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Computing faithful representations for nilpotent Lie algebras

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ABSTRACT

We describe various methods to determine a faithful representation of small dimension for a finite-dimensional nilpotent Lie algebra over an arbitrary field. We apply our methods to finding bounds for the smallest dimension $\mu(\mathfrak{g})$ of a faithful $\mathfrak{g}$-module for some nilpotent Lie algebras $\mathfrak{g}$. In particular, we introduce an infinite family of filiform nilpotent Lie algebras $\mathfrak{f}_n$ of dimension n over $\mathbb{Q}$ and conjecture that $\mu(\mathfrak{f}_n) > n+1$ holds.

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1. Introduction

The Ado–Iwasawa theorem asserts that every finite-dimensional Lie algebra over an arbitrary field has a faithful finite-dimensional representation. A constructive proof for this theorem in characteristic 0 has been given in [5]. The underlying algorithm is based on a method for extending a faithful representation of a Lie algebra $\mathfrak{g}$ to a faithful representation of a semidirect product $\mathfrak{h} \ltimes \mathfrak{g}$. It has been implemented in the library parts of the computer algebra systems GAP4 and Magma.

Here we consider a variation on this theme: we introduce methods for computing a faithful finite-dimensional representation for a finite-dimensional nilpotent Lie algebra over an arbitrary field. These take as input a structure constants table for $\mathfrak{g}$ and determine as output a faithful $\mathfrak{g}$-module given by a list of matrices describing the action of the basis elements of $\mathfrak{g}$.

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In the first part of the paper we discuss three methods for this purpose: the algorithms LLIdeal, Quotient and Dual. All three algorithms are deterministic methods. If d is the dimension of $\mathfrak{g}$ and c is its class, then the dimension of their determined representation is bounded above by

$$\nu(d,c) = \sum_{j=0}^{c} \binom{d-j}{c-j} p(j),$$

where $p(j)$ is the number of partitions of j and $p(0) = 1$. More precisely, the algorithm LLIdeal determines a (possibly large) left ideal of the universal enveloping algebra of $\mathfrak{g}$, such that the quotient is a faithful finite-dimensional module for $\mathfrak{g}$. This algorithm is based on the ideas of [2]. The resulting module has dimension at most $\nu(d,c)$. The algorithms Quotient and Dual are both based on LLIdeal. They usually determine modules of significantly smaller dimension, but require more time for this. All three methods have been implemented in GAP [4] and a report on their runtimes is included below. We note that all three methods are usually significantly faster than the method described in [5] and they usually produce faithful representations of smaller dimensions.

In the theory of Lie algebras, there is significant interested in the smallest dimension $\mu(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. This is motivated, among other things, by problems from geometry and topology. For example, Milnor's conjecture [7] stems from this area; it asserts that the Lie algebra $\mathfrak{g}$ of any solvable Lie group G satisfies $\mu(\mathfrak{g}) \leqslant \dim(\mathfrak{g}) + 1$. There are counter-examples known to Milnor's conjecture. For example, there are infinitely many filiform nilpotent Lie algebras of dimension 10 which do not have any faithful module of dimension 11. See [3] for details and background.

The three algorithms described above can be used to determine upper bounds for $\mu(\mathfrak{g})$ for a nilpotent Lie algebra $\mathfrak{g}$, but there is no method available for computing $\mu(\mathfrak{g})$. At current, it is not even possible to check if $\mu(\mathfrak{g}) \leqslant \dim(\mathfrak{g})+1$ holds for a given nilpotent Lie algebra $\mathfrak{g}$. Since we wish to study this problem, we have used a fourth method called Affine in our experiments: This tries to construct a faithful representation of $\dim(\mathfrak{g})+1$ for a given nilpotent Lie algebra $\mathfrak{g}$. It uses a randomized approach for this purpose. If it succeeds, then it returns a module in the desired dimension. However, it is possible that the method fails, even if a faithful representation of dimension $\dim(\mathfrak{g}) + 1$ exists.

As final part of this paper we consider applications of our methods. We describe an infinite family of Lie algebras $\mathfrak{f}_n$ of dimension n for $n \geqslant 13$ and we use our algorithms to study the invariant $\mu(\mathfrak{f}_n)$ for these Lie algebras. We conjecture that these Lie algebras do not have a faithful representation of dimension $n+1$. But our experiments suggest that $\mu(\mathfrak{f}_n)$ is polynomial in n for these Lie algebras.

2. Using quotients of the universal enveloping algebra

Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra over an arbitrary field. By $\mathfrak{g}^m$, $m \geqslant 1$, we denote the terms of the lower central series of $\mathfrak{g}$. If $\{x_1, \dots, x_d\}$ is a basis of $\mathfrak{g}$, then the formal products $x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ with $\alpha_i \in \mathbb{N}$ form a Poincaré–Birkhoff–Witt basis (PBW-basis for short) of the universal enveloping algebra $U(\mathfrak{g})$.

We define the *weight* $wgt(x)$ of an element $x \in \mathfrak{g}$ as the maximal m with $x \in \mathfrak{g}^m$. The *weight* of a basis element of $U(\mathfrak{g})$ is then defined by

$$wgt\big(x_1^{\alpha_1} \cdots x_d^{\alpha_d}\big) = \sum_{i=1}^{d} \alpha_i \, wgt(x_i).$$

For $m \geqslant 1$ let $U^m(\mathfrak{g})$ be the ideal in $U(\mathfrak{g})$ generated by all basis elements $x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ of weight at least m. The following theorem is proved in [2], Proposition 6 (see also [1]).

Theorem 2.1. *Let $\mathfrak{g}$ be a d-dimensional nilpotent Lie algebra of nilpotency class c. Then $\mathfrak{g}$ acts faithfully on $U(\mathfrak{g})/U^{c+1}(\mathfrak{g})$ by multiplication from the left. If the considered basis of $\mathfrak{g}$ contains bases for $\mathfrak{g}^m$ for every $m \geqslant 1$, then the resulting representation has dimension at most*

$$\nu(d,c) = \sum_{j=0}^{c} \binom{d-j}{c-j} p(j),$$

where $p(j)$ is the number of partitions of j and $p(0) = 1$.

This theorem yields a straightforward algorithm to construct a faithful module for $\mathfrak{g}$. We consider a basis of $\mathfrak{g}$ which contains bases of $\mathfrak{g}^m$ for every m. We form the space V spanned by all basis elements of weight at most c in $U(\mathfrak{g})$. An element $x \in \mathfrak{g}$ acts on V by left multiplication, where we treat any element of weight at least $c+1$ as zero. We demonstrate this algorithm in the following example.

Example 2.2. Let $\mathfrak{g}$ be the 3-dimensional Heisenberg Lie algebra spanