [6,26,27,30,52,54,55]. Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{B}(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, and write $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H},\mathcal{H})$. For $A, B \in \mathcal{B}(\mathcal{H})$, we let $[A,B] := AB - BA$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be normal if $[T^*,T] = 0$, hyponormal if $[T^*,T] \geq 0$, and subnormal if T has a normal extension, i.e., $T = N|_{\mathcal{H}}$, where N is a normal operator on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $\mathcal{H}$ is invariant for N. For an operator $T \in \mathcal{B}(\mathcal{H})$, we write $\ker T$ and $\operatorname{ran} T$ for the kernel and the range of T, respectively. For a set $\mathcal{M}$, $\operatorname{cl}\mathcal{M}$ and $\mathcal{M}^\perp$ denote the closure and the orthogonal complement of $\mathcal{M}$, respectively. Also, let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the unit circle. Recall that the Hilbert space $L^2 \equiv L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and that the Hardy space $H^2 \equiv H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \ldots\}$. An element $f \in L^2$ is said to be analytic if $f \in H^2$. Let $H^\infty \equiv H^\infty(\mathbb{T}) := L^\infty \cap H^2$, i.e., H^∞ is the set of bounded analytic functions on the open unit disk $\mathbb{D}$.

Given a bounded measurable function $\varphi \in L^\infty$, the *Toeplitz operator* T_φ and the *Hankel operator* H_φ with *symbol* φ on H^2 are defined by

$$T_\varphi g := P(\varphi g) \quad \text{and} \quad H_\varphi g := JP^\perp(\varphi g) \quad (g \in H^2), \tag{1.1}$$

where P and $P^\perp$ denote the orthogonal projections that map from L^2 onto H^2 and $(H^2)^\perp$, respectively, and J denotes the unitary operator from L^2 onto L^2 defined by $J(f)(z) = \overline{z} f(\overline{z})$ for $f \in L^2$.

To study hyponormality (resp. normality and subnormality) of the Toeplitz operator T_φ with symbol φ we may, without loss of generality, assume that $\varphi(0) = 0$; this is because hyponormality (resp. normality and subnormality) is invariant under translations by scalars.

Normal Toeplitz operators were characterized by a property of their symbols in the early 1960's by Brown and Halmos [8] and the exact nature of the relationship between the symbol $\varphi \in L^\infty$ and the hyponormality of T_φ was understood via Cowen's Theorem [14] in 1988.

Cowen's Theorem (*[14,53]*). *For each $\varphi \in L^\infty$, let*

$$\mathcal{E}(\varphi) \equiv \{k \in H^\infty : \|k\|_\infty \le 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty\}.$$

Then a Toeplitz operator T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

This elegant and useful theorem has been used in the works [21,22,28,31,32,35,42–46,50,53,59], which have been devoted to the study of hyponormality for Toeplitz operators on H^2. Particular attention has been paid to Toeplitz operators with polynomial symbols or rational symbols [45,46]. However, the case of arbitrary symbol φ, though solved in principle by Cowen's Theorem, is in practice very complicated. Indeed, it may not even be possible to find tractable necessary and sufficient condition for the hyponormality of T_φ in terms of the Fourier coefficients of the symbol φ unless certain assumptions are made about φ. To date, tractable criteria for the cases of trigonometric polynomial symbols and rational symbols were derived from a Carathéodory–Schur interpolation problem [59] and a tangential Hermite–Fejér interpolation problem [31] or the classical Hermite–Fejér interpolation problem [45], respectively.

Recall that a function $\varphi \in L^\infty$ is said to be of *bounded type* (or in the Nevanlinna class) if there are analytic functions $\psi_1, \psi_2 \in H^\infty(\mathbb{D})$ such that $\varphi = \psi_1/\psi_2$ almost everywhere on $\mathbb{T}$. To date, no tractable criterion to determine the hyponormality of T_φ when the symbol φ is of bounded type has been found.

We now introduce the notion of block Toeplitz operators. Let $M_{n\times r}$ denote the set of all $n \times r$ complex matrices and write $M_n := M_{n\times n}$. For $\mathcal{X}$ a Hilbert space, let $L^2_{\mathcal{X}} \equiv L^2_{\mathcal{X}}(\mathbb{T})$ be the Hilbert space of $\mathcal{X}$-valued norm square-integrable measurable functions on $\mathbb{T}$ and let $H^2_{\mathcal{X}} \equiv H^2_{\mathcal{X}}(\mathbb{T})$ be the corresponding Hardy space. We observe that $L^2_{\mathbb{C}^n} = L^2 \otimes \mathbb{C}^n$ and $H^2_{\mathbb{C}^n} = H^2 \otimes \mathbb{C}^n$. If Φ is a matrix-valued function in $L^\infty_{M_n} \equiv L^\infty_{M_n}(\mathbb{T})$ $(=L^\infty \otimes M_n)$ then $T_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ denotes the *block Toeplitz operator* with *symbol* Φ defined by

$$T_\Phi f := P_n(\Phi f) \quad \text{for } f \in H^2_{\mathbb{C}^n},$$

where P_n is the orthogonal projection of $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$. A *block Hankel operator* with *symbol* $\Phi \in L^\infty_{M_n}$ is the operator $H_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ defined by

$$H_\Phi f := J_n P_n^\perp(\Phi f) \quad \text{for } f \in H^2_{\mathbb{C}^n},$$

where J_n denotes the unitary operator from $L^2_{\mathbb{C}^n}$ onto $L^2_{\mathbb{C}^n}$ given by $J_n(f)(z) := \overline{z} I_n f(\overline{z})$ for $f \in L^2_{\mathbb{C}^n}$, with I_n the $n \times n$ identity matrix. If we set $H^2_{\mathbb{C}^n} = H^2 \oplus \cdots \oplus H^2$ then we see that

$$T_\Phi = \begin{bmatrix} T_{\varphi_{11}} & \cdots & T_{\varphi_{1n}} \\ & \vdots & \\ T_{\varphi_{n1}} & \cdots & T_{\varphi_{nn}} \end{bmatrix} \quad \text{and} \quad H_\Phi = \begin{bmatrix} H_{\varphi_{11}} & \cdots & H_{\varphi_{1n}} \\ & \vdots & \\ H_{\varphi_{n1}} & \cdots & H_{\varphi_{nn}} \end{bmatrix},$$

where

$$\Phi = \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ & \vdots & \\ \varphi_{n1} & \cdots & \varphi_{nn} \end{bmatrix} \in L^\infty_{M_n}.$$

For $\Phi \in L^\infty_{M_{n\times m}}$, write

$$\widetilde{\Phi}(z) := \Phi^*(\overline{z}).$$

A matrix-valued function $\Theta \in H^\infty_{M_{n\times m}}$ $(= H^\infty \otimes M_{n\times m})$ is called *inner* if $\Theta^*\Theta = I_m$ almost everywhere on $\mathbb{T}$. The following basic relations can be easily derived:

$$T^*_\Phi = T_{\Phi^*}, \qquad H^*_\Phi = H_{\widetilde{\Phi}} \quad (\Phi \in L^\infty_{M_n}); \tag{1.2}$$

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi \quad (\Phi, \Psi \in L^\infty_{M_n}); \tag{1.3}$$

$$H_\Phi T_\Psi = H_{\Phi\Psi}, \qquad H_{\Psi\Phi} = T^*_{\widetilde{\Psi}} H_\Phi \quad (\Phi \in L^\infty_{M_n}, \Psi \in H^\infty_{M_n}); \tag{1.4}$$

$$H^*_\Phi H_\Phi - H^*_{\Theta\Phi} H_{\Theta\Phi} = H^*_\Phi H_{\Theta^*} H^*_{\Theta^*} H_\Phi \quad (\Theta \in H^\infty_{M_n} \text{ inner}, \ \Phi \in L^\infty_{M_n}). \tag{1.5}$$

For a matrix-valued function $\Phi \equiv [\varphi_{ij}] \in L^\infty_{M_n}$, we say that Φ is of *bounded type* if each entry φ_{ij} is of bounded type, and we say that Φ is *rational* if each entry φ_{ij} is a rational function. A matrix-valued trigonometric polynomial $\Phi \in L^\infty_{M_n}$ is of the form

$$\Phi(z) = \sum_{j=-m}^{N} A_j z^j \quad (A_j \in M_n),$$

where A_N and A_{-m} are called the *outer* coefficients of Φ.

We recall that for matrix-valued functions $A := \sum_{j=-\infty}^{\infty} A_j z^j \in L^2_{M_n}$ and $B := \sum_{j=-\infty}^{\infty} B_j z^j \in L^2_{M_n}$, we define the inner product of A and B by

$$\langle A, B\rangle := \int_{\mathbb{T}} \operatorname{tr}(B^*A)\, d\mu = \sum_{j=-\infty}^{\infty} \operatorname{tr}(B^*_j A_j),$$

where $\operatorname{tr}(\cdot)$ denotes the trace of a matrix and define $\|A\|_2 := \langle A, A\rangle^{\frac{1}{2}}$. We also define, for $A \in L^\infty_{M_n}$,

$$\|A\|_\infty := \operatorname*{ess\,sup}_{t\in\mathbb{T}} \|A(t)\| \quad (\|\cdot\| \text{ denotes the spectral norm of a matrix}).$$

Finally, the *shift* operator S on $H^2_{\mathbb{C}^n}$ is defined by

$$S := T_{zI_n}.$$

The following fundamental result will be useful in the sequel.

The Beurling–Lax–Halmos Theorem. *A nonzero subspace M of $H^2_{\mathbb{C}^n}$ is invariant for the shift operator S on $H^2_{\mathbb{C}^n}$ if and only if $M = \Theta H^2_{\mathbb{C}^m}$, where Θ is an inner matrix function in $H^\infty_{M_{n\times m}}$ $(m \le n)$. Furthermore, Θ is unique up to a unitary constant right factor; that is, if $M = \Delta H^2_{\mathbb{C}^r}$ (for Δ an inner function in $H^\infty_{M_{n\times r}}$), then $m = r$ and $\Theta = \Delta W$, where W is a (constant in z) unitary matrix mapping $\mathbb{C}^m$ onto $\mathbb{C}^m$.*

As customarily done, we say that two matrix-valued functions A and B are *equal* if they are equal up to a unitary constant right factor. Observe by (1.4) that for $\Phi \in L^\infty_{M_n}$, $H_\Phi S = H_\Phi T_{zI_n} = H_{\Phi\cdot zI_n} = H_{zI_n\cdot\Phi} = T^*_{\overline{z}I_n} H_\Phi$, which implies that the kernel of a block Hankel operator H_Φ is an invariant subspace of the shift operator on $H^2_{\mathbb{C}^n}$. Thus, if $\ker H_\Phi \ne \{0\}$, then by the

Beurling–Lax–Halmos Theorem,

$$\ker H_{\Phi} = \Theta H^2_{\mathbb{C}^m}$$

for some inner matrix function Θ. We note that Θ need not be a square matrix.

On the other hand, recently Gu et al. [34] considered the hyponormality of block Toeplitz operators and characterized it in terms of their symbols. In particular they showed that if T_{Φ} is a hyponormal block Toeplitz operator on $H^2_{\mathbb{C}^n}$, then its symbol Φ is normal, i.e., $\Phi^*\Phi = \Phi\Phi^*$. Their characterization for hyponormality of block Toeplitz operators resembles Cowen's Theorem except for an additional condition—the normality condition of the symbol.

Hyponormality of Block Toeplitz Operators *([34]). For each $\Phi \in L^\infty_{M_n}$, let*

$$\mathcal{E}(\Phi) := \left\{ K \in H^\infty_{M_n} : \|K\|_\infty \le 1 \ \text{ and } \ \Phi - K\Phi^* \in H^\infty_{M_n} \right\}.$$

Then T_{Φ} is hyponormal if and only if Φ is normal and $\mathcal{E}(\Phi)$ is nonempty.

The hyponormality of the Toeplitz operator T_{Φ} with arbitrary matrix-valued symbol Φ, though solved in principle by Cowen's Theorem [14] and the criterion due to Gu, Hendricks and Rutherford [34], is in practice very complicated. Until now, explicit criteria for the hyponormality of block Toeplitz operators T_{Φ} with matrix-valued trigonometric polynomials or rational functions Φ were established via interpolation problems [34,47,48].

In Section 3, we obtain a tractable criterion for the hyponormality of block Toeplitz operators with bounded type symbols. To do this we employ a continuous analogue of the elementary theorem of Schur on triangularization of finite matrices: If T is a finite matrix then it can be represented as $T = D + N$, where D is a diagonal matrix and N is a nilpotent matrix. The continuous analogue is the so-called triangularization theorem for compressions of the shift operator: in this case, D and N are replaced by a certain (normal) multiplication operator and a Volterra operator of Hilbert–Schmidt class, respectively.

Section 4 deals with the gap between hyponormality and subnormality of block Toeplitz operators. The Bram–Halmos criterion for subnormality [7,10] states that an operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if $\sum_{i,j}(T^i x_j, T^j x_i) \ge 0$ for all finite collections $x_0, x_1, \ldots, x_k \in \mathcal{H}$. It is easy to see that this is equivalent to the following positivity test:

$$\begin{bmatrix} [T^*, T] & [T^{*2}, T] & \cdots & [T^{*k}, T] \\ [T^*, T^2] & [T^{*2}, T^2] & \cdots & [T^{*k}, T^2] \\ \vdots & \vdots & \ddots & \vdots \\ [T^*, T^k] & [T^{*2}, T^k] & \cdots & [T^{*k}, T^k] \end{bmatrix} \ge 0 \quad (\text{all } k \ge 1). \tag{1.6}$$

The positivity condition (1.6) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (1.6) for all $k \in \mathbb{Z}_+$. The Bram–Halmos criterion indicates that hyponormality is generally far from subnormality. But there are special classes of operators for which the positivity of (1.6) for some k and subnormality are equivalent. For example, it was shown in [21] that if $W_{\sqrt{x},(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$ is the weighted shift whose weight sequence consists of the initial weight x followed by the weight sequence of the recursively generated subnormal weighted shift $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$ with an initial segment of positive weights $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ (cf. [17–19]), then W_α is subnormal if and only if the positivity condition (1.6) is satisfied with $k = 2$. On the other hand, in [38, Problem 209], it was shown that there exists a hyponormal operator whose square is not hyponormal, e.g., $U^* + 2U$ (U is the unilateral shift on ℓ^2), which is a *trigonometric* Toeplitz operator, i.e., $U^* + 2U \equiv T_{\bar{z}+2z}$. This example addresses the gap between hyponormality and subnormality for Toeplitz operators. This matter is closely related to Halmos's Problem 5 [36,37]: Is every subnormal Toeplitz operator either normal or analytic?

In [21], as a partial answer, it was shown that every hyponormal Toeplitz operator T_φ with trigonometric polynomial symbol φ whose square is hyponormal must be either normal or analytic. In [33], Gu showed that this result still holds for Toeplitz operators T_φ with rational symbol φ (more generally, in the cases where φ is of bounded type). In Section 4 we prove the following theorem: If Φ is a matrix-valued rational function whose co-analytic part has a coprime factorization then every hyponormal Toeplitz operator T_Φ whose square is hyponormal must be either normal or analytic. This result generalizes the results in [21,33].

In Section 5, we consider a completion problem involving Toeplitz operators. Given a partially specified operator matrix, the problem of finding suitable operators to complete the given partial operator matrix so that the resulting matrix satisfies certain given properties is called a *completion problem*. The dilation problem is a special case of the completion problem: in other words, a dilation of T is a completion of the partial operator matrix $\begin{bmatrix} T & ? \\ ? & ? \end{bmatrix}$. In recent years, operator theorists have been interested in the subnormal completion problem for

$$\begin{bmatrix} U^* & ? \\ ? & U^* \end{bmatrix},$$

where U is the unilateral shift on H^2. If the unspecified entries? are Toeplitz operators this is called the *Toeplitz* subnormal completion problem. Thus this problem is related to the subnormality of block Toeplitz operators. In Section 5, we solve this Toeplitz subnormal completion problem.

Finally, in Section 6 we list some open problems.

2. Basic theory and preliminaries

We first recall [1, Lemma 3] that if $\varphi \in L^\infty$ then

$$\varphi \text{ is of bounded type} \iff \ker H_\varphi \neq \{0\}. \tag{2.1}$$

If $\varphi \in L^\infty$, we write

$$\varphi_+ \equiv P\varphi \in H^2 \quad \text{and} \quad \varphi_- \equiv \overline{P^\perp \varphi} \in zH^2.$$

Assume now that both φ and $\overline{\varphi}$ are of bounded type. Then from Beurling's Theorem, $\ker H_{\overline{\varphi_-}} = \theta_0 H^2$ and $\ker H_{\overline{\varphi_+}} = \theta_+ H^2$ for some inner functions θ_0, θ_+. We thus have $b := \overline{\varphi_-}\theta_0 \in H^2$, and hence we can write

$$\varphi_- = \theta_0 \overline{b} \quad \text{and similarly} \quad \varphi_+ = \theta_+ \overline{a} \quad \text{for some } a \in H^2. \tag{2.2}$$

In particular, if T_φ is hyponormal then since

$$[T_\varphi^*, T_\varphi] = H_{\overline{\varphi}}^* H_{\overline{\varphi}} - H_\varphi^* H_\varphi = H_{\overline{\varphi_+}}^* H_{\overline{\varphi_+}} - H_{\overline{\varphi_-}}^* H_{\overline{\varphi_-}}, \tag{2.3}$$

it follows that $\|H_{\overline{\varphi_+}} f\| \geq \|H_{\overline{\varphi_-}} f\|$ for all $f \in H^2$, and hence

$$\theta_+ H^2 = \ker H_{\overline{\varphi_+}} \subseteq \ker H_{\overline{\varphi_-}} = \theta_0 H^2,$$

which implies that θ_0 divides θ_+, i.e., $\theta_+ = \theta_0\theta_1$ for some inner function θ_1. We write, for an inner function θ,

$$\mathcal{H}(\theta) := H^2 \ominus \theta\, H^2.$$

Note that if $f = \theta\overline{a} \in L^2$, then $f \in H^2$ if and only if $a \in \mathcal{H}(z\theta)$; in particular, if $f(0) = 0$ then $a \in \mathcal{H}(\theta)$. Thus, if $\varphi = \overline{\varphi_-} + \varphi_+ \in L^\infty$ is such that φ and $\overline{\varphi}$ are of bounded type such that $\varphi_+(0) = 0$ and T_φ is hyponormal, then we can write

$$\varphi_+ = \theta_0\theta_1\bar{a} \quad \text{and} \quad \varphi_- = \theta_0\bar{b}, \quad \text{where } a \in \mathcal{H}(\theta_0\theta_1) \text{ and } b \in \mathcal{H}(\theta_0).$$

By Kronecker's Lemma [54, p. 183], if $f \in H^\infty$ then $\overline{f}$ is a rational function if and only if rank $H_{\overline{f}} < \infty$, which implies that

$$\overline{f} \text{ is rational} \iff f = \theta\overline{b} \quad \text{with a finite Blaschke product } \theta. \tag{2.4}$$

Also, from the scalar-valued case of (1.4), we can see that if $k \in \mathcal{E}(\varphi)$ then

$$[T_\varphi^*, T_\varphi] = H_{\overline{\varphi}}^* H_{\overline{\varphi}} - H_\varphi^* H_\varphi = H_{\overline{\varphi}}^* H_{\overline{\varphi}} - H_{k\overline{\varphi}}^* H_{k\overline{\varphi}} = H_{\overline{\varphi}}^*(1 - T_{\tilde{k}} T_{\tilde{k}}^*) H_{\overline{\varphi}}. \tag{2.5}$$

On the other hand, Abrahamse [1, Lemma 6] showed that if T_φ is hyponormal, if $\varphi \notin H^\infty$, and if φ or $\overline{\varphi}$ is of bounded type then both φ and $\overline{\varphi}$ are of bounded type. However, by contrast to the scalar case, Φ^* may not be of bounded type even though T_Φ is hyponormal, $\Phi \notin H^\infty_{M_n}$ and Φ is of bounded type. But we have a one-way implication: if T_Φ is hyponormal and Φ^* is of bounded type then Φ is also of bounded type (see [34, Corollary 3.5 and Remark 3.6]). Thus whenever we deal with hyponormal Toeplitz operators T_Φ with symbols Φ satisfying that both Φ and Φ^* are of bounded type (e.g., Φ is a matrix-valued rational function), it suffices to assume that only Φ^* is of bounded type. In spite of this, for convenience, we will assume that Φ and Φ^* are of bounded type whenever we deal with bounded type symbols.

For a matrix-valued function $\Phi \in H^2_{M_{n\times r}}$, we say that $\Delta \in H^2_{M_{n\times m}}$ is a *left inner divisor* of Φ if Δ is an inner matrix function such that $\Phi = \Delta A$ for some $A \in H^2_{M_{m\times r}}$ $(m \leq n)$. We also say that two matrix functions $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{n\times m}}$ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary constant and that $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{m\times r}}$ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. Two matrix functions Φ and Ψ in $H^2_{M_n}$ are said to be *coprime* if they are both left and right coprime. We note that if $\Phi \in H^2_{M_n}$ is such that det Φ is not identically zero then any left inner divisor Δ of Φ is square, i.e., $\Delta \in H^2_{M_n}$: indeed, if $\Phi = \Delta A$ with $\Delta \in H^2_{M_{n\times r}}$ $(r < n)$ then for almost all $z \in \mathbb{T}$, rank $\Phi(z) \leq$ rank $\Delta(z) \leq r < n$, so that det $\Phi(z) = 0$ for almost all $z \in \mathbb{T}$. If $\Phi \in H^2_{M_n}$ is such that det Φ is not identically zero then we say that $\Delta \in H^2_{M_n}$ is a *right inner divisor* of Φ if $\widetilde{\Delta}$ is a left inner divisor of $\widetilde{\Phi}$.

On the other hand, we have (in the Introduction) remarked that Θ need not be square in the equality ker $H_\Phi = \Theta H^2_{\mathbb{C}^n}$, which comes from the Beurling–Lax–Halmos Theorem. But it was known [34, Theorem 2.2] that for $\Phi \in L^\infty_{M_n}$, Φ is of bounded type if and only if ker $H_\Phi = \Theta H^2_{\mathbb{C}^n}$ for some square inner matrix function Θ.

Let $\{\Theta_i \in H^\infty_{M_n} : i \in J\}$ be a family of inner matrix functions. Then the greatest common left inner divisor Θ_d and the least common left inner multiple Θ_m of the family $\{\Theta_i \in H^\infty_{M_n} : i \in J\}$ are the inner functions defined by

$$\Theta_d H^2_{\mathbb{C}^p} = \bigvee_{i\in J} \Theta_i H^2_{\mathbb{C}^n} \quad \text{and} \quad \Theta_m H^2_{\mathbb{C}^q} = \bigcap_{i\in J} \Theta_i H^2_{\mathbb{C}^n}.$$

The greatest common right inner divisor Θ'_d and the least common right inner multiple Θ'_m of the family $\{\Theta_i \in H^\infty_{M_n} : i \in J\}$ are the inner functions defined by

$$\widetilde{\Theta}'_d H^2_{\mathbb{C}^r} = \bigvee_{i\in J} \widetilde{\Theta}_i H^2_{\mathbb{C}^n} \quad \text{and} \quad \widetilde{\Theta}'_m H^2_{\mathbb{C}^s} = \bigcap_{i\in J} \widetilde{\Theta}_i H^2_{\mathbb{C}^n}.$$

The Beurling–Lax–Halmos Theorem guarantees that Θ_d and Θ_m are unique up to a unitary constant right factor, and Θ'_d and Θ'_m are unique up to a unitary constant left factor. We write

$$\begin{aligned} \Theta_d &= \mathrm{GCD}_\ell\,\{\Theta_i : i \in J\}, & \Theta_m &= \mathrm{LCM}_\ell\,\{\Theta_i : i \in J\},\\ \Theta'_d &= \mathrm{GCD}_r\,\{\Theta_i : i \in J\}, & \Theta'_m &= \mathrm{LCM}_r\,\{\Theta_i : i \in J\}. \end{aligned}$$

If $n = 1$, then $\mathrm{GCD}_\ell\,\{\cdot\} = \mathrm{GCD}_r\,\{\cdot\}$ (simply denoted $\mathrm{GCD}\,\{\cdot\}$) and $\mathrm{LCM}_\ell\,\{\cdot\} = \mathrm{LCM}_r\,\{\cdot\}$ (simply denoted $\mathrm{LCM}\,\{\cdot\}$). In general, it is not true that $\mathrm{GCD}_\ell\,\{\cdot\} = \mathrm{GCD}_r\,\{\cdot\}$ and $\mathrm{LCM}_\ell\,\{\cdot\} = \mathrm{LCM}_r\,\{\cdot\}$.

However, we have:

Lemma 2.1. *Let $\Theta_i := \theta_i I_n$ for an inner function θ_i $(i \in J)$.*

(a) $\mathrm{GCD}_\ell\,\{\Theta_i : i \in J\} = \mathrm{GCD}_r\,\{\Theta_i : i \in J\} = \theta_d I_n$, *where* $\theta_d = \mathrm{GCD}\,\{\theta_i : i \in J\}$.

(b) $\mathrm{LCM}_\ell\,\{\Theta_i : i \in J\} = \mathrm{LCM}_r\,\{\Theta_i : i \in J\} = \theta_m I_n$, *where* $\theta_m = \mathrm{LCM}\,\{\theta_i : i \in J\}$.

Proof. (a) If $\Theta_d = \mathrm{GCD}_\ell\{\Theta_i : i \in J\}$, then

$$\Theta_d H^2_{\mathbb{C}^n} = \bigvee_{i\in J} \Theta_i H^2_{\mathbb{C}^n} = \bigoplus_{j=1}^n \bigvee_{i\in J} \theta_i H^2 = \bigoplus_{j=1}^n \theta_d H^2,$$

which implies that $\Theta_d = \theta_d I_n$ with $\theta_d = \mathrm{GCD}\,\{\theta_i : i \in J\}$. If instead $\Theta_d = \mathrm{GCD}_r\{\Theta_i : i \in J\}$ then $\widetilde{\Theta}_d = \mathrm{GCD}_\ell\{\widetilde{\Theta}_i : i \in J\}$. Thus we have $\widetilde{\Theta}_d = \widetilde{\theta}_d I_n$ and hence, $\Theta_d = \theta_d I_n$.

(b) If $\Theta_m = \mathrm{LCM}_\ell\{\Theta_i : i \in J\}$, then

$$\Theta_m H^2_{\mathbb{C}^n} = \bigcap_{i\in J} \Theta_i H^2_{\mathbb{C}^n} = \bigoplus_{j=1}^n \bigcap_{i\in J} \theta_i H^2 = \bigoplus_{j=1}^n \theta_m H^2,$$

which implies that $\Theta_m = \theta_m I_n$ with $\theta_m = \mathrm{LCM}\,\{\theta_i : i \in J\}$. If instead $\Theta_m = \mathrm{LCM}_r\{\Theta_i : i \in J\}$, then the same argument as in (a) gives the result. $\square$

In view of Lemma 2.1, if $\Theta_i = \theta_i I_n$ for an inner function θ_i $(i \in J)$, we can define the greatest common inner divisor Θ_d and the least common inner multiple Θ_m of the Θ_i by

$$\begin{aligned} \Theta_d &\equiv \mathrm{GCD}\,\{\Theta_i : i \in J\} := \mathrm{GCD}_\ell\,\{\Theta_i : i \in J\} = \mathrm{GCD}_r\,\{\Theta_i : i \in J\};\\ \Theta_m &\equiv \mathrm{LCM}\,\{\Theta_i : i \in J\} := \mathrm{LCM}_\ell\,\{\Theta_i : i \in J\} = \mathrm{LCM}_r\,\{\Theta_i : i \in J\} : \end{aligned}$$

they are both diagonal matrices.

For $\Phi \in L^\infty_{M_n}$ we write

$$\Phi_+ := P_n(\Phi) \in H^2_{M_n} \quad \text{and} \quad \Phi_- := \left[P_n^\perp(\Phi)\right]^* \in H^2_{M_n}.$$

Thus we can write $\Phi = \Phi_-^* + \Phi_+$. Suppose $\Phi = [\varphi_{ij}] \in L^\infty_{M_n}$ is such that Φ^* is of bounded type. Then we may write $\varphi_{ij} = \theta_{ij}\overline{b}_{ij}$, where θ_{ij} is an inner function and θ_{ij} and b_{ij} are coprime. Thus if θ is the least common inner multiple of θ_{ij}'s then we can write

$$\Phi = [\varphi_{ij}] = [\theta_{ij}\overline{b}_{ij}] = [\theta\overline{a}_{ij}] = \Theta A^* \quad (\Theta \equiv \theta I_n,\ A \equiv [a_{ij}] \in H^2_{M_n}). \tag{2.6}$$

We note that in the factorization (2.6), $A(\alpha)$ is nonzero whenever $\theta(\alpha)=0$. Let $\Phi=\Phi_-^*+\Phi_+\in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Then in view of (2.6) we can write

$$\Phi_+=\Theta_1 A^* \quad \text{and} \quad \Phi_-=\Theta_2 B^*,$$

where $\Theta_i=\theta_i I_n$ with an inner function θ_i for $i=1,2$ and $A,B\in H^2_{M_n}$. In particular, if $\Phi\in L^\infty_{M_n}$ is rational then the θ_i can be chosen as finite Blaschke products, as we observed in (2.4).

By contrast with scalar-valued functions, in (2.6) Θ and A need not be (right) coprime: for instance, if $\Phi:=\begin{bmatrix} z & z\\ z & z\end{bmatrix}$ then we can write

$$\Phi=\Theta A^*=\begin{bmatrix} z & 0\\ 0 & z\end{bmatrix}\begin{bmatrix} 1 & 1\\ 1 & 1\end{bmatrix},$$

but $\Theta:=\begin{bmatrix} z & 0\\ 0 & z\end{bmatrix}$ and $A:=\begin{bmatrix} 1 & 1\\ 1 & 1\end{bmatrix}$ are not right coprime because $\frac{1}{\sqrt{2}}\begin{bmatrix} z & -z\\ 1 & 1\end{bmatrix}$ is a common right inner divisor, i.e.,

$$\Theta=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & z\\ -1 & z\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix} z & -z\\ 1 & 1\end{bmatrix} \quad \text{and} \quad A=\sqrt{2}\begin{bmatrix} 0 & 1\\ 0 & 1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix} z & -z\\ 1 & 1\end{bmatrix}. \tag{2.7}$$

If $\Omega=\mathrm{GCD}_\ell\,\{A,\Theta\}$ in the representation (2.6):

$$\Phi=\Theta A^*=A^*\Theta \quad (\Theta\equiv\theta I_n \text{ for an inner function } \theta),$$

then $\Theta=\Omega\Omega_\ell$ and $A=\Omega A_\ell$ for some inner matrix Ω_ℓ (where $\Omega_\ell\in H^2_{M_n}$ because $\det\Theta$ is not identically zero) and some $A_l\in H^2_{M_n}$. Therefore if $\Phi^*\in L^\infty_{M_n}$ is of bounded type then we can write

$$\Phi=A_\ell{}^*\Omega_\ell, \quad \text{where } A_\ell \text{ and } \Omega_\ell \text{ are left coprime.} \tag{2.8}$$

$A_\ell^*\Omega_\ell$ is called the *left coprime factorization* of Φ; similarly, we can write

$$\Phi=\Omega_r A_r^*, \quad \text{where } A_r \text{ and } \Omega_r \text{ are right coprime.} \tag{2.9}$$

In this case, $\Omega_r A_r^*$ is called the *right coprime factorization* of Φ.

Remark 2.2 (*[34, Corollary 2.5]*)**.** As a consequence of the Beurling–Lax–Halmos Theorem, we can see that

$$\Phi=\Omega_r A_r^* \text{ (right coprime factorization)} \Longleftrightarrow \ker H_{\Phi^*}=\Omega_r H^2_{\mathbb{C}^n}. \tag{2.10}$$

In fact, if $\Phi=\Omega_r A_r^*$ (right coprime factorization) then it is evident that

$$\ker H_{\Phi^*}\supseteq\Omega_r H^2_{\mathbb{C}^n}.$$

From the Beurling–Lax–Halmos Theorem,

$$\ker H_{\Phi^*}=\Theta H^2_{\mathbb{C}^n},$$

for some inner function Θ, and hence $(I-P)(\Phi^*\Theta)=0$, i.e., $\Phi^*=D\Theta^*$, for some $D\in H^2_{\mathbb{C}^n}$. We want to show that $\Omega_r=\Theta$ up to a unitary constant right factor. Since $\Theta H^2_{\mathbb{C}^n}\supseteq\Omega_r H^2_{\mathbb{C}^n}$, we have (cf. [29, p. 240]) that $\Omega_r=\Theta\Delta$ for some square inner function Δ. Thus,

$$D\Theta^*=\Phi^*=A_r\Omega_r^*=A_r\Delta^*\Theta^*,$$

which implies $A_r = D\Delta$, so that Δ is a common right inner divisor of both A_r and Ω_r. But since A_r and Ω_r are right coprime, Δ must be a unitary constant. The proof of the converse implication is entirely similar. □

From now on, for notational convenience we write

$$I_\omega := \omega I_n \quad (\omega \in H^2) \quad \text{and} \quad H_0^2 := I_z H^2_{M_n}.$$

It is not easy to check the condition "B and Θ are coprime" in the decomposition $F = B^*\Theta$ ($\Theta \equiv I_\theta$ is inner and $B \in H^2_{M_n}$). But if F is rational (and hence Θ is given in a form $\Theta \equiv I_\theta$ with a finite Blaschke product θ) then we can obtain a more tractable criterion. To see this, we need to recall the notion of finite Blaschke–Potapov product.

Let $\lambda \in \mathbb{D}$ and write

$$b_\lambda(z) := \frac{z-\lambda}{1-\overline{\lambda}z},$$

which is called a *Blaschke factor*. If M is a closed subspace of $\mathbb{C}^n$ then the matrix function of the form

$$b_\lambda P_M + (I - P_M) \quad (P_M := \text{ the orthogonal projection of } \mathbb{C}^n \text{ onto } M)$$

is called a *Blaschke–Potapov factor*; an $n \times n$ matrix function D is called *a finite Blaschke–Potapov product* if D is of the form

$$D = \nu \prod_{m=1}^{M} \Big(b_m P_m + (I - P_m)\Big),$$

where ν is an $n \times n$ unitary constant matrix, b_m is a Blaschke factor, and P_m is an orthogonal projection in $\mathbb{C}^n$ for each $m = 1, \ldots, M$. In particular, a scalar-valued function D reduces to a finite Blaschke product $D = \nu \prod_{m=1}^{M} b_m$, where $\nu = e^{i\omega}$. It is also known (cf. [56]) that an $n \times n$ matrix function D is rational and inner if and only if it can be represented as a finite Blaschke–Potapov product.

Write $\mathcal{Z}(\theta)$ for the set of zeros of an inner function θ. We then have:

Lemma 2.3. *Let $B \in H^\infty_{M_n}$ be rational and $\Theta = I_\theta$ with a finite Blaschke product θ. Then the following statements are equivalent:*

(a) *$B(\alpha)$ is invertible for each $\alpha \in \mathcal{Z}(\theta)$;*
(b) *B and Θ are right coprime;*
(c) *B and Θ are left coprime.*

Proof. See [20, Lemma 3.10]. □

If $\Theta \in H^\infty_{M_n}$ is an inner matrix function, we write

$$\mathcal{H}(\Theta) := (\Theta H^2_{\mathbb{C}_n})^\perp;$$
$$\mathcal{H}_\Theta := (\Theta H^2_{M_n})^\perp;$$
$$\mathcal{K}_\Theta := (H^2_{M_n}\Theta)^\perp.$$

If $\Theta = I_\theta$ for an inner function θ then $\mathcal{H}_\Theta = \mathcal{K}_\Theta$ and if $n = 1$, then $\mathcal{H}(\Theta) = \mathcal{H}_\Theta = \mathcal{K}_\Theta$.

The following lemma is useful in the sequel.

Lemma 2.4. *If* $\Theta \in H^2_{M_n}$ *is an inner matrix function then*

$$\dim \mathcal{H}(\Theta) < \infty \Longleftrightarrow \Theta \text{ is a finite Blaschke–Potapov product.}$$

Proof. Let

$$\delta := \mathrm{GCD}\,\{\omega : \omega \text{ is inner, } \Theta \text{ is a left inner divisor of } \Omega = I_\omega\} \quad \text{and} \quad \Delta := I_\delta,$$

in other words, δ is a 'minimal' inner function such that $\Delta \equiv I_\delta = \Theta\Theta_1$ for some inner matrix function Θ_1. Note that

$$\begin{aligned} \Theta \text{ is a finite Blaschke–Potapov product} &\Longrightarrow \delta \text{ is a finite Blaschke product} \\ &\Longrightarrow \dim \mathcal{H}(\Delta) < \infty. \end{aligned}$$

Observe that

$$\mathcal{H}(\Delta) = \mathcal{H}(\Theta\Theta_1) = \mathcal{H}(\Theta) \bigoplus \Theta\mathcal{H}(\Theta_1).$$

Thus if Θ is a finite Blaschke–Potapov product, then $\dim \mathcal{H}(\Theta) < \infty$. Conversely, we suppose $\dim \mathcal{H}(\Theta) < \infty$. Write $\Theta := [\theta_{ij}]_{ij=1}^n$. Since

$$\operatorname{rank} H^*_{\theta_{ij}} \le \operatorname{rank} H^*_{\Theta^*} = \dim \mathcal{H}(\Theta) < \infty,$$

it follows that θ_{ij}'s are rational functions. Thus Θ is a rational inner matrix function and hence a finite Blaschke–Potapov product. □

Lemma 2.4 implies that every inner divisor of a rational inner function (i.e., a finite Blaschke–Potapov product) is also a finite Blaschke–Potapov product: indeed, if Θ is a finite Blaschke–Potapov product and Θ_1 is an inner divisor of Θ, then $\dim \mathcal{H}(\Theta_1) \le \dim \mathcal{H}(\Theta) < \infty$, and hence by Lemma 2.4, Θ_1 is a finite Blaschke–Potapov product.

From Lemma 2.4, we know that every inner divisor of $B_\lambda := I_{b_\lambda} \in H^\infty_{M_n}$ is a finite Blaschke–Potapov product. However we can say more:

Lemma 2.5. *Every inner divisor of* $B_\lambda := I_{b_\lambda} \in H^\infty_{M_n}$ *is a Blaschke–Potapov factor.*

Proof. Suppose D is an inner divisor of B_λ. By Lemma 2.4, D is a Blaschke–Potapov product of the form

$$D = \nu \prod_{i=1}^{m} D_i \quad \text{with } D_i := b_{\lambda_i} P_i + (I - P_i)\ (m \le n).$$

We write $B_\lambda = ED$ for some $E \in H^2_{M_n}$. Observe that $B_\lambda(\lambda_i)$ is not invertible, so that $\lambda_i = \lambda$ for all $i = 1, 2 \ldots, m$. We thus have

$$P_m + b_\lambda(I - P_m) = B_\lambda D_m^* = E \cdot \nu \prod_{i=1}^{m-1} D_i.$$

Then we have

$$\ker P_m = \ker(B_\lambda D_m^*)(\lambda) \supseteq \ker D_{m-1}(\lambda) = \operatorname{ran} P_{m-1},$$

which implies that $P_m P_{m-1} = 0$, and hence P_m and P_{m-1} are orthogonal. Thus $D_{m-1}D_m$ is a Blaschke–Potapov factor. Now an induction shows that D is a Blaschke–Potapov factor. □

With the aid of Lemma 2.5, we can show that the equivalence (b) $\Leftrightarrow$ (c) in Lemma 2.3 fails if Θ is not a *constant* diagonal matrix. To see this, let

$$\Theta_1 := \begin{bmatrix} b_\alpha & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Theta_2 := \frac{1}{\sqrt{2}} \begin{bmatrix} z & -z \\ 1 & 1 \end{bmatrix}.$$

Then $\Theta := \Theta_1 \Theta_2$ and Θ_1 are not left coprime. Observe that

$$\widetilde{\Theta}_1 := \begin{bmatrix} b_{\overline{\alpha}} & 0 \\ 0 & 1 \end{bmatrix}, \qquad \widetilde{\Theta} = \frac{1}{\sqrt{2}} \begin{bmatrix} z b_{\overline{\alpha}} & 1 \\ -z b_{\overline{\alpha}} & 1 \end{bmatrix}.$$

Since every right inner divisor Δ of $\widetilde{\Theta}_1$ is an inner divisor of $B_{\overline{\alpha}} := I_{b_{\overline{\alpha}}}$, it follows from Lemma 2.5 that $\Delta = \widetilde{\Theta}_1$ (up to a unitary constant right factor). Suppose that Θ and Θ_1 are not right coprime. Then $\widetilde{\Theta}$ and $\widetilde{\Theta}_1$ are not left coprime and hence $\widetilde{\Theta}_1$ is a left inner divisor of $\widetilde{\Theta}$. Write

$$\frac{1}{\sqrt{2}} \begin{bmatrix} z b_{\overline{\alpha}} & 1 \\ -z b_{\overline{\alpha}} & 1 \end{bmatrix} = \begin{bmatrix} b_{\overline{\alpha}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{21} \end{bmatrix} \quad (f_{ij} \in H^2).$$

Then we have $\frac{1}{\sqrt{2}} = b_{\overline{\alpha}} f_{12}$, so that $f_{12} = \frac{1}{\sqrt{2}} \overline{b_{\overline{\alpha}}} \notin H^2$, giving a contradiction.

For $\mathcal{X}$ a subspace of $H^2_{M_n}$, we write $P_{\mathcal{X}}$ for the orthogonal projection from $H^2_{M_n}$ onto $\mathcal{X}$.

Lemma 2.6. *Let $\Theta \in H^\infty_{M_n}$ be an inner matrix function and $A \in H^2_{M_n}$. Then the following hold:*

(a) $A \in \mathcal{K}_\Theta \Longleftrightarrow \Theta A^* \in H^2_0$;
(b) $A \in \mathcal{H}_\Theta \Longleftrightarrow A^* \Theta \in H^2_0$;
(c) $P_{H^2_0}(\Theta A^*) = \Theta \Big(P_{\mathcal{K}_\Theta} A \Big)^*$.

Proof. Let $C \in H^2_{M_n}$ be arbitrary. We then have

$$\begin{aligned} A \in \mathcal{K}_\Theta &\Longleftrightarrow \langle A, C\Theta \rangle = 0 \\ &\Longleftrightarrow \int_{\mathbb{T}} \operatorname{tr} \Big((C\Theta)^* A \Big)\, d\mu = 0 \\ &\Longleftrightarrow \int_{\mathbb{T}} \operatorname{tr} (C^* A \Theta^*)\, d\mu = 0 \quad (\text{since } \operatorname{tr}(AB) = \operatorname{tr}(BA)) \\ &\Longleftrightarrow \langle A\Theta^*, C \rangle = 0 \\ &\Longleftrightarrow \Theta A^* \in H^2_0, \end{aligned}$$

giving (a) and similarly, (b). For (c), we write $A = A_1 + A_2$, where $A_1 := P_{\mathcal{K}_\Theta} A$ and $A_2 := A_3 \Theta$ for some $A_3 \in H^2_{M_n}$. We then have

$$P_{H^2_0}(\Theta A^*) = P_{H^2_0}(\Theta A_1^* + \Theta A_2^*) = P_{H^2_0}\Big(\Theta (P_{\mathcal{K}_\Theta} A)^* + \Theta \Theta^* A_3^* \Big) = \Theta (P_{\mathcal{K}_\Theta} A)^*,$$

giving (c). $\square$

We next review the classical Hermite–Fejér interpolation problem, following [29]; this approach will be useful in the sequel. Let θ be a finite Blaschke product of degree d:

$$\theta = e^{i\xi} \prod_{i=1}^{N} (\widetilde{b}_i)^{m_i} \quad \left(\widetilde{b}_i := \frac{z - \alpha_i}{1 - \overline{\alpha}_i z}, \text{ where } \alpha_i \in \mathbb{D} \right),$$

where $d = \sum_{i=1}^{N} m_i$. For our purposes rewrite θ in the form

$$\theta = e^{i\xi} \prod_{j=1}^{d} b_j,$$

where

$$b_j := \widetilde{b}_k \quad \text{if} \sum_{l=0}^{k-1} m_l < j \le \sum_{l=0}^{k} m_l$$

and, for notational convenience, $m_0 := 0$. Let

$$\varphi_j := \frac{q_j}{1 - \overline{\alpha}_j z} b_{j-1} b_{j-2} \cdots b_1 \quad (1 \le j \le d), \tag{2.11}$$

where $\varphi_1 := q_1(1 - \overline{\alpha}_1 z)^{-1}$ and $q_j := (1 - |\alpha_j|^2)^{\frac{1}{2}}$ $(1 \le j \le d)$. It is well known (cf. [58]) that $\{\varphi_j\}_{j=1}^{d}$ is an orthonormal basis for $\mathcal{H}(\theta)$.

For our purposes we concentrate on the data given by sequences of $n \times n$ complex matrices. Given the sequence $\{K_{ij} : 1 \le i \le N,\ 0 \le j < m_i\}$ of $n \times n$ complex matrices and a set of distinct complex numbers $\alpha_1, \dots, \alpha_N$ in $\mathbb{D}$, the classical Hermite–Fejér interpolation problem entails finding necessary and sufficient conditions for the existence of a contractive analytic matrix function K in $H^\infty_{M_n}$ satisfying

$$\frac{K^{(j)}(\alpha_i)}{j!} = K_{i,j} \quad (1 \le i \le N,\ 0 \le j < m_i). \tag{2.12}$$

To construct a matrix polynomial $K(z) \equiv P(z)$ satisfying (2.12), let $p_i(z)$ be the polynomial of order $d - m_i$ defined by

$$p_i(z) := \prod_{\substack{k=1,\\ k \ne i}}^{N} \left(\frac{z - \alpha_k}{\alpha_i - \alpha_k} \right)^{m_k}.$$

Consider the matrix polynomial $P(z)$ of degree $d - 1$ defined by

$$\begin{aligned} P(z) := \sum_{i=1}^{N} \Bigg(& K'_{i,0} + K'_{i,1}(z - \alpha_i) + K'_{i,2}(z - \alpha_i)^2 \\ & + \cdots + K'_{i,m_i-1}(z - \alpha_i)^{m_i-1} \Bigg) p_i(z), \end{aligned} \tag{2.13}$$

where the $K'_{i,j}$ are obtained by the following equations:

$$K'_{i,j} = K_{i,j} - \sum_{k=0}^{j-1} \frac{K'_{i,k}\, p_i^{(j-k)}(\alpha_i)}{(j-k)!} \quad (1 \le i \le N;\ 0 \le j < m_i)$$

and $K'_{i,0} = K_{i,0}$ $(1 \le i \le N)$. Then $P(z)$ satisfies (2.12).

On the other hand, for an inner function θ, let U_θ be defined by the compression of the shift operator U: i.e.,

$$U_\theta = P_{\mathcal{H}(\theta)} U|_{\mathcal{H}(\theta)}.$$

Let $\Theta = I_\theta$ and W be the unitary operator from $\bigoplus_1^d \mathbb{C}^n$ onto $\mathcal{H}(\Theta)$ defined by

$$W := (I_{\varphi_1}, I_{\varphi_2}, \dots, I_{\varphi_d}), \tag{2.14}$$

where the φ_j are the functions in (2.11). It is known [29, Theorem X.1.5] that if θ is the finite Blaschke product of order d, then U_θ is unitarily equivalent to the lower triangular matrix M on $\mathbb{C}^d$ defined by

$$M := \begin{bmatrix} \alpha_1 & 0 & 0 & 0 & \cdots & \cdots \\ q_1q_2 & \alpha_2 & 0 & 0 & \cdots & \cdots \\ -q_1\overline{\alpha}_1q_3 & q_2q_3 & \alpha_3 & 0 & \cdots & \cdots \\ q_1\overline{\alpha}_2\overline{\alpha}_3q_4 & -q_2\overline{\alpha}_3q_4 & q_3q_4 & \alpha_4 & \cdots & \cdots \\ -q_1\overline{\alpha}_2\overline{\alpha}_3\overline{\alpha}_4q_5 & q_2\overline{\alpha}_3\overline{\alpha}_4q_5 & -q_3\overline{\alpha}_4q_5 & q_4q_5 & \alpha_5 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{2.15}$$

If $L \in M_n$ and $M = [m_{i,j}]_{d\times d}$, then the matrix $L \otimes M$ is the matrix on $\mathbb{C}^{n\times d}$ defined by the block matrix

$$L \otimes M := \begin{bmatrix} Lm_{1,1} & Lm_{1,2} & \cdots & Lm_{1,d} \\ Lm_{2,1} & Lm_{2,2} & \cdots & Lm_{2,d} \\ \vdots & \vdots & \vdots & \vdots \\ Lm_{d,1} & Lm_{d,2} & \cdots & Lm_{d,d} \end{bmatrix}.$$

Now let $P(z) \in H^\infty_{M_n}$ be a matrix polynomial of degree k. Then the matrix $P(M)$ on $\mathbb{C}^{n\times d}$ is defined by

$$P(M) := \sum_{i=0}^{k} P_i \otimes M^i, \quad \text{where } P(z) = \sum_{i=0}^{k} P_i z^i. \tag{2.16}$$

For $\Phi \in H^\infty_{M_n}$ and $\Theta := I_\theta$ with an inner function θ, we write, for brevity,

$$(T_\Phi)_\Theta := P_{\mathcal{H}(\Theta)} T_\Phi|_{\mathcal{H}(\Theta)}, \tag{2.17}$$

which is called the compression of T_Φ to $\mathcal{H}(\Theta)$. If M is given by (2.15) and P is the matrix polynomial defined by (2.13) then the matrix $P(M)$ is called the *Hermite–Fejér matrix* determined by (2.16). In particular, it is known [29, Theorem X.5.6] that

$$W^*(T_P)_\Theta W = P(M), \tag{2.18}$$

which says that $P(M)$ is a matrix representation for $(T_P)_\Theta$.

Lemma 2.7. *Let $A \in H^\infty_{M_n}$ and $\Theta = I_\theta$ for a finite Blaschke product θ. If $A(\alpha)$ is invertible for all $\alpha \in \mathcal{Z}(\theta)$ then $(T_A)_\Theta$ is invertible.*

Proof. Suppose $(T_A)_\Theta f = 0$ for some $f \in \mathcal{H}(\Theta)$, so that $P_{\mathcal{H}(\Theta)}(Af) = 0$ and hence, $Af \in \Theta H^2_{\mathbb{C}^n}$. Since $A(\alpha)$ is invertible for all $\alpha \in \mathcal{Z}(\theta)$, it follows that $f \in \Theta H^2_{\mathbb{C}^n}$ and hence, $f \in \Theta H^2_{\mathbb{C}^n} \cap \mathcal{H}(\Theta) = \{0\}$. Thus $(T_A)_\Theta$ is one–one. But since $(T_A)_\Theta$ is a finite dimensional operator (because θ is a finite Blaschke product), it follows that $(T_A)_\Theta$ is invertible. $\square$

3. Hyponormality of block Toeplitz operators

To get a tractable criterion for the hyponormality of block Toeplitz operators with bounded type symbols, we need a triangular representation for compressions of the unilateral shift operator $U \equiv T_z$. We refer to [2,54] for details on this representation. For an explicit criterion, we need to introduce the triangularization theorem concretely. To do so, recall that for an inner function θ, U_θ is defined by

$$U_\theta = P_{\mathcal{H}(\theta)} U|_{\mathcal{H}(\theta)}. \tag{3.1}$$

There are three cases to consider.

Case 1: Let B be a Blaschke product and let $\Lambda := \{\lambda_n : n \geq 1\}$ be the sequence of zeros of B counted with their multiplicities. Write

$$\beta_1 := 1, \qquad \beta_k := \prod_{n=1}^{k-1} \frac{\lambda_n - z}{1 - \overline{\lambda}_n z} \cdot \frac{|\lambda_n|}{\lambda_n} \quad (k \geq 2),$$

and let

$$\delta_j := \frac{d_j}{1 - \overline{\lambda}_j z} \beta_j \quad (j \geq 1),$$

where $d_j := (1 - |\lambda_j|^2)^{\frac{1}{2}}$. Let μ_B be a measure on $\mathbb{N}$ given by $\mu_B(\{n\}) := \frac{1}{2} d_n^2$, $(n \in \mathbb{N})$. Then the map $V_B : L^2(\mu_B) \to \mathcal{H}(B)$ defined by

$$V_B(c) := \frac{1}{\sqrt{2}} \sum_{n \geq 1} c(n) d_n \delta_n, \quad c \equiv \{c(n)\}_{n \geq 1}, \tag{3.2}$$

is unitary and U_B is mapped onto the operator

$$V_B^* U_B V_B = (I - J_B) M_B, \tag{3.3}$$

where $(M_B c)(n) := \lambda_n c(n)$ $(n \in \mathbb{N})$ is a multiplication operator and

$$(J_B c)(n) := \sum_{k=1}^{n-1} c(k) |\lambda_k|^{-2} \cdot \frac{\beta_n(0)}{\beta_k(0)} d_k d_n \quad (n \in \mathbb{N})$$

is a lower-triangular Hilbert–Schmidt operator.

Case 2: Let s be a singular inner function with continuous representing measure $\mu \equiv \mu_s$. Let μ_λ be the projection of μ onto the arc $\{\zeta : \zeta \in \mathbb{T},\ 0 < \arg \zeta \leq \arg \lambda\}$ and let

$$s_\lambda(\zeta) := \exp\left(- \int_{\mathbb{T}} \frac{t + \zeta}{t - \zeta} d\mu_\lambda(t) \right) \quad (\zeta \in \mathbb{D}).$$

Then the map $V_s : L^2(\mu) \to \mathcal{H}(s)$ defined by

$$(V_s c)(\zeta) = \sqrt{2} \int_{\mathbb{T}} c(\lambda) s_\lambda(\zeta) \frac{\lambda d\mu(\lambda)}{\lambda - \zeta} \quad (\zeta \in \mathbb{D}) \tag{3.4}$$

is unitary and U_s is mapped onto the operator

$$V_s^* U_s V_s = (I - J_s) M_s, \tag{3.5}$$

where $(M_s c)(\lambda) := \lambda c(\lambda)$ $(\lambda \in \mathbb{T})$ is a multiplication operator and

$$(J_s c)(\lambda) = 2\int_{\mathbb{T}} e^{\mu(t)-\mu(\lambda)} c(t) d_{\mu_\lambda}(t) \quad (\lambda \in \mathbb{T})$$

is a lower-triangular Hilbert–Schmidt operator.

Case 3: Let Δ be a singular inner function with pure point representing measure $\mu \equiv \mu_\Delta$. We enumerate the set $\{t \in \mathbb{T} : \mu(\{t\}) > 0\}$ as a sequence $\{t_k\}_{k\in\mathbb{N}}$. Write $\mu_k := \mu(\{t_k\})$, $k \geq 1$. Further, let μ_Δ be a measure on $\mathbb{R}_+ = [0,\infty)$ such that $d\mu_\Delta(\lambda) = \mu_{[\lambda]+1} d\lambda$ and define a function Δ_λ on the unit disk $\mathbb{D}$ by the formula

$$\Delta_\lambda(\zeta) := \exp\left\{-\sum_{k=1}^{[\lambda]} \mu_k \frac{t_k+\zeta}{t_k-\zeta} - (\lambda - [\lambda])\mu_{[\lambda]+1} \frac{t_{[\lambda]+1}+\zeta}{t_{[\lambda]+1}-\zeta}\right\},$$

where $[\lambda]$ is the integer part of λ $(\lambda \in \mathbb{R}_+)$ and by definition $\Delta_0 := 1$. Then the map $V_\Delta : L^2(\mu_\Delta) \to \mathcal{H}(\Delta)$ defined by

$$(V_\Delta c)(\zeta) := \sqrt{2}\int_{\mathbb{R}_+} c(\lambda)\Delta_\lambda(\zeta)(1 - \bar{t}_{[\lambda]+1}\zeta)^{-1} d\mu_\Delta(\lambda) \quad (\zeta \in \mathbb{D}) \tag{3.6}$$

is unitary and U_Δ is mapped onto the operator

$$V_\Delta^* U_\Delta V_\Delta = (I - J_\Delta)M_\Delta, \tag{3.7}$$

where $(M_\Delta c)(\lambda) := t_{[\lambda]+1} c(\lambda)$, $(\lambda \in \mathbb{R}_+)$ is a multiplication operator and

$$(J_\Delta c)(\lambda) := 2\int_0^\lambda c(t) \frac{\Delta_\lambda(0)}{\Delta_t(0)} d\mu_\Delta(t) \quad (\lambda \in \mathbb{R}_+)$$

is a lower-triangular Hilbert–Schmidt operator.

Collecting the above three cases we get:

Triangularization theorem (*[54, p. 123]*)**.** *Let θ be an inner function with the canonical factorization $\theta = B \cdot s \cdot \Delta$, where B is a Blaschke product, and s and Δ are singular functions with representing measures μ_s and μ_Δ respectively, with μ_s continuous and μ_Δ a pure point measure. Then the map $V : L^2(\mu_B) \times L^2(\mu_s) \times L^2(\mu_\Delta) \to \mathcal{H}(\theta)$ defined by*

$$V := \begin{bmatrix} V_B & 0 & 0 \\ 0 & BV_s & 0 \\ 0 & 0 & BsV_\Delta \end{bmatrix} \tag{3.8}$$

is unitary, where $V_B, \mu_B, V_S, \mu_S, V_\Delta, \mu_\Delta$ are defined in (3.2)–(3.7) and U_θ is mapped onto the operator

$$M := V^* U_\theta V = \begin{bmatrix} M_B & 0 & 0 \\ 0 & M_s & 0 \\ 0 & 0 & M_\Delta \end{bmatrix} + J,$$

where M_B, M_S, M_Δ are defined in (3.3), (3.5) and (3.7) and

$$J := -\begin{bmatrix} J_B M_B & 0 & 0 \\ 0 & J_s M_s & 0 \\ 0 & 0 & J_\Delta M_\Delta \end{bmatrix} + A$$

is a lower-triangular Hilbert–Schmidt operator, with $A^3 = 0, \operatorname{rank} A \leq 3$.

If $\Phi \in L^\infty_{M_n}$, then by (1.3),

$$[T^*_\Phi, T_\Phi] = H^*_{\Phi^*} H_{\Phi^*} - H^*_\Phi H_\Phi + T_{\Phi^*\Phi - \Phi\Phi^*}.$$

Since the normality of Φ is a necessary condition for the hyponormality of T_Φ, the positivity of $H^*_{\Phi^*} H_{\Phi^*} - H^*_\Phi H_\Phi$ is an essential condition for the hyponormality of T_Φ. Thus, we isolate this property as a new notion, weaker than hyponormality. The reader will notice at once that this notion is meaningful for non-scalar symbols.

Definition 3.1. Let $\Phi \in L^\infty_{M_n}$. The *pseudo-self-commutator* of T_Φ is defined by

$$[T^*_\Phi, T_\Phi]_p := H^*_{\Phi^*} H_{\Phi^*} - H^*_\Phi H_\Phi.$$

T_Φ is said to be *pseudo-hyponormal* if $[T^*_\Phi, T_\Phi]_p$ is positive semi-definite.

As in the case of hyponormality of scalar Toeplitz operators, we can see that the pseudo-hyponormality of T_Φ is independent of the constant matrix term $\Phi(0)$. Thus whenever we consider the pseudo-hyponormality of T_Φ we may assume that $\Phi(0) = 0$. Observe that if $\Phi \in L^\infty_{M_n}$ then

$$[T^*_\Phi, T_\Phi] = [T^*_\Phi, T_\Phi]_p + T_{\Phi^*\Phi - \Phi\Phi^*}.$$

We thus have

$$T_\Phi \text{ is hyponormal} \iff T_\Phi \text{ is pseudo-hyponormal and } \Phi \text{ is normal};$$

and (via [34, Theorem 3.3]) T_Φ is pseudo-hyponormal if and only if $\mathcal{E}(\Phi) \neq \emptyset$.

For $\Phi \equiv \Phi^*_- + \Phi_+ \in L^\infty_{M_n}$, we write

$$\mathcal{C}(\Phi) := \Big\{ K \in H^\infty_{M_n} : \Phi - K\Phi^* \in H^\infty_{M_n} \Big\}.$$

Thus if $\Phi \in L^\infty_{M_n}$ then

$$K \in \mathcal{E}(\Phi) \iff K \in \mathcal{C}(\Phi) \quad \text{and} \quad \|K\|_\infty \le 1.$$

Also if $K \in \mathcal{C}(\Phi)$ then $H_{\Phi^*_-} = H_{K\Phi^*_+} = T^*_{\widetilde{K}} H_{\Phi^*_+}$, which gives a necessary condition for the nonemptyness of $\mathcal{C}(\Phi)$ (and hence the hyponormality of T_Φ): in other words,

$$K \in \mathcal{C}(\Phi) \Longrightarrow \ker H_{\Phi^*_+} \subseteq \ker H_{\Phi^*_-}. \tag{3.9}$$

We begin with:

Proposition 3.2. *Let $\Phi \equiv \Phi^*_- + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Thus we may write*

$$\Phi_+ = \Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_2 B^*,$$

where $\Theta_i = I_{\theta_i}$ for an inner function θ_i $(i = 1, 2)$ and $A, B \in H^2_{M_n}$. If $\mathcal{C}(\Phi) \neq \emptyset$, then Θ_2 is an inner divisor of Θ_1, i.e., *$\Theta_1 = \Theta_0\Theta_2$ for some inner function Θ_0.*

Proof. In view of (2.6) we may write

$$\Phi_+ \equiv \big[\theta_1 \overline{a}_{ij}\big]_{n\times n} \quad \text{and} \quad \Phi_- = \big[\theta_2 \overline{b}_{ij}\big]_{n\times n} = \big[\theta_{ij} \overline{c}_{ij}\big]_{n\times n},$$

where each θ_{ij} is an inner function, $c_{ij} \in H^2$, θ_{ij} and c_{ij} are coprime, and θ_2 is the least common multiple of θ_{ij}'s. Suppose $\mathcal{C}(\Phi) \neq \emptyset$. Then there exists a matrix function $K \in H^\infty_{M_n}$ such that $\Phi_-^* - K\Phi_+^* \in H^2_{M_n}$. Thus $B\Theta_2^* - KA\Theta_1^* \in H^2_{M_n}$, which implies that

$$B\Theta_2^*\Theta_1 = \left[b_{ji}\overline{\theta}_2\theta_1\right] \in H^2_{M_n}.$$

But since $\theta_2\overline{b}_{ij} = \theta_{ij}\overline{c}_{ij}$, and hence $b_{ij} = \theta_2\overline{\theta}_{ij}c_{ij}$, it follows that

$$b_{ji}\overline{\theta}_2\theta_1 = \left(\theta_2\overline{\theta}_{ji}c_{ji}\right)\overline{\theta}_2\theta_1 = \overline{\theta}_{ji}c_{ji}\theta_1 \in H^\infty.$$

Since θ_{ji} and c_{ji} are coprime, we have that

$$\overline{\theta}_{ji}\theta_1 \in H^\infty, \quad \text{and hence} \quad \overline{\theta}_2\theta_1 \in H^\infty,$$

which implies that Θ_2 divides Θ_1. □

Proposition 3.2 shows that the hyponormality of T_φ with scalar-valued rational symbol φ implies

$$\deg(\varphi_-) \le \deg(\varphi_+),$$

which is a generalization of the well-known result for the cases of the trigonometric Toeplitz operators, i.e., if $\varphi = \sum_{n=-m}^N a_n z^n$ is such that T_φ is hyponormal then $m \le N$ (cf. [28]).

In view of Proposition 3.2, when we study the hyponormality of block Toeplitz operators with *bounded type symbols* Φ (i.e., Φ and Φ^* are of bounded type) we may assume that the symbol $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is of the form

$$\Phi_+ = \Theta_1\Theta_0A^* \quad \text{and} \quad \Phi_- = \Theta_1B^*,$$

where $\Theta_i := I_{\theta_i}$ for an inner function θ_i $(i = 0, 1)$ and $A, B \in H^2_{M_n}$.

Our criterion is as follows:

Theorem 3.3. *Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be normal such that Φ and Φ^* are of bounded type of the form*

$$\Phi_+ = \Theta_1\Theta_0A^* \quad \text{and} \quad \Phi_- = \Theta_1B^*,$$

where $\Theta_i = I_{\theta_i}$ for an inner function θ_i $(i = 0, 1)$ and $A, B \in H^2_{M_n}$. Write

$$\begin{cases} V : L \equiv L^2(\mu_B) \times L^2(\mu_s) \times L^2(\mu_\Delta) \to \mathcal{H}(\theta_1\theta_0) \text{ is unitary as in (3.8)}; \\ M := V^*U_{\theta_1\theta_0}V; \\ \mathcal{L} := L \otimes \mathbb{C}^n \\ \mathcal{V} := V \otimes I_n. \end{cases} \tag{3.10}$$

If $K \in \mathcal{C}(\Phi)$ then

$$[T_\Phi^*, T_\Phi] = (T_A)^*_{\Theta_1\Theta_0}\mathcal{V}\Big(I|_{\mathcal{L}} - K(M)^*K(M)\Big)\mathcal{V}^*(T_A)_{\Theta_1\Theta_0} \bigoplus 0|_{\Theta_1\Theta_0 H^2_{\mathbb{C}^n}},$$

where $K(M)$ is understood as an H^∞-functional calculus. Hence, in particular,

$$K(M) \text{ is contractive} \Longrightarrow T_\Phi \text{ is hyponormal};$$

the converse is also true if $(T_A)_{\Theta_1\Theta_0}$ has dense range, and in this case,

$$\operatorname{rank}\,[T_\Phi^*, T_\Phi] = \operatorname{rank}\Big(I|_{\mathcal{L}} - K(M)^*K(M)\Big).$$

Proof. Let $E, F \in \mathcal{H}(\Theta_1\Theta_0)$ and $K \in \mathcal{C}(\Phi)$. Since $\ker H_{\Theta_1^*\Theta_0^*} = \Theta_1\Theta_0 H^2_{\mathbb{C}^n}$, we have

$$H_{\Theta_1^*\Theta_0^*K}E = H_{\Theta_1^*\Theta_0^*}\big(P_{\mathcal{H}(\Theta_1\Theta_0)}(KE)\big).$$

Since $H^*_{\Theta_1^*\Theta_0^*}H_{\Theta_1^*\Theta_0^*}$ is the projection onto $\mathcal{H}(\Theta_1\Theta_0)$, it follows that

$$\begin{aligned}\Big\langle H^*_{\Theta_1^*\Theta_0^*K}H_{\Theta_1^*\Theta_0^*K}E,\ F\Big\rangle &= \big\langle P_{\mathcal{H}(\Theta_1\Theta_0)}KE,\ P_{\mathcal{H}(\Theta_1\Theta_0)}KF\big\rangle\\ &= \big\langle (T_K)_{\Theta_1\Theta_0}E,\ (T_K)_{\Theta_1\Theta_0}F\big\rangle,\end{aligned}$$

which gives

$$H^*_{\Theta_1^*\Theta_0^*K}H_{\Theta_1^*\Theta_0^*K}|_{\mathcal{H}(\Theta_1\Theta_0)} = (T_K)^*_{\Theta_1\Theta_0}(T_K)_{\Theta_1\Theta_0}.$$

Observe that $[T^*_\Phi, T_\Phi] = H^*_{A\Theta_0^*\Theta_1^*}H_{A\Theta_0^*\Theta_1^*} - H^*_{B\Theta_1^*}H_{B\Theta_1^*}$ because Φ is normal. But since

$$\operatorname{cl\,ran}\big(H^*_{A\Theta_0^*\Theta_1^*}H_{A\Theta_0^*\Theta_1^*}\big) = \Big(\ker H_{A\Theta_0^*\Theta_1^*}\Big)^\perp \subseteq \big(\Theta_1\Theta_0H^2_{\mathbb{C}^n}\big)^\perp = \mathcal{H}(\Theta_1\Theta_0)$$

and

$$\operatorname{cl\,ran}\big(H^*_{B\Theta_1^*}H_{B\Theta_1^*}\big) = \Big(\ker H_{B\Theta_1^*}\Big)^\perp \subseteq \big(\Theta_1H^2_{\mathbb{C}^n}\big)^\perp = \mathcal{H}(\Theta_1),$$

we have $\operatorname{ran}[T^*_\Phi, T_\Phi] \subseteq \mathcal{H}(\Theta_1\Theta_0)$. But since $\Phi - K\Phi^* \in H^\infty_{M_n}$, it follows that on $\mathcal{H}(\Theta_1\Theta_0)$,

$$\begin{aligned}[T^*_\Phi, T_\Phi] &= \Big(H^*_{\Phi^*}H_{\Phi^*} - H^*_{\Phi}H_{\Phi}\Big)\Big|_{\mathcal{H}(\Theta_1\Theta_0)}\\ &= \Big(H^*_{\Phi^*}H_{\Phi^*} - H^*_{K\Phi^*}H_{K\Phi^*}\Big)\Big|_{\mathcal{H}(\Theta_1\Theta_0)}\\ &= \Big(H^*_{\Theta_1^*\Theta_0^*A+\Phi_-}H_{\Theta_1^*\Theta_0^*A+\Phi_-} - H^*_{K(\Theta_1^*\Theta_0^*A+\Phi_-)}H_{K(\Theta_1^*\Theta_0^*A+\Phi_-)}\Big)\Big|_{\mathcal{H}(\Theta_1\Theta_0)}\\ &= T^*_{(A+\Theta_1\Theta_0\Phi_-)}\Big(H^*_{\Theta_1^*\Theta_0^*}H_{\Theta_1^*\Theta_0^*} - H^*_{K\Theta_1^*\Theta_0^*}H_{K\Theta_1^*\Theta_0^*}\Big)\\ &\quad\times T_{(A+\Theta_1\Theta_0\Phi_-)}\Big|_{\mathcal{H}(\Theta_1\Theta_0)}\\ &= (T_A)^*_{\Theta_1\Theta_0}\Big(I|_{\mathcal{H}(\Theta_1\Theta_0)} - (T_K)^*_{\Theta_1\Theta_0}(T_K)_{\Theta_1\Theta_0}\Big)(T_A)_{\Theta_1\Theta_0},\end{aligned}$$

where $(T_A)_{\Theta_1\Theta_0}$ is understood in the sense that the compression $(T_A)_{\Theta_1\Theta_0}$ is bounded even though T_A is possibly unbounded; in fact,

$$P_{\mathcal{H}(\Theta_1\Theta_0)}T_{(A+\Theta_1\Theta_0\Phi_-)}|_{\mathcal{H}(\Theta_1\Theta_0)} = P_{\mathcal{H}(\Theta_1\Theta_0)}T_A|_{\mathcal{H}(\Theta_1\Theta_0)}.$$

On the other hand, since $K(z) \equiv \big[k_{rs}(z)\big]_{1\le r,s\le n} \in H^\infty_{M_n}$, we may write

$$K(z) = \sum_{i=0}^\infty K_iz^i \quad (K_i \in M_n).$$

We also write $k_{rs}(z) := \sum_0^\infty c_i^{(rs)}z^i$ and then $K_i = \Big[c_i^{(rs)}\Big]_{1\le r,s\le n}$. We thus have

$$\begin{aligned}(T_K)_{\Theta_1\Theta_0} &= P_{\mathcal{H}(\Theta_1\Theta_0)}T_K|_{\mathcal{H}(\Theta_1\Theta_0)}\\ &= \big[P_{\mathcal{H}(\theta_1\theta_0)}T_{k_{rs}}|_{\mathcal{H}(\theta_1\theta_0)}\big]_{1\le r,s\le n}\end{aligned}$$

$$
\begin{aligned}
&= \left[\sum_{i=0}^{\infty} c_i^{(rs)} P_{\mathcal{H}(\theta_1\theta_0)} T_{z^i}|_{\mathcal{H}(\theta_1\theta_0)} \right]_{1\le r,s\le n} \\
&= \left[\sum_{i=0}^{\infty} c_i^{(rs)} \Big(P_{\mathcal{H}(\theta_1\theta_0)} T_z|_{\mathcal{H}(\theta_1\theta_0)} \Big)^i \right]_{1\le r,s\le n} \qquad (\text{because } \theta_1\theta_0 H^2 \subseteq \ \operatorname{Lat} T_z) \\
&= \sum_{i=0}^{\infty} \Big(U^i_{\theta_1\theta_0} \otimes \left[c_i^{(rs)} \right]_{1\le r,s\le n} \Big) \\
&= \sum_{i=0}^{\infty} \Big(U^i_{\theta_1\theta_0} \otimes K_i \Big).
\end{aligned}
$$

Let $\{\phi_j\}$ be an orthonormal basis for $\mathcal{H}(\theta_1\theta_0)$ and put $e_j := V^*\phi_j$. Then $\{e_j\}$ forms an orthonormal basis for $L^2(\mu_B) \times L^2(\mu_s) \times L^2(\mu_\Delta)$. Thus for each $f \in \mathbb{C}^n$, we have $\mathcal{V}(e_j \otimes f) = \phi_j \otimes f$. It thus follows that

$$
\begin{aligned}
\big\langle (T_K)_{\Theta_1\Theta_0}(\phi_j \otimes f), \phi_k \otimes g \big\rangle &= \sum_{i=0}^{\infty} \Big\langle (U^i_{\theta_1\theta_0} \otimes K_i)(\phi_j \otimes f),\ \phi_k \otimes g \Big\rangle \\
&= \sum_{i=0}^{\infty} \Big\langle (U^i_{\theta_1\theta_0}\phi_j) \otimes (K_i f),\ \phi_k \otimes g \Big\rangle \\
&= \sum_{i=0}^{\infty} \Big\langle U^i_{\theta_1\theta_0} V e_j,\ V e_k \Big\rangle \langle K_i f, g \rangle \\
&= \sum_{i=0}^{\infty} \Big\langle (M^i \otimes K_i)(e_j \otimes f),\ e_k \otimes g \Big\rangle \\
&= \langle K(M)(e_j \otimes f),\ e_k \otimes g \rangle,
\end{aligned}
$$

which implies that

$$
\mathcal{V}^*(T_K)_{\Theta_1\Theta_0}\mathcal{V} = K(M). \tag{3.11}
$$

Here $K(M)$ is understood as a H^∞-functional calculus (so called the Sz.-Nagy-Foiaş functional calculus) because M is an absolutely continuous contraction: in fact, we claim that

> every compression of the shift operator is completely non-unitary.

To see this, write $P_{\mathcal{X}} U P_{\mathcal{X}}$ for the compression of U to $\mathcal{X} \equiv P_{\mathcal{X}} H^2$ with some projection $P_{\mathcal{X}}$. We assume to the contrary that $P_{\mathcal{X}} U P_{\mathcal{X}}$ has a unitary summand W acting on a closed subspace $\mathcal{Y} \subseteq \mathcal{X}$. But since $\|U\| = 1$, we must have that $P_{\mathcal{Y}^\perp} U|_{\mathcal{Y}} = 0$. Thus we can see that $\mathcal{Y}$ is an invariant subspace of U. Thus, by Beurling's Theorem, $\mathcal{Y} = \theta H^2$ for some inner function θ. But then $W(\theta H^2) = z\theta H^2$, and hence W is not surjective, a contradiction. Hence every compression of the shift operator is completely non-unitary. We can therefore conclude that

$$
[T_\Phi^*,\ T_\Phi] = (T_A)^*_{\Theta_1\Theta_0} \mathcal{V}\Big(I|_{\mathcal{L}} - K(M)^* K(M) \Big) \mathcal{V}^* (T_A)_{\Theta_1\Theta_0} \bigoplus 0|_{\Theta_1\Theta_0 H^2_{\mathbb{C}^n}}.
$$

The remaining assertions follow trivially from the first assertion. □

If Φ is a scalar-valued function then Theorem 3.3 reduces to the following corollary.

Corollary 3.4. *Let $\varphi \equiv \varphi_-^* + \varphi_+ \in L^\infty$ be such that φ and $\overline{\varphi}$ are of bounded type of the form*

$$\varphi_+ = \theta_1\theta_0\overline{a} \quad \textit{and} \quad \varphi_- = \theta_1\overline{b},$$

where θ_1 and θ_0 are inner functions and $a, b \in H^2$. If $k \in \mathcal{C}(\varphi)$ then

$$T_\varphi \textit{ is hyponormal} \iff k(M) \textit{ is contractive},$$

where M is defined as in (3.10).

Proof. By Theorem 3.3, it suffices to show that $(T_a)_{\theta_1\theta_0}$ has dense range. To prove this suppose $(T_a)^*_{\theta_1\theta_0} f = 0$ for some $f \in \mathcal{H}(\theta_1\theta_0)$. Then $P_{\mathcal{H}(\theta_1\theta_0)}(\overline{a}f) = 0$, i.e., $\overline{a}f = \theta_1\theta_0 h$ for some $h \in H^2$. Thus we have $\overline{a}\overline{\theta}_1\overline{\theta}_0 f \in (H^2)^\perp \cap H^2 = \{0\}$, which implies that $f = 0$. Therefore $(T_a)^*_{\theta_1\theta_0}$ is 1–1, which gives the result. □

Remark 3.5. We note that in Corollary 3.4, if φ is a rational function then M is a finite matrix. Indeed, if φ is a rational function, and hence $\theta_1\theta_0$ is a finite Blaschke product of the form

$$\theta_1\theta_0 = \prod_{j=1}^{d} \frac{z-\alpha_j}{1-\overline{\alpha_j}z},$$

then M is obtained by (2.15). □

Remark 3.6. We mention that $K \in \mathcal{C}(\Phi)$ may not be contractive, i.e., there might exist a function $K \in \mathcal{C}(\Phi) \setminus \mathcal{E}(\Phi)$. In spite of this, Theorem 3.3 guarantees that

$$I - K(M)^*K(M)$$

is unchanged regardless of the particular choice of K in $\mathcal{C}(\Phi)$. We will illustrate this phenomenon with a scalar-valued Toeplitz operator with a trigonometric polynomial symbol. For example, let

$$\Phi(z) = z^{-2} + 2z^{-1} + z + 2z^2.$$

If $K(z) = \frac{1}{2} + \frac{3}{4}z$, then $\Phi_-^* - K\Phi_+^* = (z^{-2} + 2z^{-1}) - (\frac{1}{2} + \frac{3}{4}z)(z^{-1} + 2z^{-2}) = -\frac{3}{4} \in H^\infty$, so that $K \in \mathcal{C}(\Phi)$, but $\|K\|_\infty = \frac{5}{4} > 1$. However by (2.15) we have

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and hence,} \quad K(M) = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix},$$

so that

$$I - K(M)^*K(M) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{13}{16} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{3}{16} & -\frac{3}{8} \\ -\frac{3}{8} & \frac{3}{4} \end{bmatrix} \geq 0,$$

which implies that T_Φ is hyponormal even though $\|K\|_\infty > 1$. Of course, in view of Cowen's Theorem, there exists a function $b \in \mathcal{E}(\Phi)$: indeed, $b(z) = \frac{z+\frac{1}{2}}{1+\frac{1}{2}z} \in \mathcal{E}(\Phi)$. □

We now provide some revealing examples that illustrate Theorem 3.3.

Example 3.7. Let Δ be a singular inner function of the form

$$\Delta := \exp\left(\frac{z+1}{z-1}\right)$$

and consider the matrix-valued function

$$\Phi := \begin{bmatrix} \overline{\Delta} & \overline{z\Delta} + z\Delta \\ \overline{z\Delta} + z\Delta & \overline{\Delta} \end{bmatrix}.$$

We now use Theorem 3.3 to determine the hyponormality of T_Φ. Under the notation of Theorem 3.3 we have

$$\Theta_1 = I_{z\Delta}, \qquad \Theta_0 = I_2, \qquad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} z & 1 \\ 1 & z \end{bmatrix}.$$

If we put

$$K(z) := \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix},$$

then a straightforward calculation shows that $K \in \mathcal{C}(\Phi)$. Under the notation of Theorem 3.3 we can see that

$$(T_A)_{I_{z\Delta}} = \begin{bmatrix} 0 & I|_L \\ I|_L & 0 \end{bmatrix} : \mathcal{L} \to \mathcal{L} \text{ is invertible.}$$

But since

$$K(M) = \begin{bmatrix} I|_L & M \\ M & I|_L \end{bmatrix},$$

it follows that

$$I|_{\mathcal{L}} - K(M)^* K(M) = -\begin{bmatrix} M^*M & M + M^* \\ M + M^* & M^*M \end{bmatrix},$$

which is not positive (simply by looking at the upper-left entry). It therefore follows from Theorem 3.3 that T_Φ is not hyponormal. $\square$

Example 3.8. Let Δ be a singular inner function of the form

$$\Delta := \exp\left(\frac{z+1}{z-1}\right)$$

and let

$$\Omega := \Delta^{\frac{1}{2}} = \exp\left(\frac{1}{2} \cdot \frac{z+1}{z-1}\right).$$

Consider the function

$$\varphi := \frac{4}{5}\overline{z} + \frac{8}{5}\overline{\Delta} + \frac{37}{50}\left(z^2\overline{\Omega} - \frac{1}{\sqrt{e}}z^2 + \frac{1}{\sqrt{e}}z\right) + \frac{29}{25}(z + 2\Delta).$$

Observe that

$$\overline{\varphi_-} = \frac{4}{5}\overline{z} + \frac{8}{5}\overline{\Delta} + \frac{37}{50}(I - P)(z^2\overline{\Omega}) + c \quad (c \in \mathbb{C}).$$

We thus have

$$k := \frac{25}{29}\left(\frac{4}{5} + \frac{37}{100} z^2 \Omega\right) \in \mathcal{C}(\varphi),$$

but

$$\|k\|_\infty = \frac{117}{116} > 1.$$

Thus by the aid of such a function k, we cannot determine the hyponormality of T_φ. Using the notation in the triangularization theorem we can show that

(i) $L^2(\mu_\Delta) = L^2(0,1)$ and $L^2(\mu_B) = \mathbb{C}$;
(ii) $M_\Delta = I$ and $M_B = 0$;
(iii) $(J_\Delta c)(\lambda) = 2\int_0^\lambda e^{t-\lambda} c(t)\,dt$ for $\lambda \in (0,1)$ and $c \in L^2(0,1)$;
(iv) $V_z = \frac{1}{\sqrt{2}}$ and $(V_\Delta c)(\zeta) = \sqrt{2}\int_0^1 c(\lambda)\exp\bigl(-\lambda\frac{1+\zeta}{1-\zeta}\bigr)(1-\zeta)^{-1} d\lambda,\ \zeta \in \mathbb{D},\ c \in L^2(0,1)$;
(v) $M \equiv M_B \times M_\Delta + J : L \to L$, where $L \equiv \mathbb{C} \oplus L^2(0,1) \cong \mathcal{H}(I_{z\Delta})$.

Note that $\mathcal{H}(z\Delta) = \mathcal{H}(z) \oplus z\mathcal{H}(\Delta)$, so that $P_{\mathcal{H}(z\Delta)} = P_{\mathcal{H}(z)} + zP_{\mathcal{H}(\Delta)}\overline{z}$. We then have

$$U_{z\Delta} = \begin{bmatrix} U_z & 0 \\ a & zU_\Delta \overline{z} \end{bmatrix} : \begin{bmatrix} \mathcal{H}(z) \\ z\mathcal{H}(\Delta) \end{bmatrix} \to \begin{bmatrix} \mathcal{H}(z) \\ z\mathcal{H}(\Delta) \end{bmatrix}.$$

Since $U_z = 0$, it follows that

$$M = \begin{bmatrix} V_z & 0 \\ 0 & zV_\Delta \end{bmatrix}^* \begin{bmatrix} 0 & 0 \\ a & zU_\Delta \overline{z} \end{bmatrix} \begin{bmatrix} V_z & 0 \\ 0 & zV_\Delta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ (zV_\Delta)^* a V_z & I - J_\Delta \end{bmatrix}.$$

By using the fact that $P_{\mathcal{H}(\theta)} = I - \theta P \overline{\theta}$, we can compute:

$$a = P_{z\mathcal{H}(\Delta)}(z \cdot 1) = zP_{\mathcal{H}(\Delta)}\overline{z}(z \cdot 1) = zP_{\mathcal{H}(\Delta)}(1) = z(I - \Delta P \overline{\Delta})(1) = z\left(1 - \frac{\Delta}{e}\right).$$

Thus we have

$$M = \begin{bmatrix} 0 & 0 \\ V_\Delta^* \left(1 - \frac{\Delta}{e}\right) V_z & I - J_\Delta \end{bmatrix}.$$

Write

$$A := V_\Delta^* \left(1 - \frac{\Delta}{e}\right) V_z.$$

Then we have

$$\begin{aligned} \|A\|^2 &= \left\|1 - \frac{\Delta}{e}\right\|^2 = \left\|1 - \frac{\Delta(0)}{e} - \frac{1}{e}\bigl(\Delta - \Delta(0)\bigr)\right\|^2 \\ &= \left(1 - \frac{\Delta(0)}{e}\right)^2 + \frac{1}{e^2}\left(\|\Delta\|^2 - |\Delta(0)|^2\right) \\ &= \left(1 - \frac{1}{e^2}\right)^2 + \frac{1}{e^2}\left(1 - \frac{1}{e^2}\right) = 1 - \frac{1}{e^2}. \end{aligned}$$

A straightforward calculation shows that

$$k(M) = \frac{25}{29}\begin{bmatrix} \frac{4}{5} & 0 \\ S & \left(\frac{4}{5} + \frac{37}{100}z^2\Omega\right)(I - J_\Delta) \end{bmatrix},$$

where

$$S := \left(\left(\frac{37}{100}z\Omega\right)(I - J_\Delta)\right)A.$$

On the other hand, consider the function

$$\varphi_1(z) := (I - P)(z\overline{\Omega}) + \Delta.$$

Then $q := z\Omega \in \mathcal{E}(\varphi_1)$. Since $M_\Delta = 1$, it follows from Corollary 3.4 that $q(M) = (z\Omega)(I - J_\Delta)$ is a contraction. Thus we have

$$\|S\| = \frac{37}{100}\|(z\Omega)(I - J_\Delta)A\| \le \frac{37}{100}\sqrt{1 - \frac{1}{e^2}}.$$

Also we consider the function

$$\varphi_2(z) := \frac{4}{5}\overline{\Delta} + \frac{9}{25}(I - P)(z^2\,\overline{\Omega}) + \Delta.$$

Put

$$B(z) := \frac{z^2\Omega + \frac{4}{5}}{1 + \frac{4}{5}z^2\Omega}.$$

Since $B(z) = \frac{4}{5} + \frac{9}{25}z^2\Omega + \Delta g$ for some $g \in H^2$, we have

$$\begin{aligned}(\varphi_2)_- - B\,\overline{(\varphi_2)_+} &= \frac{4}{5}\overline{\Delta} + \frac{9}{25}(I - P)(z^2\,\overline{\Omega}) - \left(\frac{4}{5} + \frac{9}{25}z^2\Omega + \Delta g\right)\overline{\Delta} \\ &= -\frac{9}{25}P(z^2\Omega) - g \in H^2.\end{aligned}$$

Since $\|B\|_\infty = 1$, it follows that T_{φ_2} is hyponormal. In particular, since

$$r := \frac{4}{5} + \frac{9}{25}z^2\Omega \in \mathcal{C}(\varphi_2),$$

it follows from Corollary 3.4 that $r(M) = (\frac{4}{5} + \frac{9}{25}z^2\Omega)(I - J_\Delta)$ is a contraction. Thus we have that

$$\begin{aligned}\left\|\left(\frac{4}{5} + \frac{37}{100}z^2\Omega\right)(I - J_\Delta)\right\| &\le \left\|\left(\frac{4}{5} + \frac{9}{25}z^2\Omega\right)(I - J_\Delta)\right\| + \frac{1}{100}\|(z^2\Omega)(I - J_\Delta)\| \\ &\le \frac{101}{100}.\end{aligned}$$

Using the observation that if A, B, C and D are operators then

$$\left\|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right\| \le \left\|\begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix}\right\|,$$

we can see that

$$\|k(M)\| \le \frac{25}{29}\left\|\begin{bmatrix} \frac{4}{5} & 0 \\ \|S\| & \left\|\left(\frac{4}{5}+\frac{9}{25}z^2\Omega\right)(I-J_\Delta)\right\| \end{bmatrix}\right\|$$

$$\le \frac{25}{29}\left\|\begin{bmatrix} \frac{4}{5} & 0 \\ \frac{37}{100}\sqrt{1-\frac{1}{e^2}} & \frac{101}{100} \end{bmatrix}\right\| \approx 0.968 < 1,$$

which, by Corollary 3.4, implies that T_φ is hyponormal. □

We now consider the condition "$\mathcal{C}(\Phi) \neq \emptyset$", i.e., the existence of a function $K \in H^\infty_{M_n}$ such that $\Phi - K\Phi^* \in H^\infty_{M_n}$. In view of (3.9), we may assume that

$$\ker H_{\Phi_+^*} \subseteq \ker H_{\Phi_-^*} \tag{3.12}$$

whenever we study the hyponormality of T_Φ. Recall [31, Corollary 2] that

$$H^*_{\Phi^*}H_{\Phi^*} - H^*_\Phi H_\Phi \ge 0 \iff \exists\, K \in H^\infty_{M_n} \text{ with } \|K\|_\infty \le 1 \text{ such that } H_{\Phi_-^*} = T^*_{\widetilde{K}} H_{\Phi_+^*}. \tag{3.13}$$

We thus have

$$\begin{aligned} \mathcal{C}(\Phi) \neq \emptyset &\iff \exists\, K \in H^\infty_{M_n} \quad \text{such that } \Phi_-^* - K\Phi_+^* \in H^2_{M_n} \\ &\iff H_{\Phi_-^*} = T^*_{\widetilde{K}}H_{\Phi_+^*} \quad \text{for some } K \in H^\infty_{M_n} \\ &\iff H^*_{\alpha\Phi_+^*}H_{\alpha\Phi_+^*} - H^*_{\Phi_-^*}H_{\Phi_-^*} \ge 0 \quad \text{for some } \alpha > 0 \text{ (by (3.13))} \\ &\iff \ker H_{\Phi_+^*} \subseteq \ker H_{\Phi_-^*} \quad \text{and} \\ &\sup\left\{\frac{\|H_{\Phi_-^*}F\|}{\|H_{\Phi_+^*}F\|} : F \in \ker(H_{\Phi_+^*})^\perp,\ \|F\| = 1\right\} \le \alpha. \end{aligned} \tag{3.14}$$

If $\Phi \in L^\infty_{M_n}$ is a rational function then by Kronecker's Lemma (cf. [54, p. 183]), ran $H_{\Phi_+^*}$ is finite dimensional. Thus by (3.14) we can see that

$$\mathcal{C}(\Phi) \neq \emptyset \iff \ker H_{\Phi_+^*} \subseteq \ker H_{\Phi_-^*}.$$

Consequently, if $\Phi \in L^\infty_{M_n}$ is a rational function then there always exists a function $K \in \mathcal{C}(\Phi)$ under the kernel assumption (3.12). We record this in

Proposition 3.9. *If $\Phi \in L^\infty_{M_n}$ is a rational function satisfying $\ker H_{\Phi_+^*} \subseteq \ker H_{\Phi_-^*}$, then $\mathcal{C}(\Phi) \neq \emptyset$.*

We remark that there is an explicit way to find a function (in fact, a matrix-valued polynomial) K in $\mathcal{C}(\Phi)$ for the rational symbol case. To see this, in view of Proposition 3.2, suppose $\Phi \in L^\infty_{M_n}$ is of the form

$$\Phi_+ = \Theta_1\Theta_0A^* \quad \text{and} \quad \Phi_- = \Theta_1B^*,$$

where $\Theta_i = I_{\theta_i}$ for a finite Blaschke product θ_i $(i = 0, 1)$. We observe first that

$$K \in \mathcal{C}(\Phi) \Longleftrightarrow \Phi - K\Phi^* \in H^\infty_{M_n} \Longleftrightarrow \Theta_0 B - KA \in \Theta_1\Theta_0 H^\infty_{M_n}. \tag{3.15}$$

Suppose $\theta_1\theta_0$ is a finite Blaschke product of degree d of the form

$$\theta_1\theta_0 = \prod_{i=1}^{N}\left(\frac{z-\alpha_i}{1-\overline{\alpha}_i z}\right)^{m_i} \quad \left(d := \sum_{i=1}^{N} m_i\right).$$

Then the last assertion in (3.15) holds if and only if the following equations hold: for each $i = 1, \ldots, N$,

$$\begin{bmatrix} B_{i,0} \\ B_{i,1} \\ B_{i,2} \\ \vdots \\ B_{i,m_i-1} \end{bmatrix} = \begin{bmatrix} K_{i,0} & 0 & 0 & \cdots & 0 \\ K_{i,1} & K_{i,0} & 0 & \cdots & 0 \\ K_{i,2} & K_{i,1} & K_{i,0} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ K_{i,m_i-1} & K_{i,m_i-2} & \cdots & K_{i,1} & K_{i,0} \end{bmatrix} \begin{bmatrix} A_{i,0} \\ A_{i,1} \\ A_{i,2} \\ \vdots \\ A_{i,m_i-1} \end{bmatrix}, \tag{3.16}$$

where

$$K_{i,j} := \frac{K^{(j)}(\alpha_i)}{j!}, \qquad A_{i,j} := \frac{A^{(j)}(\alpha_i)}{j!} \quad \text{and} \quad B_{i,j} := \frac{(\theta_0 B)^{(j)}(\alpha_i)}{j!}.$$

Thus K is a function in $H^\infty_{M_n}$ for which

$$\frac{K^{(j)}(\alpha_i)}{j!} = K_{i,j} \quad (1 \le i \le N,\ 0 \le j < m_i), \tag{3.17}$$

where the $K_{i,j}$ are determined by the Eq. (3.16). This is exactly the classical Hermite–Fejér interpolation problem which we have introduced in Section 2. Thus the solution (2.13) for the classical Hermite–Fejér interpolation problem provides a polynomial $K \in \mathcal{C}(\Phi)$.

Therefore we get:

Proposition 3.10. *If $\Phi \in L^\infty_{M_n}$ is a rational function such that $\mathcal{C}(\Phi) \neq \emptyset$, then $\mathcal{C}(\Phi)$ contains a polynomial.*

However, by comparison with the rational symbol case, there may not exist a function $K \in \mathcal{C}(\Phi)$ if Φ is of bounded type. But we guarantee the existence of a function $K \in \mathcal{C}(\Phi)$ if the bounded type symbol Φ satisfies a certain determinant property. To see this, we recall the notion of the reduced minimum modulus. If $T \in \mathcal{B}(\mathcal{H})$ then the *reduced minimum modulus* of T is defined by

$$\gamma(T) = \begin{cases} \inf\bigl\{\|Tx\| : x \in (\ker T)^\perp,\ \|x\| = 1\bigr\} & \text{if } T \neq 0 \\ 0 & \text{if } T = 0. \end{cases}$$

It is well known [4,39] that if $T \neq 0$ then $\gamma(T) > 0$ if and only if T has closed range. We can easily show that if $S, T \in \mathcal{B}(\mathcal{H})$ and S is one–one then

$$\gamma(ST) \ge \gamma(S)\gamma(T). \tag{3.18}$$

We then have:

Proposition 3.11. *Let $\Phi \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type satisfying*

$$\ker H_{\Phi_+^*} \subseteq \ker H_{\Phi_-^*}.$$

If there exists $\delta > 0$ such that $\mathcal{M} := \left\{t : |\det \Phi_+(e^{it})| < \delta\right\}$ has measure zero then $\mathcal{C}(\Phi) \neq \emptyset$.

Proof. Suppose $\mathcal{M}$ has measure zero for some $\delta > 0$. Write

$$\Phi_+ = \Theta A^* \quad \text{(right coprime factorization)}.$$

Since $\det \Theta$ is inner, we have $|\det \Phi_+| = |\det A|$ a.e. on $\mathbb{T}$. Then by the well-known result [30, Theorem XXIII.2.4], our determinant condition shows that the multiplication operator M_A is invertible and $\gamma(M_A) > 0$, where $M_A f := Af$ for $f \in L^2_{\mathbb{C}^n}$. Since $A \in H^\infty_{M_n}$, the Toeplitz operator T_A is a restriction of M_A. Thus it follows that $\gamma(T_A) \geq \gamma(M_A) > 0$. Since $\operatorname{ran} H_{\Theta^*} = \mathcal{H}(\widetilde{\Theta})$, it follows that

$$H_{\Phi_+^*} = H_{A\Theta^*} = T^*_{\widetilde{A}} H_{\Theta^*} = T^*_{\widetilde{A}}|_{\mathcal{H}(\widetilde{\Theta})} H_{\Theta^*}.$$

Observe that

$$\begin{aligned}\gamma(H_{\Theta^*}) &= \inf\left\{\|H_{\Theta^*}F\| : F \in \mathcal{H}(\Theta), \|F\| = 1\right\}\\ &= \inf\left\{\|\Theta^* F\| : F \in \mathcal{H}(\Theta), \|F\| = 1\right\}\\ &= 1.\end{aligned}$$

We now claim that

$$T^*_{\widetilde{A}}|_{\mathcal{H}(\widetilde{\Theta})} \text{ is one–one.} \tag{3.19}$$

Indeed, since

$$\Theta H^2_{\mathbb{C}^n} = \ker H_{A\Theta^*} = \ker T^*_{\widetilde{A}} H_{\Theta^*} \quad \text{and} \quad \ker H_{\Theta^*} = \Theta H^2_{\mathbb{C}^n},$$

it follows that $T^*_{\widetilde{A}}|_{\operatorname{ran} H_{\Theta^*}} = T^*_{\widetilde{A}}|_{\mathcal{H}(\widetilde{\Theta})}$ is one–one, which gives (3.19). Now since $\gamma(T_A) > 0$ it follows from (3.18) and (3.19) that

$$\begin{aligned}\gamma(H_{\Phi^*}) = \gamma(H_{\Phi_+^*}) = \gamma(T^*_{\widetilde{A}}|_{\mathcal{H}(\widetilde{\Theta})} H_{\Theta^*}) &\geq \gamma(T^*_{\widetilde{A}}|_{\mathcal{H}(\widetilde{\Theta})})\\ \geq \gamma(T^*_{\widetilde{A}}) = \gamma(T_{\widetilde{A}}) = \gamma(T_A) &> 0.\end{aligned}$$

We thus have

$$\begin{aligned}\sup\left\{\frac{\|H_{\Phi_-^*}F\|}{\|H_{\Phi_+^*}F\|} : F \in \left(\ker H_{\Phi_+^*}\right)^\perp, \|F\| = 1\right\} &\leq \frac{\|H_\Phi\|}{\gamma(H_{\Phi^*})}\\ &< \alpha \quad \text{for some } \alpha > 0.\end{aligned} \tag{3.20}$$

Therefore, by (3.14) we can conclude that $\mathcal{C}(\Phi) \neq \emptyset$. □

4. Subnormality of block Toeplitz operators

As we saw in introduction, the Bram–Halmos criterion on subnormality [7,10] says that $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if the positive test (1.6) holds. It is easy to see that (1.6) is equivalent to the following positivity test:

$$\begin{bmatrix} I & T^* & \cdots & T^{*k} \\ T & T^*T & \cdots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \cdots & T^{*k}T^k \end{bmatrix} \geq 0 \quad (\text{all } k \geq 1). \tag{4.1}$$

Condition (4.1) provides a measure of the gap between hyponormality and subnormality. In fact the positivity condition (4.1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (4.1) for all k. For $k \geq 1$, an operator T is said to be *k-hyponormal* if T satisfies the positivity condition (4.1) for a fixed k [16]. Thus the Bram–Halmos criterion can be stated as: T is subnormal if and only if T is k-hyponormal for all $k \geq 1$. The notion of k-hyponormality has been considered by many authors with an aim at understanding the gap between hyponormality and subnormality. For instance, the Bram–Halmos criterion on subnormality indicates that 2-hyponormality is generally far from subnormality. There are special classes of operators, however, for which these two notions are equivalent. A trivial example is given by the class of operators whose square is compact (e.g., compact perturbations of nilpotent operators of nilpotency 2). Also in [21, Example 3.1], it was shown that there is no gap between 2-hyponormality and subnormality for back-step extensions of recursively generated subnormal weighted shifts.

On the other hand, in 1970, Halmos posed the following problem, listed as Problem 5 in his lectures “Ten problems in Hilbert space” [36,37]:

> Is every subnormal Toeplitz operator either normal or analytic?

A Toeplitz operator T_φ is called *analytic* if $\varphi \in H^\infty$. Any analytic Toeplitz operator is easily seen to be subnormal: indeed, $T_\varphi h = P(\varphi h) = \varphi h = M_\varphi h$ for $h \in H^2$, where M_φ is the normal operator of multiplication by φ on L^2. The question is natural because the two classes, the normal and analytic Toeplitz operators, are fairly well understood and are subnormal. Halmos’s Problem 5 has been partially answered in the affirmative by many authors (cf. [1,3,12,13,21, 22,53], and etc.). In 1984, Halmos’s Problem 5 was answered in the negative by Cowen and Long [15]: they found an analytic function ψ for which $T_{\psi+\alpha\overline{\psi}}$ $(0 < \alpha < 1)$ is subnormal—in fact, this Toeplitz operator is unitarily equivalent to a subnormal weighted shift W_β with weight sequence $\beta \equiv \{\beta_n\}$, where $\beta_n = (1 - \alpha^{2n+2})^{\frac{1}{2}}$ for $n = 0, 1, 2, \ldots$. Unfortunately, Cowen and Long’s construction does not provide an intrinsic connection between subnormality and the theory of Toeplitz operators. Until now researchers have been unable to characterize subnormal Toeplitz operators in terms of their symbols. Thus the following question is very interesting and challenging:

$$\text{Which Toeplitz operators are subnormal?} \tag{4.2}$$

The most interesting partial answer to Halmos’s Problem 5 was given by Abrahamse [1]. Abrahamse gave a general sufficient condition for the answer to Halmos’s Problem 5 to be affirmative. Abrahamse’s Theorem can be then stated as follows: *Let $\varphi = \overline{g} + f \in L^\infty$ $(f, g \in H^2)$ be such that φ or $\overline{\varphi}$ is of bounded type. If T_φ is subnormal then T_φ is normal or analytic.* In fact, it was also shown (cf. [22,23]) that every 2-hyponormal Toeplitz operator with a bounded type symbol is normal or analytic, and hence subnormal. On the other hand, very recently, the authors of [20] have extended Abrahamse’s Theorem to block Toeplitz operators.

Theorem 4.1 (*Extension of Abrahamse's Theorem; Curto et al. [20]*). *Suppose $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type of the form*

$$\Phi_- = B^*\Theta \quad (B \in H^2_{M_n};\ \Theta = I_\theta \text{ with an inner function } \theta),$$

where B and Θ are coprime. If T_Φ is hyponormal and $\ker[T_\Phi^*, T_\Phi]$ is invariant under T_Φ then T_Φ is normal or analytic. Hence, in particular, if T_Φ is subnormal then T_Φ is normal or analytic.

We note that if $n = 1$ (i.e., T_Φ is a scalar-valued Toeplitz operator), then $\Phi_- = \overline{b}\theta$ with $b \in H^2$. Thus, it automatically holds that b and θ are coprime. Consequently, if $n = 1$ then Theorem 4.1 reduces to Abrahamse's Theorem.

On the other hand, the study of square-hyponormality originated in [38, Problem 209]. It is easy to see that every power of a normal operator is normal and the same statement is true for every subnormal operator. How about hyponormal operators? [38, Problem 209] shows that there exists a hyponormal operator whose square is not hyponormal (e.g., $U^* + 2U$ for the unilateral shift U). However, as we remarked in the preceding, there exist special classes of operators for which square-hyponormality and subnormality coincide. For those classes of operators, it suffices to check the square-hyponormality to show subnormality. This certainly gives a nice answer to question (4.2). Indeed, in [21], it was shown that every hyponormal trigonometric Toeplitz operator whose square is hyponormal must be either normal or analytic, and hence subnormal. In [33], Gu showed that this result still holds for Toeplitz operators T_φ with rational symbols φ (more generally, the cases where both φ and $\overline{\varphi}$ are of bounded type).

The aim of this section is to prove that this result can be extended to the block Toeplitz operators whose symbols are matrix-valued rational functions.

We begin with:

Lemma 4.2. *Suppose $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is a matrix-valued rational function of the form*

$$\Phi_- = B^*\Theta \quad (\text{coprime factorization}) \quad \text{and} \quad \Phi_+ = \Theta\Theta_0 A^*,$$

where $\Theta = I_\theta$ and $\Theta_0 = I_{\theta_0}$ with finite Blaschke products θ, θ_0 and $A, B \in H^2_{M_n}$. If T_Φ is hyponormal then $A(\alpha)$ is invertible for each $\alpha \in \mathcal{Z}(\theta) \setminus \mathcal{Z}(\theta_0)$.

Proof. Assume to the contrary that $A(\alpha)$ is not invertible for some $\alpha \in \mathcal{Z}(\theta) \setminus \mathcal{Z}(\theta_0)$. Then by Lemma 2.3, A and $B_\alpha := I_{b_\alpha}$ are not right coprime. Thus there exists a nonconstant inner matrix function Δ such that

$$B_\alpha = \Delta_1\Delta = \Delta\Delta_1 \quad \text{and} \quad A = A_1\Delta.$$

Write $\Theta := B_\alpha\Theta' = \Theta' B_\alpha$. Then we may write $\Phi_+ = \Theta_0\Theta'\Delta_1 A_1^*$. Since T_Φ is hyponormal, it follows that

$$\Theta_0\Theta'\Delta_1 H^2_{\mathbb{C}^n} \subseteq \ker H_{\Phi_+^*} \subseteq \ker H_{\Phi_-^*} = \Theta H^2_{\mathbb{C}^n},$$

which implies that Θ is a (left) inner divisor of $\Theta_0\Theta'\Delta_1$ (cf. [29, Corollary IX.2.2]). Observe that

$$\begin{aligned}\Theta \text{ is a (left) inner divisor of } \Theta_0\Theta'\Delta_1 &\Longrightarrow \Theta^*\Theta_0\Theta'\Delta_1 \in H^2_{M_n}\\ &\Longrightarrow \Theta_0\Delta_1\Theta'\Theta^* \in H^2_{M_n}\\ &\Longrightarrow \Theta_0\Delta^* \in H^2_{M_n},\end{aligned}$$

which implies that Δ is a (right) inner divisor of Θ_0. But since B_α and Θ_0 are coprime, it follows that Δ and Θ_0 are coprime. Thus Δ is a constant unitary, a contradiction. This completes the proof. □

Lemma 4.3. *Suppose $F, B_\lambda \in H^\infty_{M_n}$ ($B_\lambda := I_{b_\lambda}$). If $G = \mathrm{GCD}_\ell\{F, B_\lambda\}$, then G is a Blaschke–Potapov factor of the form $G = b_\lambda P_N + (I - P_N)$ with*

$$N := \big(\operatorname{ran} F(\lambda)\big)^\perp.$$

Proof. By assumption, $\widetilde{G} = \mathrm{GCD}_r\{\widetilde{F}, \widetilde{B}_\lambda\}$. Then by Lemma 2.5,

$$\widetilde{G} = b_{\overline{\lambda}} P_N + (I - P_N) \quad \text{for a closed subspace } N.$$

Thus $\widetilde{F} = \widetilde{L}\widetilde{G}$ for some $\widetilde{L} \in H^2_{M_n}$, where $\widetilde{L}$ and $\widetilde{B}_\lambda \widetilde{G}^* = P_N + b_{\overline{\lambda}}(I - P_N)$ are right coprime. We argue that $\ker \widetilde{L}(\overline{\lambda}) \cap \operatorname{ran}(I - P_N) = \{0\}$. Indeed, if $\ker \widetilde{L}(\overline{\lambda}) \cap \operatorname{ran}(I - P_N) =: N_0 \neq \{0\}$ then $P_{N_0^\perp} + b_{\overline{\lambda}}(I - P_{N_0^\perp})$ would be a right inner divisor of $\widetilde{L}$ and $P_N + b_{\overline{\lambda}}(I - P_N)$ as follows:

$$\widetilde{L}(\overline{\lambda}) = \begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix} \begin{matrix} N_0^\perp \\ N_0 \end{matrix} = \begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & b_{\overline{\lambda}} \end{bmatrix} \begin{matrix} N_0^\perp \\ N_0 \end{matrix},$$

and hence,

$$\widetilde{L} = \widetilde{L}(\overline{\lambda}) + C B_{\overline{\lambda}} = D \begin{bmatrix} 1 & 0 \\ 0 & b_{\overline{\lambda}} \end{bmatrix} \begin{matrix} N_0^\perp \\ N_0 \end{matrix} \quad (\text{some } C, D \in H^2_{M_n})$$

and

$$P_N + b_{\overline{\lambda}}(I - P_N) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & b_{\overline{\lambda}} & 0 \\ 0 & 0 & b_{\overline{\lambda}} \end{bmatrix} \begin{matrix} N \\ N' \\ N_0 \end{matrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & b_{\overline{\lambda}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b_{\overline{\lambda}} \end{bmatrix} \begin{matrix} N \\ N', \\ N_0 \end{matrix}$$

where $N' := N^\perp \ominus N_0$. But since $\widetilde{F}(\overline{\lambda}) = \widetilde{L}(\overline{\lambda})\Big(b_{\overline{\lambda}} P_N + (I - P_N)\Big)(\overline{\lambda}) = \widetilde{L}(\overline{\lambda})(I - P_N)$, it follows that

$$N = \ker(I - P_N) = \ker \widetilde{F}(\overline{\lambda}) = \ker F^*(\lambda) = \big(\operatorname{ran} F(\lambda)\big)^\perp.$$

Therefore $G = b_\lambda P_N + (I - P_N)$ with $N := \big(\operatorname{ran} F(\lambda)\big)^\perp$. □

Lemma 4.4. *Let $\Phi \equiv \Phi_+ \in H^\infty_{M_n}$ be a matrix-valued rational function of the form*

$$\begin{aligned} \Phi &= \Theta \Delta_r A_r^*, \quad (\textit{right coprime factorization}), \\ &= A_\ell^* \Omega, \quad (\textit{left coprime factorization}), \end{aligned}$$

where $\Theta = I_\theta$ with a finite Blaschke product θ and Δ_r, Ω are inner matrix functions. Then Θ is an inner divisor of Ω.

Proof. Since Δ_r is a finite Blaschke–Potapov product, we may write

$$\Delta_r = \nu \prod_{m=1}^{M} \Big(b_m P_m + (I - P_m)\Big) \quad \left(b_m := \frac{z - \alpha_m}{1 - \overline{\alpha}_m z}\right).$$

Without loss of generality we may assume that $\nu = I_n$. Define

$$\theta_0 := \mathrm{GCD}\Big\{\omega : \omega \text{ is inner, } \Delta_r \text{ is an inner divisor of } \Omega = \omega I_n\Big\}.$$

Then $\theta_0 = \prod_{m=1}^{M} b_m$. Observe that

$$\begin{aligned}
\Phi &= \Theta \Delta_r A_r^* \\
&= \Theta \prod_{m=1}^{M} \Big(b_m P_m + (I - P_m) \Big) A_r^* \\
&= \prod_{m=1}^{M-1} \Big(b_m P_m + (I - P_m) \Big) B_M \Big(P_M + b_M (I - P_M) \Big)^* A_r^* \Theta \quad (B_m := I_{b_m}) \\
&= \prod_{m=1}^{M-1} \Big(b_m P_m + (I - P_m) \Big) \Big[A_r \Big(P_M + b_M (I - P_M) \Big) \Big]^* B_M \Theta.
\end{aligned}$$

If $P_M = I$, then

$$\Phi = \prod_{m=1}^{M-1} \Big(b_m P_m + (I - P_m) \Big) A_r^* B_M \Theta,$$

where Θ and A_r are coprime. If instead $P_M \neq I$, then there are two cases to consider.

Case 1: Let $\alpha_M \notin \mathcal{Z}(\theta)$. Then

$$\Phi = \prod_{m=1}^{M-1} \Big(b_m P_m + (I - P_m) \Big) A_1^* B_M \Theta \quad (\text{with } A_1 := A_r (P_M + b_M (I - P_M))),$$

where Θ and A_1 are coprime (by passing to Lemma 2.3).

Case 2: Let $\alpha_M \in \mathcal{Z}(\theta)$. Write $\Omega_M := \mathrm{GCD}_\ell \{ B_M,\ A_r \big(P_M + b_M (I - P_M) \big) \}$. Then we can write

$$B_M = \Omega_M \Omega_M' \quad \text{and} \quad A_r \Big(P_M + b_M (I - P_M) \Big) = \Omega_M \Gamma_M \tag{4.3}$$

for some $\Omega_M', \Gamma_M \in H^\infty_{M_n}$. By Lemma 4.3, $\Omega_M = b_M P_N + (I - P_N)$ with $N := \big(\mathrm{ran}\, (A_r (\alpha_M) P_M) \big)^\perp$. We now claim that

$$\Gamma_M (\alpha_M) \text{ is invertible.} \tag{4.4}$$

Since

$$\det \Big[A_r \Big(P_M + b_M (I - P_M) \Big) \Big] = b_M^{\mathrm{rank}(I - P_M)} \cdot \det A_r$$

and

$$\det \Omega_M \Gamma_M = (b_M)^{\dim N} \cdot \det \Gamma_M,$$

it follows from (4.3) that

$$\det A_r \cdot (b_M)^{\mathrm{rank}(I - P_M)} = (b_M)^{\dim N} \cdot \det \Gamma_M. \tag{4.5}$$

But since A_r and Θ are right coprime, and hence $A_r(\alpha_M)$ is invertible, it follows that

$$\dim N = \dim \big(\mathrm{ran}\, (A_r (\alpha_M) P_M) \big)^\perp = \dim (\mathrm{ran}\, P_M)^\perp = \mathrm{rank}(I - P_M),$$

which together with (4.5) implies that $\det \Gamma_M = \det A_r$. This proves (4.4). Therefore we have

$$\Phi = \prod_{m=1}^{M-1} \Big(b_m P_m + (I - P_m) \Big) \Gamma_M^* \Omega_M' \Theta,$$

where $\Gamma_M(\alpha_M)$ is invertible. Thus $\Gamma_M(\alpha)$ is invertible for all $\alpha \in \mathcal{Z}(\theta)$, and hence by Lemma 2.3, Θ and Γ_M are coprime.

If we repeat this argument then after M steps we get the left coprime factorization of $\Phi = A_l^* \Omega$, where Ω still has Θ as an inner divisor. □

Our main theorem of this section now follows:

Theorem 4.5. *Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued rational function. Then we may write*

$$\Phi_- = B^* \Theta,$$

where $B \in H^2_{M_n}$ and $\Theta := I_\theta$ with a finite Blaschke product θ. Suppose B and Θ are coprime. If both T_Φ and T_Φ^2 are hyponormal then T_Φ is either normal or analytic.

Proof. Suppose Φ is not analytic. Then Θ is not constant unitary. Since T_Φ is hyponormal, it follows that $\ker H_{\Phi_+^*} \subseteq \ker H_{\Phi_-^*} = \Theta H^2_{\mathbb{C}^n}$. Thus we can write

$$\Phi_+ = \Theta \Delta_r A_r^* \quad \text{(right coprime factorization)},$$

where Δ_r is an inner matrix function. Let θ_0 be a minimal inner function such that $\Theta_0 \equiv I_{\theta_0} = \Delta_r \Theta_1$ for some inner matrix function Θ_1. We also write $A := A_r \Theta_1$, and hence

$$\Phi_+ = \Theta \Theta_0 A^*.$$

On the other hand, we need to keep in mind that $\Theta = I_\theta$ and $\Theta_0 = I_{\theta_0}$ are inner functions, *constant* along the diagonal, so that these factors commute with all other matrix functions in the computations below. Note that $\Phi^2 \Theta^2 \Theta_0^2 \in H^\infty_{M_n}$ and $\Phi^{*2} \Theta^2 \Theta_0^2 \in H^\infty_{M_n}$. We thus have

$$\begin{aligned}
T^*_{\Theta^2 \Theta_0^2} [T_\Phi^{2*}, T_\Phi^2] T_{\Theta^2 \Theta_0^2} &= T^*_{\Theta^2 \Theta_0^2} T_\Phi^{2*} T_\Phi^2 T_{\Theta^2 \Theta_0^2} - T^*_{\Theta^2 \Theta_0^2} T_\Phi^2 T_\Phi^{2*} T_{\Theta^2 \Theta_0^2} \\
&= T_{\Phi^{*2} \Theta^{*2} \Theta_0^{*2}} T_{\Phi^2 \Theta^2 \Theta_0^2} - T_{\Phi^2 \Theta^{*2} \Theta_0^{*2}} T_{\Phi^{*2} \Theta^2 \Theta_0^2} \\
&= T_{\Phi^{*2} \Theta^{*2} \Theta_0^{*2} \Phi^2 \Theta^2 \Theta_0^2} - T_{\Phi^2 \Theta^{*2} \Theta_0^{*2} \Phi^{*2} \Theta^2 \Theta_0^2} \\
&= T_{\Phi^{*2} \Phi^2} - T_{\Phi^2 \Phi^{*2}} = 0 \quad (\text{since } \Phi \text{ is normal}).
\end{aligned}$$

The positivity of $[T_\Phi^{2*}, T_\Phi^2]$ implies that $[T_\Phi^{2*}, T_\Phi^2] T_{\Theta^2 \Theta_0^2} = 0$. We thus have

$$\begin{aligned}
0 &= [T_\Phi^{2*}, T_\Phi^2] T_{\Theta^2 \Theta_0^2} \\
&= T_{\Phi^*}^2 T_{\Phi^2 \Theta^2 \Theta_0^2} - T_\Phi^2 T_{\Phi^{*2} \Theta^2 \Theta_0^2} \\
&= T_{\Phi^*} T_{\Phi^* \Phi^2 \Theta^2 \Theta_0^2} - T_\Phi T_{\Phi \Phi^{*2} \Theta^2 \Theta_0^2} \\
&= \Big(T_{\Phi^{*2} \Phi^2 \Theta^2 \Theta_0^2} - H_\Phi^* H_{\Phi^* \Phi^2 \Theta^2 \Theta_0^2} \Big) \\
&\quad - \Big(T_{\Phi^2 \Phi^{*2} \Theta^2 \Theta_0^2} - H_{\Phi^*}^* H_{\Phi \Phi^{*2} \Theta^2 \Theta_0^2} \Big) \quad (\text{by } (1.3)) \\
&= H_{\Phi^*}^* H_{\Phi \Phi^{*2} \Theta^2 \Theta_0^2} - H_\Phi^* H_{\Phi^* \Phi^2 \Theta^2 \Theta_0^2} \\
&= H_{\Phi_+^*}^* H_{\Phi \Phi^{*2} \Theta^2 \Theta_0^2} - H_{\Phi_-^*}^* H_{\Phi^* \Phi^2 \Theta^2 \Theta_0^2}.
\end{aligned} \tag{4.6}$$

Let $\Omega := \mathrm{GCD}(\Theta_0, \Theta)$. Then by Lemma 2.1, $\Omega = I_\omega$ for an inner function ω. Thus we can write

$$\Theta = \Theta' \Omega \quad \text{and} \quad \Theta_0 = \Theta_0' \Omega,$$

where $\Theta' = I_{\theta'}$ and $\Theta_0' = I_{\theta_0'}$ for some inner functions θ' and θ_0'. Observe that

$$\begin{aligned} H_{\Phi^*\Phi^2\Theta^2\Theta_0^2} &= H_{(\Theta B^*+\Theta^*\Theta_0^*A)(\Theta^*B+\Theta\Theta_0A^*)^2\Theta^2\Theta_0^2} \\ &= H_{(\Theta B^*+\Theta^*\Theta_0^*A)(B+\Theta^2\Theta_0A)^2\Theta_0^2} \\ &= H_{\Theta^*\Theta_0^*A(B+\Theta^2\Theta_0A)^2\Theta_0^2} \\ &= H_{A(B+\Theta^2\Theta_0A)^2\Theta_0\Theta^*} \\ &= H_{A(B+\Theta^2\Theta_0A)^2\Theta_0'\Theta'^*}. \end{aligned} \tag{4.7}$$

Since Φ is normal we also have

$$\begin{aligned} H_{\Phi\Phi^{*2}\Theta^2\Theta_0^2} &= H_{(\Theta B^*+\Theta^*\Theta_0^*A)^2(\Theta^*B+\Theta\Theta_0A^*)\Theta^2\Theta_0^2} \\ &= H_{(\Theta^2\Theta_0B^*+A)(\Theta^*B+\Theta\Theta_0A^*)} \\ &= H_{(\Theta^2\Theta_0B^*+A)B\Theta^*} \\ &= H_{AB\Theta^*}. \end{aligned} \tag{4.8}$$

We now claim that

$$\theta' \text{ is not constant.} \tag{4.9}$$

Toward (4.9) we assume to the contrary that θ' is a constant. Then by (4.7) we have

$$H_{\Phi^*\Phi^2\Theta^2\Theta_0^2} = H_{A(B+\Theta^2\Theta_0A)^2\Theta_0'\Theta'^*} = 0,$$

so that $H^*_{\Phi_-^*}H_{\Phi^*\Phi^2\Theta^2\Theta_0^2} = 0$, and by (4.6) we have

$$H^*_{\Phi_+^*}H_{\Phi\Phi^{*2}\Theta^2\Theta_0^2} = 0. \tag{4.10}$$

Observe that

$$\begin{aligned} H_{AB\Theta^*} = \quad 0 &\iff AB\Theta^* \in H^2_{M_n} \\ &\iff AB \in \Theta H^2_{M_n} \\ &\iff A = \Theta A' \quad \text{(since } B \text{ and } \Theta \text{ are coprime)}, \end{aligned}$$

which implies that $A(\alpha) = 0$ for each $\alpha \in \mathcal{Z}(\theta)$, a contradiction. Therefore

$$H_{\Phi\Phi^{*2}\Theta^2\Theta_0^2} = H_{AB\Theta^*} \neq 0 \quad \text{and} \quad \text{cl ran}\, H_{\Phi\Phi^{*2}\Theta^2\Theta_0^2} = \mathcal{H}(\widetilde{\Delta})$$

for some nonconstant (left) inner divisor Δ of Θ. Thus it follows from Lemma 4.4 and (4.10) that

$$\mathcal{H}(\widetilde{\Delta}) = \text{cl ran}\, H_{\Phi\Phi^{*2}\Theta^2\Theta_0^2} \subseteq \ker H^*_{\Phi_+^*} \subseteq \widetilde{\Theta} H^2_{\mathbb{C}^n},$$

giving a contradiction. This proves (4.9). Observe

$$\text{cl ran}\, H_{\Phi^*\Phi^2\Theta^2\Theta_0^2} = \text{cl ran}\, H_{A(B+\Theta^2\Theta_0A)^2\Theta_0'\Theta'^*} \subseteq \mathcal{H}(\widetilde{\Theta}') \perp \widetilde{\Theta} H^2_{\mathbb{C}^n} = \ker H^*_{\Phi_-^*},$$

and

$$\text{cl ran}\, H_{\Phi\Phi^{*2}\Theta^2\Theta_0^2} = \text{cl ran}\, H_{AB\Theta^*} \subseteq \mathcal{H}(\widetilde{\Theta}) \perp \widetilde{\Theta} H^2_{\mathbb{C}^n} \supseteq \ker H^*_{\Phi_+^*}.$$

Thus by (4.6) we have

$$\begin{aligned}\ker H_{\Phi^*\Phi^2\Theta^2\Theta_0^2} &= \ker H^*_{\Phi^*_-} H_{\Phi^*\Phi^2\Theta^2\Theta_0^2}\\ &= \ker H^*_{\Phi^*_+} H_{\Phi\Phi^{*2}\Theta^2\Theta_0^2}\\ &= \ker H_{\Phi\Phi^{*2}\Theta^2\Theta_0^2}.\end{aligned} \tag{4.11}$$

Observe that for all $\alpha \in \mathcal{Z}(\theta)$,

$$\Big(A(B+\Theta^2\Theta_0 A)^2\Theta_0'\Big)(\alpha) = A(\alpha)B(\alpha)^2\Theta_0'(\alpha).$$

Since $B(\alpha)$ and $\Theta_0'(\alpha)$ are invertible, we have

$$\dim\ker\big(A(B+\Theta^2\Theta_0 A)^2\Theta_0'\big)(\alpha) = \dim\ker A(\alpha) = \dim\ker(AB)(\alpha).$$

By (4.7), (4.8) and (4.11), we have that $A(\alpha) = 0$ for all $\alpha \in \mathcal{Z}(\omega)$ and hence ω is a constant. Thus Ω is a constant unitary, and hence $\Theta = \Theta'$ and $\Theta_0 = \Theta_0'$. Therefore $\mathcal{Z}(\theta) = \mathcal{Z}(\theta) \setminus \mathcal{Z}(\theta_0)$ and hence, by Lemma 4.2, $A(\alpha)$ is invertible for each $\alpha \in \mathcal{Z}(\theta)$. Since for each $\alpha \in \mathcal{Z}(\theta)$,

$$\Big(A(B+\Theta^2\Theta_0 A)^2\Theta_0\Big)(\alpha) = A(\alpha)B(\alpha)^2\Theta_0(\alpha) \text{ and } (AB)(\alpha) \text{ are invertible,}$$

it follows that $A(B+\Theta^2\Theta_0 A)^2\Theta_0$ and Θ are coprime, and AB and Θ are coprime. Thus by (4.7), (4.8) and (4.11) we have

$$\operatorname{cl}\operatorname{ran} H_{\Phi^*\Phi^2\Theta^2\Theta_0^2} = \operatorname{cl}\operatorname{ran} H_{\Phi\Phi^{*2}\Theta^2\Theta_0^2} = \mathcal{H}(\widetilde{\Theta}). \tag{4.12}$$

By the well-known result of Cowen [11, Theorem 1]— if $\varphi \in L^\infty$ and b is a finite Blaschke product of degree n then $T_{\varphi\circ b} \cong \oplus_n T_\varphi$, we may, without loss of generality, assume that $0 \in \mathcal{Z}(\theta)$. Since T_Φ is hyponormal, by [34, cf. p. 4] there exists $K \in H^\infty_{M_n}$ with $\|K\|_\infty \le 1$ such that

$$H_{\Phi^*_-} = H_{K\Phi^*_+} = T^*_{\widetilde{K}} H_{\Phi^*_+}.$$

Since $\Phi\Phi^*\Theta^2\Theta_0^2 \in H^\infty_{M_n}$, we have

$$\begin{aligned}T_{\widetilde{K}} H_{\Phi^2\Phi^*\Theta^2\Theta_0^2} &= T_{\widetilde{K}} H_\Phi T_{\Phi\Phi^*\Theta^2\Theta_0^2} = T_{\widetilde{K}} H_{\Phi^*_-} T_{\Phi\Phi^*\Theta^2\Theta_0^2}\\ &= T_{\widetilde{K}}(T^*_{\widetilde{K}} H_{\Phi^*_+}) T_{\Phi\Phi^*\Theta^2\Theta_0^2} = T_{\widetilde{K}} T^*_{\widetilde{K}} H_{\Phi\Phi^{*2}\Theta^2\Theta_0^2}.\end{aligned}$$

Thus by (4.6) we have

$$\begin{aligned}0 &= H^*_{\Phi^*_+} H_{\Phi\Phi^{*2}\Theta^2\Theta_0^2} - H^*_{\Phi^*_-} H_{\Phi^*\Phi^2\Theta^2\Theta_0^2}\\ &= H^*_{\Phi^*_+}\Big(H_{\Phi\Phi^{*2}\Theta^2\Theta_0^2} - T_{\widetilde{K}} H_{\Phi^*\Phi^2\Theta^2\Theta_0^2}\Big)\\ &= H^*_{\Phi^*_+}(I - T_{\widetilde{K}} T^*_{\widetilde{K}}) H_{\Phi\Phi^{*2}\Theta^2\Theta_0^2}.\end{aligned} \tag{4.13}$$

It thus follows from (4.12), (4.13) and Lemma 4.4 that

$$(I - T_{\widetilde{K}} T^*_{\widetilde{K}})\big(\mathcal{H}(\widetilde{\Theta})\big) = \operatorname{cl}\operatorname{ran}\Big((I - T_{\widetilde{K}} T^*_{\widetilde{K}}) H_{\Phi^{*2}\Phi\Theta^2\Theta_0^2}\Big) \subseteq \ker H^*_{\Phi^*_+} \subseteq \widetilde{\Theta} H^2_{\mathbb{C}^n}. \tag{4.14}$$

Since $\|K\|_\infty = \|\widetilde{K}\|_\infty = \|\widetilde{K}^*\|_\infty$, it follows that $\|T_{\widetilde{K}}\| = \|T_{\widetilde{K}^*}\| \le 1$. For each $i = 1, 2, \ldots, n$, put

$$E_i := (0, 0, \ldots, 1, 0, \ldots, 0, 0)^t.$$

Since $0 \in \mathcal{Z}(\theta) \cap \mathcal{Z}(\widetilde{\theta})$, we have $E_i \in \mathcal{H}(\Theta) \cap \mathcal{H}(\widetilde{\Theta})$ and by (4.14),

$$E_i - T_{\widetilde{K}} T^*_{\widetilde{K}} E_i = \widetilde{\Theta} F_i \quad (\text{some } F_i \in H^2_{\mathbb{C}^n}).$$

Observe that

$$\begin{aligned} E_i - T_{\widetilde{K}} T^*_{\widetilde{K}} E_i = \ & \widetilde{\Theta} F_i \Longrightarrow T_{\widetilde{K}} T^*_{\widetilde{K}} E_i = E_i - \widetilde{\Theta} F_i \\ & \Longrightarrow \| T_{\widetilde{K}} T^*_{\widetilde{K}} E_i \|_2^2 = \| E_i - \widetilde{\Theta} F_i \|_2^2 \\ & \Longrightarrow \| T_{\widetilde{K}} T^*_{\widetilde{K}} E_i \|_2^2 = \| E_i \|_2^2 + \| \widetilde{\Theta} F_i \|_2^2 \quad (\text{since } E_i \in \mathcal{H}(\widetilde{\Theta})) \\ & \Longrightarrow \| T_{\widetilde{K}} T^*_{\widetilde{K}} E_i \|_2^2 = 1 + \| \widetilde{\Theta} F_i \|_2^2. \end{aligned}$$

But since $\| T_{\widetilde{K}} T^*_{\widetilde{K}} \| \le 1$, it follows that $\| T_{\widetilde{K}} T^*_{\widetilde{K}} E_i \|_2 = 1$ and $F_i = 0$ for all $i = 1, 2, \ldots, n$. We thus have $\| T^*_{\widetilde{K}} E_i \|_2 = 1$ for all $i = 1, 2, \ldots, n$. Write

$$K(z) := \bigl[k_{ij}(z)\bigr] \quad \text{and} \quad k_{ij}(z) = \sum_{m=0}^{\infty} k_{ij}^{(m)} z^m.$$

Then $\widetilde{K}^*(z) = K(\overline{z}) = \bigl[k_{ij}(\overline{z})\bigr]$, and hence

$$T^*_{\widetilde{K}} E_i = [P(k_{1i}(\overline{z})), P(k_{2i}(\overline{z})), \ldots, P(k_{ni}(\overline{z}))]^t = [k_{1i}^{(0)}, k_{2i}^{(0)}, \ldots, k_{ni}^{(0)}]^t.$$

But since $\| T^*_{\widetilde{K}} E_i \|_2 = 1$ for all $i = 1, 2, \ldots, n$, it follows that

$$\| [k_{1i}^{(0)}, k_{2i}^{(0)}, \ldots, k_{ni}^{(0)}]^t \|_2 = 1 \quad \text{for all } i = 1, 2, \ldots, n.$$

Therefore

$$1 = \frac{1}{n} \| [k_{ij}^{(0)}] \|_2^2 \le \frac{1}{n} \sum_{m=0}^{\infty} \| [k_{ij}^{(m)}] \|_2^2 = \frac{1}{n} \| K \|_2^2 \le \| K \|_\infty^2 \le 1,$$

which implies that $\bigl[k_{ij}^{(m)}\bigr] = 0$ for all $m \ge 1$. Hence $K = \bigl[k_{ij}^{(0)}\bigr]$, so that $\widetilde{K} = K^*$. Observe that

$$\begin{aligned} (I - T_{\widetilde{K}} T^*_{\widetilde{K}}) \mathcal{H}(\widetilde{\Theta}) = \ & 0 \\ & \Longrightarrow (I - T_{K^*K}) \mathcal{H}(\widetilde{\Theta}) = 0 \\ & \Longrightarrow K^*K = I_n \quad (\text{since } 0 \in \mathcal{Z}(\widetilde{\theta})). \end{aligned}$$

Therefore $\widetilde{K} = K^*$ is a constant unitary and hence we have

$$[T^*_\Phi, T_\Phi] = H^*_{\Phi^*} H_{\Phi^*} - H^*_{\Phi} H_{\Phi} = H^*_{\Phi^*} H_{\Phi^*} - H^*_{\Phi^*} T_{\widetilde{K}\widetilde{K}^*} H_{\Phi^*} = 0,$$

which implies that T_Φ is normal. □

Corollary 4.6. *Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued trigonometric polynomial whose co-analytic outer coefficient is invertible. If T_Φ and T^2_Φ are hyponormal then T_Φ is normal.*

Proof. Immediate from Theorem 4.5 together with the observation that $\Phi_- = B^*\Theta$ with $\Theta = I_{z^m}$ is a coprime factorization if and only if $B(0)$ is a co-analytic outer coefficient and is invertible. □

Remark 4.7. In Theorem 4.5, the "coprime" condition is essential. To see this, let

$$T_\Phi := \begin{bmatrix} T_b + T_b^* & 0 \\ 0 & T_b \end{bmatrix} \quad (b \text{ is a finite Blaschke product}).$$

Since $T_b + T_b^*$ is normal and T_b is analytic, it follows that T_Φ and T_Φ^2 are both hyponormal. Obviously, T_Φ is neither normal nor analytic. Note that $\Phi_- \equiv \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^* \cdot I_b$, where $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and I_b are not coprime. $\square$

On the other hand, we have not been able to determine whether this phenomenon is quite accidental. In fact we would guess that if $\Phi \in L^\infty_{M_n}$ is a matrix-valued rational function such that T_Φ is subnormal then $T_\Phi = T_A \oplus T_B$, where T_A is normal and T_B is analytic.

5. Subnormal Toeplitz completions

Given a partially specified operator matrix with some known entries, the problem of finding suitable operators to complete the given partial operator matrix so that the resulting matrix satisfies certain given properties is called a *completion problem*. Dilation problems are special cases of completion problems: in other words, the dilation of T is a completion of the partial operator matrix $\begin{bmatrix} T & ? \\ ? & ? \end{bmatrix}$. In recent years, operator theorists have been interested in the subnormal completion problem for $\begin{bmatrix} U^* & ? \\ ? & U^* \end{bmatrix}$, where U is the shift on H^2. In this section, we solve this completion problem.

A *partial block Toeplitz matrix* is simply an $n \times n$ matrix some of whose entries are specified Toeplitz operators and whose remaining entries are unspecified. A *subnormal completion* of a partial operator matrix is a particular specification of the unspecified entries resulting in a subnormal operator. For example

$$\begin{bmatrix} T_z & 1 - T_z T_{\overline{z}} \\ 0 & T_{\overline{z}} \end{bmatrix} \tag{5.1}$$

is a subnormal (even unitary) completion of the 2×2 partial operator matrix

$$\begin{bmatrix} T_z & ? \\ ? & T_{\overline{z}} \end{bmatrix}.$$

A *subnormal Toeplitz completion* of a partial block Toeplitz matrix is a subnormal completion whose unspecified entries are Toeplitz operators. Then the following question comes up at once: Does there exist a subnormal Toeplitz completion of $\begin{bmatrix} T_z & ? \\ ? & T_{\overline{z}} \end{bmatrix}$? Evidently, (5.1) is not such a completion. To answer this question, let

$$\Phi \equiv \begin{bmatrix} z & \varphi \\ \psi & \overline{z} \end{bmatrix} \quad (\varphi, \psi \in L^\infty).$$

If T_Φ is hyponormal then by [34, cf. p. 4], Φ should be normal. Thus a straightforward calculation shows that

$$|\varphi| = |\psi| \quad \text{and} \quad \overline{z}(\varphi + \overline{\psi}) = z(\varphi + \overline{\psi}),$$

which implies that $\varphi = -\overline{\psi}$. Thus a direct calculation shows that

$$[T_\Phi^*, T_\Phi] = \begin{bmatrix} * & * \\ * & T_z T_{\overline{z}} - 1 \end{bmatrix},$$

which is not positive semi-definite because $T_z T_{\overline{z}} - 1$ is not. Therefore, there are no hyponormal Toeplitz completions of $\begin{bmatrix} T_z & ? \\ ? & T_{\overline{z}} \end{bmatrix}$. However the following problem has remained unsolved until now:

Problem A. Let U be the shift on H^2. Complete the unspecified Toeplitz entries of the partial block Toeplitz matrix $A := \begin{bmatrix} U^* & ? \\ ? & U^* \end{bmatrix}$ to make A subnormal.

In this section we give a complete answer to Problem A.

Theorem 5.1. *Let $\varphi, \psi \in L^\infty$ and consider*

$$A := \begin{bmatrix} T_{\overline{z}} & T_\varphi \\ T_\psi & T_{\overline{z}} \end{bmatrix}.$$

The following statements are equivalent.

(i) *A is normal.*
(ii) *A is subnormal.*
(iii) *A is 2-hyponormal.*
(iv) *One of the following conditions holds:*
  1. $$\varphi = e^{i\theta} z + \beta \quad \textit{and} \quad \psi = e^{i\omega}\varphi \quad (\beta \in \mathbb{C}; \theta, \omega \in [0, 2\pi));$$
  2. $$\varphi = \alpha\,\overline{z} + e^{i\theta}\sqrt{1 + |\alpha|^2}\, z + \beta \quad \textit{and}$$ $$\psi = e^{i\,(\pi - 2\arg\alpha)}\varphi \quad (\alpha, \beta \in \mathbb{C}, \alpha \neq 0; \theta \in [0, 2\pi)).$$

Theorem 5.1 says that the unspecified entries of the matrix $\begin{bmatrix} T_{\overline{z}} & ? \\ ? & T_{\overline{z}} \end{bmatrix}$ are Toeplitz operators with symbols which are both analytic or trigonometric polynomials of degree 1. In fact, as we will see in the proof of Theorem 5.1, our solution is just the normal completion. However the solution is somewhat more intricate than one would expect.

To prove Theorem 5.1 we need several technical lemmas.

Lemma 5.2. *For $j = 1, 2, 3$, let θ_j be an inner function. If $\theta_1\mathcal{H}(\theta_2) \subseteq \mathcal{H}(\theta_3)$ then either θ_2 is constant or $\theta_1\theta_2$ is a divisor of θ_3. In particular, if $\theta_1\mathcal{H}(\theta_2) \subseteq \mathcal{H}(\theta_1)$ then θ_1 or θ_2 is constant.*

Proof. Suppose θ_2 is not constant. If $\theta_1\mathcal{H}(\theta_2) \subseteq \mathcal{H}(\theta_3)$ then by Lemma 2.6, for all $f \in \mathcal{H}(\theta_2)$, $\overline{\theta}_1 \overline{f} \theta_3 \in zH^2$, and hence $\overline{f}\theta_3 \in zH^2$, so that $f \in \mathcal{H}(\theta_3)$, which implies that $\mathcal{H}(\theta_2) \subseteq \mathcal{H}(\theta_3)$, and therefore $\theta_3 H^2 \subseteq \theta_2 H^2$. Thus θ_2 is a divisor of θ_3. We can then write $\theta_3 = \theta_0\theta_2$ for some inner function θ_0. It suffices to show that θ_1 is a divisor of θ_0. Observe that

$$\begin{aligned} \theta_1\mathcal{H}(\theta_2) \subseteq \quad & \mathcal{H}(\theta_0\theta_2) \Longrightarrow \operatorname{ran}(T_{\theta_1} H^*_{\overline{\theta}_2}) \subseteq \mathcal{H}(\theta_0\theta_2) \\ & \Longrightarrow \theta_0\theta_2 H^2 \subseteq \ker H_{\overline{\theta}_2} T_{\overline{\theta}_1} \\ & \Longrightarrow H_{\overline{\theta}_2} T_{\overline{\theta}_1\theta_0\theta_2} = 0 \\ & \Longrightarrow H_{\overline{\theta}_1\theta_0} - T_{\widetilde{\theta}_2} H_{\overline{\theta}_1\theta_0\theta_2} = 0 \\ & \Longrightarrow H_{\overline{\theta}_1\theta_0} - T_{\widetilde{\theta}_2} T_{\overline{\widetilde{\theta}}_2} H_{\overline{\theta}_1\theta_0} = 0 \\ & \Longrightarrow H_{\overline{\widetilde{\theta}}_2} H_{\theta_0\overline{\theta}_1} = 0, \end{aligned}$$

where the fourth implication follows from the fact that $H_{\varphi\psi} = T^*_{\widetilde{\varphi}} H_\psi + H_\varphi T_\psi$ for any $\varphi, \psi \in L^\infty$. But since θ_2 is not constant it follows that θ_1 is a divisor of θ_0. The second assertion follows at once from the first. $\square$

Suppose $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type, with

$$\Phi_+ = A^*\Theta \quad \text{and} \quad \Phi_- = B_\ell^*\Omega_2 \quad \text{(left coprime factorization)},$$

where $\Theta = I_\theta$ for an inner function θ. If T_Φ is hyponormal, then in view of Proposition 3.2, Φ can be written as:

$$\Phi_+ = A^*\Omega_1\Omega_2 \quad \text{and} \quad \Phi_- = B_\ell^*\Omega_2, \tag{5.2}$$

where $\Omega_1\Omega_2 = \Theta = I_\theta$. We also note that $\Omega_1\Omega_2 = \Theta = \Omega_2\Omega_1$.

The following lemma will be extensively used in the proof of Theorem 5.1.

Lemma 5.3. *Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type of the form (5.2):*

$$\Phi_+ = A^*\Omega_1\Omega_2 = A^*\Theta \quad \textit{and} \quad \Phi_- = B_\ell^*\Omega_2 \quad (\textit{left coprime factorization}),$$

where $\Theta = I_\theta$ for an inner function θ. If $\ker[T_\Phi^*, T_\Phi]$ is invariant under T_Φ, then

$$\Omega_1 H^2_{\mathbb{C}^n} \subseteq \ker[T_\Phi^*, T_\Phi],$$

and therefore

$$\operatorname{cl}\operatorname{ran}[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}(\Omega_1).$$

Assume instead that we decompose $\Phi \in L^\infty_{M_n}$ as:

$$\Phi_+ = \Delta_2\Delta_0 A_r^* \quad (\textit{right coprime factorization})$$

and

$$\Phi_- = \Delta_2 B_r^* \quad (\textit{right coprime factorization}).$$

If T_Φ is hyponormal then

$$\Delta_2\mathcal{H}(\Delta_0) \subseteq \operatorname{cl}\operatorname{ran}[T_\Phi^*, T_\Phi].$$

Hence, in particular, if T_Φ is hyponormal and $\ker[T_\Phi^*, T_\Phi]$ is invariant under T_Φ, then

$$\Delta_2\mathcal{H}(\Delta_0) \subseteq \operatorname{cl}\operatorname{ran}[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}(\Omega_1). \tag{5.3}$$

Proof. See [20, Lemma 3.2 and Theorem 3.7]. □

Lemma 5.4. *Let*

$$\Phi_- = \begin{bmatrix} z & \theta_1\overline{b} \\ \theta_0\overline{a} & z \end{bmatrix} \quad (a \in \mathcal{H}(\theta_0),\ b \in \mathcal{H}(\theta_1) \textit{ and } \theta_j \textit{ inner } (j = 0, 1)).$$

If $\theta_0 = z^n\theta_0'$ ($n \ge 1$; $\theta_0'(0) \neq 0$) and $\theta_1(0) \neq 0$, then $\ker H_{\Phi_-^*} = \Delta H^2_{\mathbb{C}^2}$, *where*

$$\Delta = \begin{cases} \begin{bmatrix} z\theta_1 & 0 \\ 0 & \theta_0 \end{bmatrix} & (n = 1); \\ \dfrac{1}{\sqrt{|\alpha|^2+1}} \begin{bmatrix} z\theta_1 & \alpha\theta_1 \\ -\overline{\alpha}\theta_0 & z^{n-1}\theta_0' \end{bmatrix} & (n \ge 2) \ \left(\alpha := -\dfrac{a(0)}{\theta_1(0)}\right). \end{cases}$$

Proof. Observe that for $f, g \in H^2$,

$$\Phi_-^* \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \overline{z} & \overline{z}^n\overline{\theta_0'}a \\ \overline{\theta}_1 b & \overline{z} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \in H^2_{\mathbb{C}^2} \iff \begin{bmatrix} \overline{z}f + \overline{z}^n\overline{\theta_0'}ag \\ \overline{\theta}_1 bf + \overline{z}g \end{bmatrix} \in H^2_{\mathbb{C}^2}.$$

Thus if $\Phi_-^* \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in H^2_{\mathbb{C}^2}$, then $\overline{\theta}_1 bf + \overline{z}g \in H^2$. Since $\theta_1(0) \neq 0$, we have $\overline{\theta}_1 bfz \in H^2$, and hence $f = \theta_1 f_1$ for some $f_1 \in H^2$. In turn, $bf_1 + \overline{z}g \in H^2$, so that $g = zg_1$ for some $g_1 \in H^2$. We therefore have

$$\overline{z}\theta_1 f_1 + \overline{z}^{n-1}\overline{\theta_0'}ag_1 \in H^2. \tag{5.4}$$

If $n = 1$, then (5.4) implies $g_1 = \theta_0' g_2$ and $f_1 = zf_2$ for some $g_2, f_2 \in H^2$. Thus $f = z\theta_1 f_2$ and $g = \theta_0 g_2$, which implies

$$\ker H_{\Phi_-^*} = \begin{bmatrix} z\theta_1 & 0 \\ 0 & \theta_0 \end{bmatrix} H^2_{\mathbb{C}^2}.$$

If instead $n \geq 2$, then (5.4) implies that $\overline{z}^{n-2}\overline{\theta_0'}ag_1 \in H^2$, so that $g_1 = z^{n-2}\theta_0' g_2$ for some $g_2 \in H^2$. We thus have

$$\begin{aligned} \overline{z}\theta_1 f_1 + \overline{z}^{n-1}\overline{\theta_0'}ag_1 \in H^2 &\iff \overline{z}\theta_1 f_1 + \overline{z}ag_2 \in H^2 \\ &\iff \theta_1(0)f_1(0) + a(0)g_2(0) = 0 \\ &\iff g_2(0) = \frac{1}{\alpha}f_1(0) \quad \left(\text{recall that } \alpha = -\frac{a(0)}{\theta_1(0)}\right). \end{aligned}$$

Therefore we have

$$\begin{bmatrix} f \\ g \end{bmatrix} \in \ker H_{\Phi_-^*} \iff f = \theta_1 f_1, \quad g = z^{n-1}\theta_0' g_2, \quad \text{and} \quad g_2(0) = \frac{1}{\alpha}f_1(0). \tag{5.5}$$

Put

$$\Delta := \frac{1}{\sqrt{|\alpha|^2+1}} \begin{bmatrix} z\theta_1 & \alpha\theta_1 \\ -\overline{\alpha}\theta_0 & z^{n-1}\theta_0' \end{bmatrix}.$$

Then Δ is inner, and for $h_1, h_2 \in H^2$,

$$\begin{aligned} \Delta \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} &= \frac{1}{\sqrt{|\alpha|^2+1}} \begin{bmatrix} z\theta_1 h_1 + \alpha\theta_1 h_2 \\ -\overline{\alpha}z^n\theta_0' h_1 + z^{n-1}\theta_0' h_2 \end{bmatrix} \\ &= \frac{1}{\sqrt{|\alpha|^2+1}} \begin{bmatrix} \theta_1(zh_1 + \alpha h_2) \\ z^{n-1}\theta_0'(-\overline{\alpha}zh_1 + h_2) \end{bmatrix}. \end{aligned}$$

But since $\frac{1}{\alpha}(zh_1+\alpha h_2)(0) = (-\overline{\alpha}zh_1+h_2)(0)$, it follows from (5.5) that $\ker H_{\Phi_-^*} = \Delta H^2_{\mathbb{C}^2}$. $\square$

Lemma 5.5. *Let*

$$\Phi_- = \begin{bmatrix} z & \theta_1\overline{b} \\ \theta_0\overline{a} & z \end{bmatrix} \quad (a \in \mathcal{H}(\theta_0),\ b \in \mathcal{H}(\theta_1) \text{ and } \theta_j \text{ inner } (j = 0, 1)).$$

If $\theta_1 = z^n\theta_1'$ $(n \geq 1;\, \theta_1'(0) \neq 0)$ *and* $\theta_0(0) \neq 0$, *then* $\ker H_{\Phi_-^*} = \Delta H^2_{\mathbb{C}^2}$, *where*

$$\Delta = \begin{cases} \begin{bmatrix} \theta_1 & 0 \\ 0 & z\theta_0 \end{bmatrix} & (n = 1); \\ \dfrac{1}{\sqrt{|\alpha|^2+1}} \begin{bmatrix} z^{n-1}\theta_1' & -\overline{\alpha}\theta_1 \\ \alpha\theta_0 & z\theta_0 \end{bmatrix} & (n \geq 2) \ \left(\alpha := -\dfrac{b(0)}{\theta_0(0)}\right). \end{cases}$$

Proof. Same as the proof of Lemma 5.4. □

Lemma 5.6. *Let*

$$\Phi_- = \begin{bmatrix} z & \theta_1\overline{b} \\ \theta_0\overline{a} & z \end{bmatrix} \quad (a \in \mathcal{H}(\theta_0),\ b \in \mathcal{H}(\theta_1) \text{ and } \theta_j \text{ inner } (j = 0, 1)).$$

If $\theta_0 = z\theta_0'$ *and* $\theta_1 = z\theta_1'$ *then* $\ker H_{\Phi_-^*} = \Delta H^2_{\mathbb{C}^2}$, *where*

$$\Delta = \begin{cases} \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_0 \end{bmatrix} & \Big((ab)(0) \neq (\theta_0'\theta_1')(0)\Big); \\ \dfrac{1}{\sqrt{|\alpha|^2+1}} \begin{bmatrix} \theta_1 & \alpha\theta_1' \\ -\overline{\alpha}\theta_0 & \theta_0' \end{bmatrix} & \Big((ab)(0) = (\theta_0'\theta_1')(0)\Big) \ \left(\alpha := -\dfrac{a(0)}{\theta_1'(0)}\right). \end{cases}$$

Remark 5.7. Since $a(0), b(0) \neq 0$, the second part of the above assertion makes sense because by assumption, $\theta_0'(0), \theta_1'(0) \neq 0$. □

Proof of Lemma 5.6. Observe that for $f, g \in H^2$,

$$\begin{aligned} \Phi_-^* \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \overline{z} & \overline{z\theta_0' a} \\ \overline{z\theta_1' b} & \overline{z} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \in H^2_{\mathbb{C}^2} &\iff \begin{bmatrix} \overline{z}(f + \overline{\theta_0'} a g) \\ \overline{z}(g + \overline{\theta_1'} b f) \end{bmatrix} \in H^2_{\mathbb{C}^2} \\ &\Longrightarrow \begin{bmatrix} f + \overline{\theta_0'} a g \\ g + \overline{\theta_1'} b f \end{bmatrix} \in zH^2_{\mathbb{C}^2} \\ &\Longrightarrow \begin{bmatrix} \overline{\theta_0'} a g \\ \overline{\theta_1'} b f \end{bmatrix} \in H^2_{\mathbb{C}^2} \\ &\Longrightarrow g = \theta_0' g_1 \text{ and } f = \theta_1' f_1 \text{ for some } g_1, f_1 \in H^2. \end{aligned}$$

Thus if $\Phi_-^* \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in H^2_{\mathbb{C}^2}$ then $\overline{z}(\theta_1' f_1 + a g_1) \in H^2$ and $\overline{z}(\theta_0' g_1 + b f_1) \in H^2$, so that

$$\theta_1'(0) f_1(0) = -a(0) g_1(0) \quad \text{and} \quad \theta_0'(0) g_1(0) = -b(0) f_1(0).$$

If $(ab)(0) = (\theta_0'\theta_1')(0)$, then

$$\theta_1'(0) f_1(0) = -a(0) g_1(0) \iff \theta_0'(0) g_1(0) = -b(0) f_1(0).$$

Put

$$\Delta := \frac{1}{\sqrt{|\alpha|^2+1}} \begin{bmatrix} \theta_1 & \alpha\theta_1' \\ -\overline{\alpha}\theta_0 & \theta_0' \end{bmatrix}.$$

Then we can see that Δ is inner and $\ker H_{\Phi_-^*} = \Delta H^2_{\mathbb{C}^2}$.

If $(ab)(0) \neq (\theta_0'\theta_1')(0)$, put $\theta_0 = z^m\theta_0''$ and $\theta_1 = z^n\theta_1''$ $(\theta_0''(0), \theta_1''(0) \neq 0)$. If $n = m$ then

$$\Phi_- = \begin{bmatrix} z & z^n\theta_1''\overline{b} \\ z^m\theta_0''\overline{a} & z \end{bmatrix} = I_{z^n\theta_0''\theta_1''} \begin{bmatrix} \overline{z}^{n-1}\overline{\theta_0''\theta_1''} & \overline{\theta_0''b} \\ \overline{\theta_1''a} & \overline{z}^{n-1}\overline{\theta_0''\theta_1''} \end{bmatrix} \equiv I_{z^n\theta_0''\theta_1''}\, B^*.$$

Since $B(0)$ is invertible, it follows from Lemma 2.3 that I_{z^n} and B are coprime. Observe that

$$\Phi_-^* \begin{bmatrix} z^n f \\ z^n g \end{bmatrix} = \begin{bmatrix} \overline{z} & \overline{z}^n\overline{\theta_0''}a \\ \overline{z}^n\overline{\theta_1''}b & \overline{z} \end{bmatrix} \begin{bmatrix} z^n f \\ z^n g \end{bmatrix} \in H^2_{\mathbb{C}^2} \iff \begin{bmatrix} z^{n-1}f + \overline{\theta_0''}ag \\ \overline{\theta_1''}bf + z^{n-1}g \end{bmatrix} \in H^2_{\mathbb{C}^2},$$

which implies

$$f = \theta_1'' f_1 \quad \text{and} \quad g = \theta_0'' g_1 \quad \text{for some } f_1, g_1 \in H^2.$$

We thus have

$$\ker H_{\Phi_-^*} = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_0 \end{bmatrix} H^2_{\mathbb{C}^2}.$$

If instead $n \neq m$, then

$$\begin{aligned} \Phi_-^* \begin{bmatrix} f \\ g \end{bmatrix} = & \begin{bmatrix} \overline{z} & \overline{z}^m\overline{\theta_0''}a \\ \overline{z}^n\overline{\theta_1''}b & \overline{z} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \overline{z}(f + \overline{z}^{m-1}\overline{\theta_0''}ag) \\ \overline{z}(g + \overline{z}^{n-1}\overline{\theta_1''}bf) \end{bmatrix} \in H^2_{\mathbb{C}^2} \\ & \iff \begin{cases} f + \overline{z}^{m-1}\overline{\theta_0''}ag \in zH^2 \\ g + \overline{z}^{n-1}\overline{\theta_1''}bf \in zH^2, \end{cases} \end{aligned}$$

which implies

$$f = z^{n-1}\theta_1'' f_1 \quad \text{and} \quad g = z^{m-1}\theta_0'' g_1.$$

Suppose $n > m$, and hence $n \geq 2$. We thus have

$$z^{n-2}\theta_1'' f_1 + \overline{z}ag_1 \in H^2 \implies \overline{z}ag_1 \in H^2 \implies g_1 = zg_2 \implies g = \theta_0 g_2.$$

In turn,

$$g + \overline{z}^{n-1}\overline{\theta_1''}bf \in zH^2 \implies z^m\theta_0'' g_2 + bf_1 \in zH^2 \implies f_1 = zf_2,$$

which implies

$$\ker H_{\Phi_-^*} = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_0 \end{bmatrix} H^2_{\mathbb{C}^2}.$$

If $m > n$, a similar argument gives the result. $\square$

We are ready for:

Proof of Theorem 5.1. Clearly (i) ⇒ (ii) and (ii) ⇒ (iii). Moreover, a simple calculation shows that (iv) ⇒ (i).

(iii) ⇒ (iv): Write

$$\Phi \equiv \begin{bmatrix} \overline{z} & \varphi \\ \psi & \overline{z} \end{bmatrix} \equiv \Phi_-^* + \Phi_+ = \begin{bmatrix} z & \psi_- \\ \varphi_- & z \end{bmatrix}^* + \begin{bmatrix} 0 & \varphi_+ \\ \psi_+ & 0 \end{bmatrix}$$

and assume that T_Φ is 2-hyponormal. Since $\ker[T^*, T]$ is invariant under T for every 2-hyponormal operator $T \in \mathcal{B}(\mathcal{H})$, we note that Theorem 4.1 and Lemma 5.3 hold for 2-hyponormal operators T_Φ. We claim that

$$|\varphi| = |\psi|, \quad \text{and} \tag{5.6}$$

$$\Phi \text{ and } \Phi^* \text{ are of bounded type.} \tag{5.7}$$

Indeed, if T_Φ is hyponormal then Φ is normal, so that a straightforward calculation gives (5.6). Also there exists a matrix function $K \equiv \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \in H^\infty_{M_2}$ with $\|K\|_\infty \le 1$ such that $\Phi - K\Phi^* \in H^\infty_{M_2}$, i.e.,

$$\begin{bmatrix} \overline{z} & \overline{\varphi_-} \\ \overline{\psi_-} & \overline{z} \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \begin{bmatrix} 0 & \overline{\psi_+} \\ \overline{\varphi_+} & 0 \end{bmatrix} \in H^2_{M_2},$$

which implies that

$$\begin{cases} H_{\overline{z}} = H_{k_2\overline{\varphi_+}} = H_{\overline{\varphi_+}} T_{k_2}; \\ H_{\overline{\varphi_-}} = H_{k_1\overline{\psi_+}} = H_{\overline{\psi_+}} T_{k_1}; \\ H_{\overline{\psi_-}} = H_{k_4\overline{\varphi_+}} = H_{\overline{\varphi_+}} T_{k_4}; \\ H_{\overline{z}} = H_{k_3\overline{\psi_+}} = H_{\overline{\psi_+}} T_{k_3}. \end{cases}$$

If $\overline{\varphi_+}$ is not of bounded type then $\ker H_{\overline{\varphi_+}} = 0$, so that $k_2 = 0$, a contradiction; and if $\overline{\psi_+}$ is not of bounded type then $\ker H_{\overline{\psi_+}} = 0$, so that $k_3 = 0$, a contradiction. Therefore we should have Φ^* of bounded type. Since T_Φ is hyponormal, Φ is also of bounded type, giving (5.7). Thus we can write

$$\varphi_- := \theta_0 \overline{a} \quad \text{and} \quad \psi_- := \theta_1 \overline{b} \quad (a \in \mathcal{H}(\theta_0),\ b \in \mathcal{H}(\theta_1)).$$

Put

$$\theta_0 = z^m \theta_0' \quad \text{and} \quad \theta_1 = z^n \theta_1' \quad (m, n \ge 0;\ \theta_0'(0) \ne 0 \ne \theta_1'(0)).$$

We now claim that

$$m = n = 0 \quad \text{or} \quad m = n = 1. \tag{5.8}$$

We split the proof of (5.8) into three cases.

Case 1 ($m \ne 0$ and $n = 0$) In this case, we have $a(0) \ne 0$ because $\theta_0(0) = 0$ and θ_0 and a are coprime. We first claim that $m = 1$. To show this we assume to the contrary that $m \ge 2$. Write

$$\alpha := -\frac{a(0)}{\theta_1(0)} \quad \text{and} \quad \nu := \frac{1}{\sqrt{|\alpha|^2 + 1}}.$$

By Lemma 5.4, we can write:

$$\Phi_- = \begin{bmatrix} z & \theta_1 \overline{b} \\ \theta_0 \overline{a} & z \end{bmatrix} = \Delta_2 B_r^* \quad \text{(right coprime factorization)},$$

where

$$\Delta_2 := \nu \begin{bmatrix} z\theta_1 & \alpha\theta_1 \\ -\overline{\alpha}\theta_0 & z^{m-1}\theta_0' \end{bmatrix} \quad \text{and} \quad B_r := \nu \begin{bmatrix} \theta_1 - \overline{\alpha} a & \alpha\theta_1 \overline{z} + a\overline{z} \\ zb - \overline{\alpha} z^{m-1}\theta_0' & \alpha b + z^{m-2}\theta_0' \end{bmatrix}.$$

To get the left coprime factorization of Φ_-, applying Lemma 5.4 for $\widetilde{\Phi_-}$ gives

$$\widetilde{\Phi_-} = \begin{bmatrix} z & \widetilde{\theta}_0\widetilde{\overline{a}} \\ \widetilde{\theta}_1\widetilde{\overline{a}} & z \end{bmatrix} = \widetilde{\Omega}_2 \widetilde{B}_\ell^* \quad \text{(right coprime factorization)},$$

where

$$\Omega_2 := \nu \begin{bmatrix} z^{m-1}\theta_0' & \alpha\theta_1 \\ -\overline{\alpha}\theta_0 & z\theta_1 \end{bmatrix} \quad \text{and} \quad B_\ell := \nu \begin{bmatrix} z^{m-2}\theta_0' + \alpha b & a\overline{z} + \alpha\theta_1\overline{z} \\ -\overline{\alpha}z^{m-1}\theta_0' + zb & -\overline{\alpha}a + \theta_1 \end{bmatrix},$$

which gives

$$\Phi_- = \begin{bmatrix} z & \theta_1\overline{b} \\ \theta_0\overline{a} & z \end{bmatrix} = B_\ell^* \Omega_2 \quad \text{(left coprime factorization)}.$$

On the other hand, since $\Phi_-^* - K\Phi_+^* \in H^2_{M_n}$, and hence

$$\begin{bmatrix} \overline{z} & \overline{\theta}_0 a \\ \overline{\theta}_1 b & \overline{z} \end{bmatrix} - \begin{bmatrix} k_2\overline{\varphi_+} & k_1\overline{\psi_+} \\ k_4\overline{\varphi_+} & k_3\overline{\psi_+} \end{bmatrix} \in H^2_{M_2},$$

we have

$$\begin{cases} \overline{z} - k_2\overline{\varphi_+} \in H^2, & \overline{\theta}_1 b - k_4\overline{\varphi_+} \in H^2 \\ \overline{z} - k_3\overline{\psi_+} \in H^2, & \overline{\theta}_0 a - k_1\overline{\psi_+} \in H^2, \end{cases}$$

which via Cowen's Theorem gives that the following Toeplitz operators are all hyponormal:

$$T_{\overline{z}+\varphi_+}, \ T_{\overline{\theta}_1 b+\varphi_+}, \ T_{\overline{z}+\psi_+}, \ T_{\overline{\theta}_0 a+\psi_+}.$$

Thus by Proposition 3.2 we can see that

$$\varphi_+ = z\theta_1\theta_3\overline{d} \quad \text{and} \quad \psi_+ = \theta_0\theta_2\overline{c} \quad \text{for some inner functions } \theta_2, \theta_3.$$

We thus have

$$\Phi_+ = \begin{bmatrix} z\theta_1\theta_3 & 0 \\ 0 & \theta_0\theta_2 \end{bmatrix} \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}^* \equiv \Delta_2\Delta_0 A_r^* \quad \text{(right coprime factorization)},$$

where

$$A_r := \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix};$$

$$\Delta_0 := \nu \begin{bmatrix} 1 & -\alpha \\ \overline{\alpha}z & z \end{bmatrix} \begin{bmatrix} \theta_3 & 0 \\ 0 & \theta_2 \end{bmatrix};$$

$$\Delta_2 := \nu \begin{bmatrix} z\theta_1 & \alpha\theta_1 \\ -\overline{\alpha}\theta_0 & z^{m-1}\theta_0' \end{bmatrix} = \begin{bmatrix} \theta_1 & 0 \\ 0 & z^{m-1}\theta_0' \end{bmatrix} \cdot \nu \begin{bmatrix} z & \alpha \\ -\overline{\alpha}z & 1 \end{bmatrix}.$$

Write

$$\theta_2 = z^p\theta_2' \quad \text{and} \quad \theta_3 = z^q\theta_3' \quad (p, q \geq 0;\ \theta_2'(0), \theta_3'(0) \neq 0).$$

If $q+1 \geq m+p$, then LCM $(z\theta_1\theta_3, \theta_0\theta_2)$ is an inner divisor of $z^{q+1}\theta_0'\theta_1\theta_2'\theta_3'$. Thus we can write

$$\Phi_+ = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}^* I_{z^{q+1}\theta_0'\theta_1\theta_2'\theta_3'} \equiv A^*\Omega_1\Omega_2,$$

where

$$A := \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \quad (\text{some } x, y \in H^2);$$

$$\Omega_1 := \begin{bmatrix} z^{q+1-m}\theta_1\theta_2'\theta_3' & 0 \\ 0 & z^q\theta_0'\theta_2'\theta_3' \end{bmatrix} \cdot \nu \begin{bmatrix} z & -\alpha \\ \overline{\alpha} z & 1 \end{bmatrix}.$$

It thus follows from Lemma 5.3 that

$$\Delta_2\mathcal{H}(\Delta_0) \subseteq \operatorname{cl\,ran}\,[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}(\Omega_1). \tag{5.9}$$

But since in general, $\Theta_2\mathcal{H}(\Theta_1) \subseteq \mathcal{H}(\Theta_1\Theta_2)$ for inner matrix functions Θ_1, Θ_2, we have

$$\nu \begin{bmatrix} 1 & -\alpha \\ \overline{\alpha} z & z \end{bmatrix} \mathcal{H}\left(\begin{bmatrix} \theta_3 & 0 \\ 0 & \theta_2 \end{bmatrix}\right) \subseteq \mathcal{H}(\Delta_0).$$

Thus by (5.9), we have

$$\Delta_2 \cdot \nu \begin{bmatrix} 1 & -\alpha \\ \overline{\alpha} z & z \end{bmatrix} \mathcal{H}\left(\begin{bmatrix} \theta_3 & 0 \\ 0 & \theta_2 \end{bmatrix}\right) \subseteq \mathcal{H}(\Omega_1),$$

or equivalently,

$$\begin{bmatrix} z\theta_1 & 0 \\ 0 & z^m\theta_0' \end{bmatrix} \mathcal{H}\left(\begin{bmatrix} \theta_3 & 0 \\ 0 & \theta_2 \end{bmatrix}\right) \subseteq \mathcal{H}(\Omega_1). \tag{5.10}$$

Since in general, $F \in \mathcal{H}(\Theta)$ if and only if $\Theta^* F \in (H^2_{\mathbb{C}^n})^\perp$, (5.10) implies

$$\nu \begin{bmatrix} 1 & \alpha \\ -\overline{\alpha} z & z \end{bmatrix} \mathcal{H}\left(\begin{bmatrix} \theta_3 & 0 \\ 0 & \theta_2 \end{bmatrix}\right) \subseteq \mathcal{H}\left(\begin{bmatrix} z^{q+1-m}\theta_2'\theta_3' & 0 \\ 0 & z^{q+1-m}\theta_2'\theta_3' \end{bmatrix}\right). \tag{5.11}$$

Also since for inner matrix functions Θ_1, Θ_2 and any closed subspace F of $H^2_{\mathbb{C}^n}$,

$$\Theta_1 F \subseteq \mathcal{H}(\Theta_1\Theta_2) \quad \text{and} \quad \Theta_1\Theta_2 = \Theta_2\Theta_1 \Longrightarrow F \subseteq \mathcal{H}(\Theta_1\Theta_2),$$

it follows from (5.11) that

$$\begin{bmatrix} \mathcal{H}(\theta_3) \\ \mathcal{H}(\theta_2) \end{bmatrix} \subseteq \begin{bmatrix} \mathcal{H}(z^{q+1-m}\theta_2'\theta_3') \\ \mathcal{H}(z^{q+1-m}\theta_2'\theta_3') \end{bmatrix}.$$

But since $\theta_3 = z^q\theta_3'$, it follows that $q + 1 - m \geq q$, giving a contradiction.

If instead $q + 1 < m + p$, then LCM $(z\theta_1\theta_3, \theta_0\theta_2)$ is an inner divisor of $z^{m+p}\theta_0'\theta_1\theta_2'\theta_3'$. Thus we can write

$$\Phi_+ = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}^* I_{z^{m+p}\theta_0'\theta_1\theta_2'\theta_3'} \equiv A_1^*\Omega_1'\Omega_2,$$

where

$$A_1 := \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \quad (\text{some } x, y \in H^2);$$

$$\Omega_1' := \begin{bmatrix} z^p\theta_1\theta_2'\theta_3' & 0 \\ 0 & z^{m+p-1}\theta_0'\theta_2'\theta_3' \end{bmatrix} \cdot \nu \begin{bmatrix} z & -\alpha \\ \overline{\alpha} z & 1 \end{bmatrix}.$$

It thus follows from Lemma 5.3 with Ω_1' in place of Ω_1 that

$$\Delta_2\mathcal{H}(\Delta_0) \subseteq \operatorname{cl\,ran}\,[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}(\Omega_1').$$

Since

$$\begin{aligned}\mathcal{H}(\Delta_0) &= \mathcal{H}\left(\nu\begin{bmatrix}1 & -\alpha\\ \overline{\alpha}z & z\end{bmatrix}\begin{bmatrix}\theta_3 & 0\\ 0 & \theta_2\end{bmatrix}\right)\\ &= \mathcal{H}\left(\nu\begin{bmatrix}1 & -\alpha\\ \overline{\alpha}z & z\end{bmatrix}\right)\bigoplus \nu\begin{bmatrix}1 & -\alpha\\ \overline{\alpha}z & z\end{bmatrix}\mathcal{H}\left(\begin{bmatrix}\theta_3 & 0\\ 0 & \theta_2\end{bmatrix}\right),\end{aligned}$$

we have

$$\Delta_2\mathcal{H}\left(\nu\begin{bmatrix}1 & -\alpha\\ \overline{\alpha}z & z\end{bmatrix}\right)\bigoplus\begin{bmatrix}z\theta_1 & 0\\ 0 & z^m\theta_0'\end{bmatrix}\mathcal{H}\left(\begin{bmatrix}\theta_3 & 0\\ 0 & \theta_2\end{bmatrix}\right)\subseteq\mathcal{H}(\Omega_1'). \tag{5.12}$$

Then by the same argument as (5.10) and (5.11), we can see that

$$\nu\begin{bmatrix}1 & \alpha\\ -\overline{\alpha}z & z\end{bmatrix}\mathcal{H}\left(\begin{bmatrix}\theta_3 & 0\\ 0 & \theta_2\end{bmatrix}\right)\subseteq\mathcal{H}\left(\begin{bmatrix}z^p\theta_2'\theta_3' & 0\\ 0 & z^p\theta_2'\theta_3'\end{bmatrix}\right), \tag{5.13}$$

which gives

$$z\mathcal{H}(\theta_2)\subseteq\mathcal{H}(z^p\theta_2'\theta_3'),$$

which by Lemma 5.2 implies that θ_2 should be a constant. Thus (5.13) can be written as

$$\nu\begin{bmatrix}1 & \alpha\\ -\overline{\alpha}z & z\end{bmatrix}\mathcal{H}\left(\begin{bmatrix}\theta_3 & 0\\ 0 & 1\end{bmatrix}\right)\subseteq\mathcal{H}\left(\begin{bmatrix}\theta_3' & 0\\ 0 & \theta_3'\end{bmatrix}\right),$$

which gives $z\mathcal{H}(\theta_3)\subseteq\mathcal{H}(\theta_3')$. It follows from again Lemma 5.2 that θ_3 is a constant. Thus by (5.12), we have

$$\begin{bmatrix}\theta_1 & 0\\ 0 & z^{m-1}\theta_0'\end{bmatrix}\nu\begin{bmatrix}z & \alpha\\ -\overline{\alpha}z & 1\end{bmatrix}\mathcal{H}\left(\nu\begin{bmatrix}1 & -\alpha\\ \overline{\alpha}z & z\end{bmatrix}\right)\subseteq\mathcal{H}\left(\begin{bmatrix}\theta_1 & 0\\ 0 & z^{m-1}\theta_0'\end{bmatrix}\nu\begin{bmatrix}z & -\alpha\\ \overline{\alpha}z & 1\end{bmatrix}\right),$$

so that

$$\nu\begin{bmatrix}z & \alpha\\ -\overline{\alpha}z & 1\end{bmatrix}\mathcal{H}\left(\nu\begin{bmatrix}1 & -\alpha\\ \overline{\alpha}z & z\end{bmatrix}\right)\subseteq\mathcal{H}\left(\nu\begin{bmatrix}z & -\alpha\\ \overline{\alpha}z & 1\end{bmatrix}\right),$$

giving a contradiction by Lemma 5.3. Therefore we should have

$$m=1,\quad \text{i.e.,}\quad \theta_0=z\theta_0'.$$

Thus, by Lemmas 5.5 and 5.6, we have

$$\Phi_-=\begin{bmatrix}z & \theta_1\overline{b}\\ z\theta_0'\overline{a} & z\end{bmatrix}=\begin{bmatrix}z\theta_1 & 0\\ 0 & \theta_0\end{bmatrix}\begin{bmatrix}\theta_1 & a\\ zb & \theta_0'\end{bmatrix}^*\quad\text{(right coprime factorization)}$$

and

$$\Phi_-=\begin{bmatrix}z & \theta_1\overline{b}\\ z\theta_0'\overline{a} & z\end{bmatrix}=\begin{bmatrix}\theta_0' & a\\ zb & \theta_1\end{bmatrix}^*\begin{bmatrix}\theta_0 & 0\\ 0 & z\theta_1\end{bmatrix}\quad\text{(left coprime factorization).}$$

Recall that

$$\psi_+=\theta_0\theta_2\overline{c}\quad\text{and}\quad\varphi_+=z\theta_1\theta_3\overline{d}\quad\text{for some inner functions }\theta_2\text{ and }\theta_3.$$

We can thus write

$$\Phi_+=\begin{bmatrix}0 & z\theta_1\theta_3\overline{d}\\ \theta_0\theta_2\overline{c} & 0\end{bmatrix}=\begin{bmatrix}z\theta_1\theta_3 & 0\\ 0 & \theta_0\theta_2\end{bmatrix}\begin{bmatrix}0 & c\\ d & 0\end{bmatrix}^*\quad\text{(right coprime factorization).}$$

Note that LCM $(z\theta_1\theta_3, \theta_0\theta_2)$ is an inner divisor of $\theta_0\theta_1\theta_2\theta_3$. Thus we can write

$$\Phi_+ = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}^* I_{\theta_0\theta_1\theta_2\theta_3} \quad (x, y \in H^2).$$

It follows from Lemma 5.3 that

$$\begin{bmatrix} z\theta_1\mathcal{H}(\theta_3) \\ \theta_0\mathcal{H}(\theta_2) \end{bmatrix} \subseteq \operatorname{cl}\operatorname{ran}[T_\Phi^*, T_\Phi] \subseteq \begin{bmatrix} \mathcal{H}(\theta_1\theta_2\theta_3) \\ \mathcal{H}(\theta_0'\theta_2\theta_3) \end{bmatrix},$$

which implies

$$z\mathcal{H}(\theta_3) \subseteq \mathcal{H}(\theta_2\theta_3) \quad \text{and} \quad z\mathcal{H}(\theta_2) \subseteq \mathcal{H}(\theta_2\theta_3).$$

By Lemma 5.2,

$$\begin{cases} \text{either } \theta_3 \text{ is constant or } z\theta_3 \text{ is a divisor of } \theta_2\theta_3; \\ \text{either } \theta_2 \text{ is constant or } z\theta_2 \text{ is a divisor of } \theta_2\theta_3. \end{cases} \tag{5.14}$$

If θ_2 or θ_3 is not constant then it follows from (5.14) that z is a divisor of θ_2 and θ_3. Thus we have $p, q \geq 1$. Let $N := \max(p, q)$. Then LCM $(z\theta_1\theta_3, \theta_0\theta_2)$ is an inner divisor of $z^N\theta_0\theta_1\theta_2'\theta_3'$. Thus we can write

$$\Phi_+ = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}^* I_{z^N\theta_0\theta_1\theta_2'\theta_3'} \quad (x, y \in H^2).$$

It follows from Lemma 5.3 that

$$\begin{bmatrix} z\theta_1\mathcal{H}(z^q\theta_3') \\ \theta_0\mathcal{H}(z^p\theta_2') \end{bmatrix} \subseteq \quad \operatorname{cl}\operatorname{ran}[T_\Phi^*, T_\Phi] \subseteq \begin{bmatrix} \mathcal{H}(z^N\theta_1\theta_2'\theta_3') \\ \mathcal{H}(z^N\theta_0'\theta_2'\theta_3') \end{bmatrix}$$
$$\Longrightarrow \begin{cases} z^q \text{ is a divisor of } z^{N-1}\theta_2' \\ z^p \text{ is a divisor of } z^{N-1}\theta_3', \end{cases}$$

giving a contradiction. Therefore

$$\theta_2 \text{ and } \theta_3 \text{ are constant.}$$

We observe that $\operatorname{LCM}(z\theta_1, \theta_0)$ is an inner divisor of $z\theta_0'\theta_1$. It follows from Lemma 5.3 that

$$\begin{bmatrix} \theta_1 H^2 \\ \theta_0' H^2 \end{bmatrix} \subseteq \ker[T_\Phi^*, T_\Phi].$$

In particular, $\left[\begin{smallmatrix} 0 \\ \theta_0' \end{smallmatrix}\right] \in \ker[T_\Phi^*, T_\Phi]$. Observe that

$$\Phi_-^* \begin{bmatrix} 0 \\ \theta_0' \end{bmatrix} = \begin{bmatrix} \overline{z} & \overline{\theta}_0 a \\ \overline{\theta}_1 b & \overline{z} \end{bmatrix} \begin{bmatrix} 0 \\ \theta_0' \end{bmatrix} = \begin{bmatrix} \overline{z}a \\ \overline{z}\theta_0' \end{bmatrix},$$

so that

$$H_{\Phi_-^*} \begin{bmatrix} 0 \\ \theta_0' \end{bmatrix} = \begin{bmatrix} a(0) \\ \theta_0'(0) \end{bmatrix}.$$

We thus have

$$H_{\Phi_-^*}^* H_{\Phi_-^*} \begin{bmatrix} 0 \\ \theta_0' \end{bmatrix} = \begin{bmatrix} \theta_0'(0) J(I-P)(\overline{\widetilde{\theta}}_1 \widetilde{b}) + a(0) \\ * \end{bmatrix}.$$

A similar calculation shows that

$$H^*_{\Phi^*_+} H_{\Phi^*_+} \begin{bmatrix} 0 \\ \theta_0' \end{bmatrix} = \begin{bmatrix} 0 \\ * \end{bmatrix}.$$

Since $[T^*_\Phi, T_\Phi] = H^*_{\Phi^*_+} H_{\Phi^*_+} - H^*_{\Phi^*_-} H_{\Phi^*_-}$, it follows that

$$\theta_0'(0) J(I - P)(\overline{\widetilde{\theta}_1} \widetilde{b}) = -a(0).$$

But since $a(0) \neq 0$, we must have that $\overline{\widetilde{\theta}_1} \widetilde{b} \in \overline{z} H^2 \cap (H^2)^\perp$, which implies that $\theta_1 = cz$ for a nonzero constant c, giving a contradiction because $\theta_1(0) \neq 0$. Therefore this case cannot occur.

Case 2 ($m = 0$ and $n \neq 0$) This case is symmetrical to Case 1. Thus the proof is identical to that of Case 1. Therefore this case cannot occur either.

Case 3 ($m \neq 0, n \neq 0$ and $m \geq 2$ or $n \geq 2$) In this case we have $a(0) \neq 0$ and $b(0) \neq 0$ because $\theta_0(0) = \theta_1(0) = 0$, θ_0 and a are coprime, and θ_1 and b are coprime. By Lemma 5.6 we have

$$\Phi_- = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_0 \end{bmatrix} \begin{bmatrix} z^{n-1}\theta_1' & a \\ b & z^{m-1}\theta_0' \end{bmatrix}^* \quad \text{(right coprime factorization).}$$

Similarly, we have

$$\Phi_- = \begin{bmatrix} z^{m-1}\theta_0' & a \\ b & z^{n-1}\theta_1' \end{bmatrix}^* \begin{bmatrix} \theta_0 & 0 \\ 0 & \theta_1 \end{bmatrix} \quad \text{(left coprime factorization)}$$

and

$$\Phi_+ = \begin{bmatrix} \theta_1\theta_3 & 0 \\ 0 & \theta_0\theta_2 \end{bmatrix} \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}^* \quad \text{(right coprime factorization).}$$

We then claim that

$$\theta_2 \text{ and } \theta_3 \text{ are constant.} \tag{5.15}$$

Assume θ_2 is not constant. Put $\theta_2 = z^p \theta_2'$ and $\theta_3 = z^q \theta_3'$ ($\theta_2' \neq 0 \neq \theta_3'(0)$ and $p, q \geq 0$) and let $N := \max(m + p, n + q)$. Then $\mathrm{LCM}(\theta_0\theta_2, \theta_1\theta_3)$ is a divisor of $z^N \theta_0'\theta_1'\theta_2'\theta_3'$. Thus we can write

$$\Phi_+ = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}^* I_{z^N\theta_0'\theta_1'\theta_2'\theta_3'} \quad (x, y \in H^2).$$

By Lemma 5.3 we have

$$\begin{bmatrix} \theta_1 \mathcal{H}(\theta_3) \\ \theta_0 \mathcal{H}(\theta_2) \end{bmatrix} \subseteq \begin{bmatrix} \mathcal{H}(z^{N-m}\theta_1'\theta_2'\theta_3') \\ \mathcal{H}(z^{N-n}\theta_0'\theta_2'\theta_3') \end{bmatrix}.$$

If θ_3 is a constant, then $q = 0$ and $m + p \leq N - n$, giving a contradiction because $m, n \geq 2$. If θ_3 is not constant, then $n + q \leq N - m$ and $m + p \leq N - n$, giving a contradiction because $m, n \geq 2$. Therefore we should have that θ_2 is constant. Similarly, we can show that θ_3 is also constant. This proves (5.15).

We now suppose $n \leq m$. By Lemma 5.3 we have

$$\begin{bmatrix} \theta_1' H^2 \\ z^{m-n}\theta_0' H^2 \end{bmatrix} \subseteq \ker[T^*_\Phi, T_\Phi].$$

In particular, $\begin{bmatrix} \theta_1' \\ 0 \end{bmatrix} \in \ker[T_\Phi^*, T_\Phi]$. Observe that

$$\Phi_-^* \begin{bmatrix} \theta_1' \\ 0 \end{bmatrix} = \begin{bmatrix} \overline{z} & \overline{\theta}_0 a \\ \overline{\theta}_1 b & \overline{z} \end{bmatrix} \begin{bmatrix} \theta_1' \\ 0 \end{bmatrix} = \begin{bmatrix} \overline{z}\theta_1' \\ \overline{z}^n b \end{bmatrix},$$

so that

$$H_{\Phi_-^*} \begin{bmatrix} \theta_1' \\ 0 \end{bmatrix} = \begin{bmatrix} \theta_1'(0) \\ z^{n-1}\overline{\widetilde{b}}_1 \end{bmatrix} \quad (b_1 := P_{\mathcal{H}(z^n)}(b)).$$

Put $b_3 := z^{n-1}\overline{\widetilde{b}}_1$. We also have

$$\widetilde{\Phi_-^*} \begin{bmatrix} \theta_1'(0) \\ b_3 \end{bmatrix} = \begin{bmatrix} \overline{z} & \overline{\widetilde{\theta}}_1 \widetilde{b} \\ \overline{\widetilde{\theta}}_0 \widetilde{a} & \overline{z} \end{bmatrix} \begin{bmatrix} \theta_1'(0) \\ b_3 \end{bmatrix} = \begin{bmatrix} * \\ \theta_1'(0)\overline{\widetilde{\theta}}_0 \widetilde{a} + b_3 \overline{z} \end{bmatrix},$$

so that

$$H_{\Phi_-^*}^* H_{\Phi_-^*} \begin{bmatrix} \theta_1' \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ \theta_1'(0) J(I-P)(\overline{\widetilde{\theta}}_0 \widetilde{a}) + b_3(0) \end{bmatrix}.$$

A similar calculation shows that

$$H_{\Phi_+^*}^* H_{\Phi_+^*} \begin{bmatrix} \theta_1' \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix}.$$

It thus follows that

$$\begin{aligned} \theta_1'(0) J(I-P)(\overline{\widetilde{\theta}}_0 \widetilde{a}) = \quad & -b_3(0) \Longrightarrow \overline{\widetilde{\theta}}_0 \widetilde{a} \in \overline{z} H^2 \\ & \Longrightarrow \overline{\theta}_0 a \in \overline{z} H^2 \Longrightarrow \overline{\theta}_0 a \in \overline{z} H^2 \cap \left(H^2\right)^\perp. \end{aligned}$$

Since $n \le m$ and ($m \ge 2$ or $n \ge 2$), it follows that $m \ge 2$. Thus $\overline{\theta_0}\, a \in \overline{z} H^2 \cap (H^2)^\perp$ implies $\overline{z}^{m-1}\overline{\theta_0'}\, a = c$ (a constant), which forces $a = 0$, giving a contradiction. If instead $n > m$ then the same argument leads a contradiction. Therefore this case cannot occur. This completes the proof of (5.8).

Now in view of (5.8) it suffices to consider the case $m = n = 0$ and the case $m = n = 1$.

Case A ($m = n = 0$). In this case, we first claim that

$$\varphi_- = \psi_- = 0, \quad \text{i.e.,} \quad \varphi \text{ and } \psi \text{ are analytic.} \tag{5.16}$$

Put $\theta := \mathrm{GCD}(\theta_0, \theta_1)$. Then $\theta_0 = \theta_0'\theta$ and $\theta_1 = \theta_1'\theta$ for some inner functions θ_0', θ_1', and hence $\mathrm{LCM}(\theta_0, \theta_1) = \theta\theta_0'\theta_1'$. We thus have

$$\Phi_- = I_{z\theta\theta_0'\theta_1'} \begin{bmatrix} \theta\theta_0'\theta_1' & z\theta_1' a \\ z\theta_0' b & \theta\theta_0'\theta_1' \end{bmatrix}^* \equiv I_{z\theta\theta_0'\theta_1'} B^*.$$

Since $B(0)$ is invertible it follows from Lemma 2.3 that I_z and B are coprime. Observe that

$$\begin{aligned} \Phi_-^* \begin{bmatrix} zf \\ zg \end{bmatrix} = \quad & \begin{bmatrix} \overline{z} & \overline{\theta}_0 a \\ \overline{\theta}_1 b & \overline{z} \end{bmatrix} \begin{bmatrix} zf \\ zg \end{bmatrix} \in H^2_{\mathbb{C}^2} \Longleftrightarrow \begin{bmatrix} \overline{\theta}_0 azg \\ \overline{\theta}_1 bzf \end{bmatrix} \in H^2_{\mathbb{C}^2} \\ & \Longleftrightarrow g \in \theta_0 H^2, \quad f \in \theta_1 H^2, \end{aligned}$$

which implies

$$\ker H_{\Phi_-^*} = \begin{bmatrix} z\theta_1 & 0 \\ 0 & z\theta_0 \end{bmatrix} H^2_{\mathbb{C}^2}.$$

We thus have

$$\Phi_- = \begin{bmatrix} z\theta_1 & 0 \\ 0 & z\theta_0 \end{bmatrix} \begin{bmatrix} \theta_1 & za \\ zb & \theta_0 \end{bmatrix}^* \quad \text{(right coprime factorization).}$$

To get the left coprime factorization of Φ_-, we take

$$\widetilde{\Phi_-} = \begin{bmatrix} z\widetilde{\theta_0} & 0 \\ 0 & z\widetilde{\theta_1} \end{bmatrix} \begin{bmatrix} \widetilde{\theta_0} & z\widetilde{b} \\ z\widetilde{a} & \widetilde{\theta_1} \end{bmatrix}^* \quad \text{(right coprime factorization),}$$

which implies

$$\Phi_- = \begin{bmatrix} \theta_0 & za \\ zb & \theta_1 \end{bmatrix}^* \begin{bmatrix} z\theta_0 & 0 \\ 0 & z\theta_1 \end{bmatrix} \quad \text{(left coprime factorization).}$$

Since T_Φ is hyponormal and hence,

$$\ker H_{\Phi_+^*} \subseteq \ker H_{\Phi_-^*} = \begin{bmatrix} z\theta_1 H^2 \\ z\theta_0 H^2 \end{bmatrix},$$

it follows that

$$\psi_+ = z\theta_0\theta_2\overline{c} \quad \text{and} \quad \varphi_+ = z\theta_1\theta_3\overline{d} \quad \text{for some inner functions } \theta_2, \theta_3.$$

We can thus write

$$\Phi_+ = \begin{bmatrix} 0 & z\theta_1\theta_3\overline{d} \\ z\theta_0\theta_2\overline{c} & 0 \end{bmatrix} = \begin{bmatrix} z\theta_1\theta_3 & 0 \\ 0 & z\theta_0\theta_2 \end{bmatrix} \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}^* \quad \text{(right coprime factorization).}$$

Observe that LCM $(z\theta_1\theta_3, z\theta_0\theta_2)$ is an inner divisor of $z\theta_0\theta_1\theta_2\theta_3$. Thus we can write

$$\Phi_+ = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}^* I_{z\theta_0\theta_1\theta_2\theta_3} \quad (x, y \in H^2).$$

It follows from Lemma 5.3 that

$$\begin{bmatrix} z\theta_1\mathcal{H}(\theta_3) \\ z\theta_0\mathcal{H}(\theta_2) \end{bmatrix} \subseteq \operatorname{cl}\operatorname{ran}\,[T_\Phi^*, T_\Phi] \subseteq \begin{bmatrix} \mathcal{H}(\theta_1\theta_2\theta_3) \\ \mathcal{H}(\theta_0\theta_2\theta_3) \end{bmatrix},$$

which implies that

$$z\mathcal{H}(\theta_3) \subseteq \mathcal{H}(\theta_2\theta_3) \quad \text{and} \quad z\mathcal{H}(\theta_2) \subseteq \mathcal{H}(\theta_2\theta_3).$$

Thus the same argument as in (5.14) shows that

$$\theta_2 \text{ and } \theta_3 \text{ are constant.}$$

We now observe that LCM$(z\theta_1, z\theta_0)$ is an inner divisor of $z\theta_0\theta_1$. Thus we can write

$$\Phi_+ = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}^* I_{z\theta_0\theta_1} \quad (x, y \in H^2).$$

It follows from Lemma 5.3 that $\begin{bmatrix}\theta_1 H^2\\ \theta_0 H^2\end{bmatrix} \subseteq \ker[T^*_\Phi, T_\Phi]$. In particular, $\begin{bmatrix}\theta_1\\ 0\end{bmatrix} \in \ker[T^*_\Phi, T_\Phi]$. Observe that

$$\Phi^*_- \begin{bmatrix}\theta_1\\ 0\end{bmatrix} = \begin{bmatrix}\overline{z} & \overline{\theta}_0 a\\ \overline{\theta}_1 b & \overline{z}\end{bmatrix}\begin{bmatrix}\theta_1\\ 0\end{bmatrix} = \begin{bmatrix}\overline{z}\theta_1\\ b\end{bmatrix},$$

so that

$$H_{\Phi^*_-}\begin{bmatrix}\theta_1\\ 0\end{bmatrix} = \begin{bmatrix}\theta_1(0)\\ 0\end{bmatrix}.$$

We thus have

$$H^*_{\Phi^*_-} H_{\Phi^*_-}\begin{bmatrix}\theta_1\\ 0\end{bmatrix} = \begin{bmatrix}\theta_1(0)\\ \theta_1(0)J(I-P)(\overline{\widetilde{\theta}}_0\widetilde{a})\end{bmatrix}.$$

A similar calculation shows that

$$H^*_{\Phi^*_+} H_{\Phi^*_+}\begin{bmatrix}\theta_1\\ 0\end{bmatrix} = \begin{bmatrix}*\\ 0\end{bmatrix}.$$

It thus follows that

$$\theta_1(0)J(I-P)(\overline{\widetilde{\theta}}_0\widetilde{a}) = 0 \Longrightarrow \overline{\widetilde{\theta}}_0\widetilde{a} \in H^2 \Longrightarrow \overline{\theta}_0 a \in H^2 \Longrightarrow \overline{\theta}_0 a \in H^2 \cap \left(H^2\right)^\perp,$$

which implies that $a = 0$ and hence φ is analytic. Similarly, we can show that ψ is also analytic. This gives (5.16).

Now since by (5.16), $\varphi, \psi \in H^\infty$ and $|\varphi| = |\psi|$, we can write $\varphi = \theta_1 a$ and $\psi = \theta_2 a$, where the θ_i are inner functions and a is an outer function. Observe that

$$\Phi_- \equiv B^*\Theta_2,$$

where $B \equiv \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $\Theta_2 \equiv \begin{bmatrix}z & 0\\ 0 & z\end{bmatrix}$ are coprime. Thus our symbol satisfies all the assumptions of Theorem 4.1. Thus by Theorem 4.1, since T_Φ is 2-hyponormal then T_Φ must be normal. We thus have

$$H^*_{\Phi^*_+} H_{\Phi^*_+} = H^*_{\Phi^*_-} H_{\Phi^*_-}. \tag{5.17}$$

Now observe that

$$\Phi_+ = \begin{bmatrix}0 & \varphi\\ \psi & 0\end{bmatrix} \quad \text{and} \quad \Phi_- = \begin{bmatrix}z & 0\\ 0 & z\end{bmatrix}.$$

Since T_Φ is normal we have

$$\begin{bmatrix}H^*_{\overline{\varphi}} H_{\overline{\varphi}} & 0\\ 0 & H^*_{\overline{\psi}} H_{\overline{\psi}}\end{bmatrix} = \begin{bmatrix}H_{\overline{z}} & 0\\ 0 & H_{\overline{z}}\end{bmatrix},$$

which implies

$$H^*_{\overline{\varphi}} H_{\overline{\varphi}} = H_{\overline{z}} = H^*_{\overline{\psi}} H_{\overline{\psi}}, \tag{5.18}$$

which says that $H_{\overline{\varphi}}$ and $H_{\overline{\psi}}$ are both rank-one operators. Now remember that if T is a rank-one Hankel operator then there exist $\omega \in \mathbb{D}$ and a constant c such that $T = c\,(k_{\overline{\omega}} \otimes k_\omega)$, where

$k_\omega := \frac{1}{1-\overline{\omega}z}$ is the reproducing kernel for ω. Note that $k_{\overline{\omega}} \otimes k_\omega$ is represented by the matrix

$$\begin{bmatrix} 1 & \omega & \omega^2 & \omega^3 & \cdots \\ \omega & \omega^2 & \omega^3 & \cdots & \\ \omega^2 & \omega^3 & \cdots & & \\ \omega^3 & & & & \\ \vdots & & & & \end{bmatrix}.$$

By (5.18) we have that $\omega = 0$. We thus have

$$\varphi = e^{i\theta_1}z + \beta_1 \quad \text{and} \quad \psi = e^{i\theta_2}z + \beta_2 \quad (\beta_1, \beta_2 \in \mathbb{C},\ \theta_1, \theta_2 \in [0, 2\pi)). \tag{5.19}$$

But since $|\varphi| = |\psi|$, we have

$$\varphi = e^{i\theta}z + \beta \quad \text{and} \quad \psi = e^{i\omega}\varphi \quad (\beta \in \mathbb{C},\ \theta, \omega \in [0, 2\pi)). \tag{5.20}$$

Case B ($m = n = 1$) In this case, $\theta_0 = z\theta_0'$ and $\theta_1 = z\theta_1'$ ($\theta_0'(0), \theta_1'(0) \neq 0$). We thus have

$$\varphi_- = z\theta_0'\overline{a} \quad \text{and} \quad \psi_- = z\theta_1'\overline{b},$$

so that

$$\Phi_- = \begin{bmatrix} z & z\theta_1'\overline{b} \\ z\theta_0'\overline{a} & z \end{bmatrix}.$$

There are two subcases to consider.

Case B-1 $\big((ab)(0) \neq (\theta_0'\theta_1')(0)\big)$. In this case, we have, by Lemma 5.6,

$$\begin{aligned} \Phi_- &= \begin{bmatrix} z & z\theta_1'\overline{b} \\ z\theta_0'\overline{a} & z \end{bmatrix} = \begin{bmatrix} z\theta_1' & 0 \\ 0 & z\theta_0' \end{bmatrix} \begin{bmatrix} \theta_1' & a \\ b & \theta_0' \end{bmatrix}^* \quad \text{(right coprime decomposition)} \\ &= \begin{bmatrix} \theta_0' & a \\ b & \theta_1' \end{bmatrix}^* \begin{bmatrix} z\theta_0' & 0 \\ 0 & z\theta_1' \end{bmatrix} \quad \text{(left coprime factorization)} \end{aligned}$$

and

$$\Phi_+ = \begin{bmatrix} z\theta_1'\theta_3 & 0 \\ 0 & z\theta_0'\theta_2 \end{bmatrix} \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}^* \quad \text{(right coprime factorization)}.$$

Suppose θ_2 is not constant. Put $\theta_2 = z^p\theta_2'$ and $\theta_3 = z^q\theta_3'$ ($p, q \in \mathbb{N} \cup \{0\}$). Let $N := \max(p, q)$. Then $\text{LCM}(\theta_1\theta_3, \theta_0\theta_2)$ is a divisor of $z^{N+1}\theta_0'\theta_1'\theta_2'\theta_3'$. Thus we can write

$$\Phi_+ = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}^* I_{z^{N+1}\theta_0'\theta_1'\theta_2'\theta_3'} \quad (x, y \in H^2).$$

By Lemma 5.3 we have

$$\begin{bmatrix} z\theta_1'\mathcal{H}(\theta_3) \\ z\theta_0'\mathcal{H}(\theta_2) \end{bmatrix} \subseteq \begin{bmatrix} \mathcal{H}(z^N\theta_1'\theta_2'\theta_3') \\ \mathcal{H}(z^N\theta_0'\theta_2'\theta_3') \end{bmatrix}.$$

If θ_3 is a constant then $p + 1 \leq N = p$, giving a contradiction. If instead θ_3 is not constant then $q + 1 \leq N$ and $p + 1 \leq N$, giving a contradiction. The same argument gives θ_3 is a constant. Therefore θ_2 and θ_3 should be constant. Note that $\text{LCM}(z\theta_1', z\theta_0')$ is an inner divisor of $z\theta_0'\theta_1'$. Thus we can write

$$\Phi_+ = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}^* I_{z\theta_0'\theta_1'} \quad (x, y \in H^2).$$

It follows from Lemma 5.3 that

$$\begin{bmatrix}\theta_1' H^2\\ \theta_0' H^2\end{bmatrix} \subseteq \ker[T_\Phi^*, T_\Phi]. \tag{5.21}$$

In particular, $\left[\begin{smallmatrix}\theta_1'\\ 0\end{smallmatrix}\right] \in \ker[T_\Phi^*, T_\Phi]$. Observe that

$$\Phi_-^*\begin{bmatrix}\theta_1'\\ 0\end{bmatrix} = \begin{bmatrix}\overline{z} & \overline{\theta}_0 a\\ \overline{\theta}_1 b & \overline{z}\end{bmatrix}\begin{bmatrix}\theta_1'\\ 0\end{bmatrix} = \begin{bmatrix}\overline{z}\theta_1'\\ \overline{z}b\end{bmatrix},$$

so that

$$H_{\Phi_-^*}\begin{bmatrix}\theta_1'\\ 0\end{bmatrix} = \begin{bmatrix}\theta_1'(0)\\ b(0)\end{bmatrix}.$$

We thus have

$$\widetilde{\Phi_-^*}\begin{bmatrix}\theta_1'(0)\\ b(0)\end{bmatrix} = \begin{bmatrix}\overline{z} & \widetilde{\overline{\theta}}_1\widetilde{b}\\ \widetilde{\overline{\theta}}_0\widetilde{a} & \overline{z}\end{bmatrix}\begin{bmatrix}\theta_1'(0)\\ b(0)\end{bmatrix} = \begin{bmatrix}*\\ \theta_1'(0)\widetilde{\overline{\theta}}_0\widetilde{a} + b(0)\overline{z}\end{bmatrix},$$

so that

$$H_{\Phi_-^*}^* H_{\Phi_-^*}\begin{bmatrix}\theta_1'\\ 0\end{bmatrix} = \begin{bmatrix}*\\ \theta_1'(0)J(I-P)(\widetilde{\overline{\theta}}_0\widetilde{a}) + b(0)\end{bmatrix}.$$

A similar calculation shows that

$$H_{\Phi_+^*}^* H_{\Phi_+^*}\begin{bmatrix}\theta_1'\\ 0\end{bmatrix} = \begin{bmatrix}*\\ 0\end{bmatrix}.$$

It thus follows that

$$\theta_1'(0)J(I-P)(\widetilde{\overline{\theta}}_0\widetilde{a}) = -b(0).$$

Since $b(0) \neq 0$, we have that $\widetilde{\overline{\theta}}_0\widetilde{a} \in \overline{z}H^2$, which implies that $\overline{\theta_0}a = \alpha\overline{z}$ for a nonzero constant α. Therefore we must have that θ_0' is a constant. Similarly, we can show that $\overline{\theta_1}b = \beta\overline{z}$ for a nonzero constant β, and hence θ_1' is also a constant. Therefore by (5.21), T_Φ is normal. Now observe that

$$\Phi_+ = \begin{bmatrix}0 & \varphi_+\\ \psi_+ & 0\end{bmatrix} \quad \text{and} \quad \Phi_-^* = \begin{bmatrix}\overline{z} & \alpha\overline{z}\\ \beta\overline{z} & \overline{z}\end{bmatrix} \quad (\alpha \neq 0 \neq \beta).$$

Since T_Φ is normal we have

$$\begin{bmatrix}H_{\overline{\varphi_+}}^* H_{\overline{\varphi_+}} & 0\\ 0 & H_{\overline{\psi_+}}^* H_{\overline{\psi_+}}\end{bmatrix} = \begin{bmatrix}(1+|\beta|^2)H_{\overline{z}} & (\alpha+\overline{\beta})H_{\overline{z}}\\ (\overline{\alpha}+\beta)H_{\overline{z}} & (1+|\alpha|^2)H_{\overline{z}}\end{bmatrix},$$

which implies that

$$\begin{cases}\beta = -\overline{\alpha}\\ H_{\overline{\varphi_+}}^* H_{\overline{\varphi_+}} = (1+|\beta|^2)H_{\overline{z}}\\ H_{\overline{\psi_+}}^* H_{\overline{\psi_+}} = (1+|\alpha|^2)H_{\overline{z}}.\end{cases} \tag{5.22}$$

By the case assumption, $1 \neq |ab| = |\alpha\beta| = |\alpha|^2$, i.e., $|\alpha| \neq 1$. By the same argument as in (5.18) we have

$$\varphi_+ = e^{i\theta_1}\sqrt{1+|\alpha|^2}\,z + \beta_1 \quad \text{and} \quad \psi_+ = e^{i\theta_2}\sqrt{1+|\alpha|^2}\,z + \beta_2,$$

($\beta_1, \beta_2 \in \mathbb{C}$; $\theta_1, \theta_2 \in [0, 2\pi)$) which implies that

$$\varphi = \alpha\,\overline{z} + e^{i\theta_1}\sqrt{1+|\alpha|^2}\,z + \beta_1 \quad \text{and} \quad \psi = -\overline{\alpha}\,\overline{z} + e^{i\theta_2}\sqrt{1+|\alpha|^2}\,z + \beta_2.$$

Since $|\varphi| = |\psi|$, it follows that

$$\left| e^{i\theta_1}\sqrt{1+|\alpha|^2}\,z^2 + \beta_1 z + \alpha \right| = \left| e^{i\theta_2}\sqrt{1+|\alpha|^2}\,z^2 + \beta_2 z - \overline{\alpha} \right| \quad \text{for all } z \text{ on } \mathbb{T}.$$

We argue that if p and q are polynomials having the same degree and the outer coefficients of the same modulus then

$$|p(z)| = |q(z)| \quad \text{on } |z| = 1 \Longrightarrow p(z) = e^{i\omega}q(z) \text{ for some } \omega \in [0, 2\pi).$$

Indeed, if $|p(z)| = |q(z)|$ on $|z| = 1$, then $p = \theta q$ for a finite Blaschke product θ, i.e., $p = \prod_{j=1}^{n} \frac{z-\alpha_j}{1-\overline{\alpha}_j z} q$ ($|\alpha_j| \le 1$). But since the modulus of the outer coefficients are same, it follows that $\prod_{j=1}^{n} |\alpha_j| = 1$ and therefore, $p = e^{i\omega}q$ for some ω. Using this fact we can see that $\psi = e^{i\omega}\varphi$ for some $\omega \in [0, 2\pi)$. But then a straightforward calculation shows that $\omega = \pi - 2\arg\alpha$, and hence

$$\varphi = \alpha\,\overline{z} + e^{i\theta}\sqrt{1+|\alpha|^2}\,z + \beta \quad \text{and} \quad \psi = e^{i(\pi - 2\arg\alpha)}\varphi,$$

where $\alpha \neq 0$, $|\alpha| \neq 1$, $\beta \in \mathbb{C}$, and $\theta \in [0, 2\pi)$.

Case B-2 $\big((ab)(0) = (\theta_0'\theta_1')(0)\big)$. A similar argument as in Case 1 shows that θ_2 and θ_3 are constant and the same argument as in Case B-1 gives that

$$\varphi = \alpha\,\overline{z} + e^{i\theta}\sqrt{1+|\alpha|^2}\,z + \beta \quad \text{and} \quad \psi = e^{i(\pi - 2\arg\alpha)}\varphi,$$

where $\alpha \neq 0$, $|\alpha| = 1$, $\beta \in \mathbb{C}$, and $\theta \in [0, 2\pi)$. We here note that the condition $|\alpha| = 1$ comes from the case assumption $1 = |\theta_0'\theta_1'| = |ab| = |\alpha|^2$.

Therefore if we combine the two subcases of Case B-1 and B-2 then we can conclude that

$$\varphi = \alpha\,\overline{z} + e^{i\theta}\sqrt{1+|\alpha|^2}\,z + \beta \quad \text{and} \quad \psi = e^{i(\pi - 2\arg\alpha)}\varphi, \tag{5.23}$$

where $\alpha \neq 0$, $\beta \in \mathbb{C}$, and $\theta \in [0, 2\pi)$. This completes the proof. □

Remark 5.8. We would also ask whether there is a subnormal *non*-Toeplitz completion of $\left[\begin{smallmatrix} T_{\overline{z}} & ? \\ ? & T_{\overline{z}} \end{smallmatrix}\right]$. Unexpectedly, there is a normal non-Toeplitz completion of $\left[\begin{smallmatrix} T_{\overline{z}} & ? \\ ? & T_{\overline{z}} \end{smallmatrix}\right]$. To see this, let B be a selfadjoint operator and put

$$T = \begin{bmatrix} T_{\overline{z}} & T_z + B \\ T_z + B & T_{\overline{z}} \end{bmatrix}.$$

Then

$$[T^*, T] = \begin{bmatrix} T_{\overline{z}}B + BT_z - (T_zB + BT_{\overline{z}}) & T_zB + BT_{\overline{z}} - (T_{\overline{z}}B + BT_z) \\ BT_{\overline{z}} + T_zB - (BT_z + T_{\overline{z}}B) & T_{\overline{z}}B + BT_z - (T_zB + BT_{\overline{z}}) \end{bmatrix},$$

so that T is normal if and only if

$$T_{\overline{z}}B + BT_z = T_zB + BT_{\overline{z}}, \quad \text{i.e.,} \quad [T_z, B] = [T_{\overline{z}}, B]. \tag{5.24}$$

We define

$$\alpha_1 := 0 \quad \text{and} \quad \alpha_n := -\frac{2}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right) \quad \text{for } n \geq 2.$$

Let $D \equiv \operatorname{diag}(\alpha_n)$, i.e., a diagonal operator whose diagonal entries are α_n $(n = 1, 2, \ldots)$ and for each $n = 1, 2, \ldots$, let B_n be defined by

$$B_n = -\frac{1}{2^{n-1}} \operatorname{diag}(\alpha_{n-1}) T^*_{z^{2n}}.$$

Then

$$\|B_n\| \le \frac{1}{2^{n-1}} \sup\{\alpha_{n-1}\} < \frac{1}{2^{n-1}},$$

which implies that

$$\left\| \sum_{n=1}^{\infty} B_n \right\| \le 2.$$

We define C by

$$C := \sum_{n=1}^{\infty} B_n.$$

Then C looks like:

$$C = \begin{bmatrix}
0 & 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2^2} & 0 & \cdots \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2^2} & 0 & \frac{1}{2^3} & \cdots \\
0 & 0 & 0 & 0 & \frac{3}{2^2} & 0 & \frac{3}{2^3} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \frac{5}{2^3} & 0 & \frac{5}{2^4} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{11}{2^4} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{21}{2^5} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.$$

Note that C is bounded. If we define B by

$$B := D + C + C^*,$$

then a straightforward calculation shows that B satisfies Eq. (5.24). Therefore the operator

$$T = \begin{bmatrix} T_{\bar{z}} & T_z + B \\ T_z + B & T_{\bar{z}} \end{bmatrix}$$

is normal. We note that $T_z + B$ is not a Toeplitz operator. $\square$

Remark 5.9. In Theorem 5.1 we have seen that a 2-hyponormal Toeplitz completion of $\left[\begin{smallmatrix} T_{\bar{z}} & ? \\ ? & T_{\bar{z}} \end{smallmatrix}\right]$ is automatically normal. Consequently, from the viewpoint of k-hyponormality as a bridge between hyponormality and subnormality, there is no gap between the 2-hyponormality and the subnormality of $\begin{bmatrix} T_{\overline{z}} & T_\varphi \\ T_\psi & T_{\overline{z}} \end{bmatrix}$ $(\varphi, \psi \in H^2)$. Of course there does exist a gap between the hyponormality and the 2-hyponormality of $\begin{bmatrix} T_{\overline{z}} & T_\varphi \\ T_\psi & T_{\overline{z}} \end{bmatrix}$. To see this, let

$$\Phi := \begin{bmatrix} \overline{z} & \overline{z}^2 + 2z^2 \\ \overline{z}^2 + 2z^2 & \overline{z} \end{bmatrix}.$$

Then Φ is normal and if we put $K := \begin{bmatrix} \frac{1}{2} & \frac{z}{2} \\ \frac{z}{2} & \frac{1}{2} \end{bmatrix}$, then $\Phi - K\,\Phi^* \in H^2_{M_2}$ and $\|K\|_\infty = 1$, so that T_Φ is hyponormal. But by Theorem 5.1, T_Φ is not 2-hyponormal. However, we have not been able to characterize all hyponormal completions of $\begin{bmatrix} T_{\overline{z}} & ? \\ ? & T_{\overline{z}} \end{bmatrix}$; this completion problem appears to be quite difficult. $\square$

6. Open problems

1. *Nakazi–Takahashi's Theorem for matrix-valued symbols*. Nakazi and Takahashi [53] have shown that if $\varphi \in L^\infty$ is such that T_φ is a hyponormal operator whose self-commutator $[T_\varphi^*, T_\varphi]$ is of finite rank then there exists a finite Blaschke product $b \in \mathcal{E}(\varphi)$ such that

$$\deg(b) = \operatorname{rank}\,[T_\varphi^*, T_\varphi].$$

What is the matrix-valued version of Nakazi and Takahashi's Theorem? A candidate is as follows: If $\Phi \in L^\infty_{M_n}$ is such that T_Φ is a hyponormal operator whose self-commutator $[T_\Phi^*, T_\Phi]$ is of finite rank then there exists a finite Blaschke–Potapov product $B \in \mathcal{E}(\Phi)$ such that $\deg(B) = \operatorname{rank}\,[T_\Phi^*, T_\Phi]$. We note that the degree of the finite Blaschke–Potapov product B is defined by

$$\deg(B) := \dim \mathcal{H}(B) = \deg(\det B), \tag{6.1}$$

where the second equality follows from the well-known Fredholm theory of block Toeplitz operators [27] that

$$\begin{aligned} \dim \mathcal{H}(\Theta) &= \dim\ker T_{\Theta^*} = -\operatorname{index} T_\Theta \\ &= -\operatorname{index} T_{\det\Theta} = \dim\ker T_{\overline{\det\Theta}} \\ &= \dim\Big(\mathcal{H}(\det\Theta)\Big) = \deg\Big(\det\Theta\Big). \end{aligned}$$

Thus we conjecture the following:

Conjecture 6.1. *If $\Phi \in L^\infty_{M_n}$ is such that T_Φ is a hyponormal operator whose self-commutator $[T_\Phi^*, T_\Phi]$ is of finite rank then there exists a finite Blaschke–Potapov product $B \in \mathcal{E}(\Phi)$ such that* $\operatorname{rank}\,[T_\Phi^*, T_\Phi] = \deg\Big(\det B\Big)$.

On the other hand, in [53], it was shown that if $\varphi \in L^\infty$ is such that T_φ is subnormal and $\varphi = q\overline{\varphi}$, where q is a finite Blaschke product then T_φ is normal or analytic. We now we pose its block version:

Problem 6.2. If $\Phi \in L^\infty_{M_n}$ is such that T_Φ is subnormal and $\Phi = B\Phi^*$, where B is a finite Blaschke–Potapov product, does it follow that T_Φ is normal or analytic?

2. *Subnormality of block Toeplitz operators*. In Remark 4.7 we have shown that if the "coprime" condition of Theorem 4.5 is dropped, then Theorem 4.5 may fail. However we note that the example given in Remark 4.7 is a direct sum of a normal Toeplitz operator and an analytic Toeplitz operator. Based on this observation, we have:

Problem 6.3. Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued rational function. If T_Φ and T^2_Φ are hyponormal, but T_Φ is neither normal nor analytic, does it follow that T_Φ is of the form

$$T_\Phi = \begin{bmatrix} T_A & 0 \\ 0 & T_B \end{bmatrix} \quad \text{(where } T_A \text{ is normal and } T_B \text{ is analytic)?}$$

It is well-known that if $T \in \mathcal{B}(\mathcal{H})$ is subnormal then $\ker[T^*, T]$ is invariant under T. Thus we might be tempted to guess that if the condition "T_Φ and T^2_Φ are hyponormal" is replaced by "T_Φ is hyponormal and $\ker[T^*_\Phi, T_\Phi]$ is invariant under T_Φ", then the answer to Problem 6.3 is affirmative. But this is not the case. Indeed, consider

$$T_\Phi = \begin{bmatrix} 2U + U^* & U^* \\ U^* & 2U + U^* \end{bmatrix}.$$

Then a straightforward calculation shows that T_Φ is hyponormal and $\ker[T^*_\Phi, T_\Phi]$ is invariant under T_Φ, but T_Φ is never normal (cf. [20, Remark 3.9]). However, if the condition "T_Φ and T^2_Φ are hyponormal" is strengthened to "T_Φ is subnormal", what conclusion do you draw?

3. *Subnormal completion problem*. Theorem 5.1 provides the subnormal Toeplitz completion of

$$\begin{bmatrix} U^* & ? \\ ? & U^* \end{bmatrix} \quad (U \text{ is the shift on } H^2). \tag{6.2}$$

Moreover Remark 5.8 shows that there is a normal non-Toeplitz completion of (6.2). However we were unable to find all subnormal completions of (6.2).

Problem 6.4. Let U be the shift on H^2. Complete the unspecified entries of the partial block matrix $\left[\begin{smallmatrix} U^* & ? \\ ? & U^* \end{smallmatrix}\right]$ to make it subnormal.

On the other hand, Theorem 5.1 shows that the solution of the subnormal Toeplitz completion of $\left[\begin{smallmatrix} U^* & ? \\ ? & U^* \end{smallmatrix}\right]$ consists of Toeplitz operators with symbols which are both analytic or trigonometric polynomials of degree 1. Hence we might expect that if the symbols of the specified Toeplitz operators of (6.2) are co-analytic polynomials of degree two then the non-analytic solution of the unspecified entries consists of trigonometric polynomials of degree ≤ 2.

More generally, we have:

Problem 6.5. If Φ and Ψ are co-analytic polynomials of degree n, does it follow that the non-analytic solution of the subnormal Toeplitz completion of the partial Toeplitz matrix $\left[\begin{smallmatrix} T_\Phi & ? \\ ? & T_\Psi \end{smallmatrix}\right]$ consists of Toeplitz operators whose symbols are trigonometric polynomials of degree $\le n$?

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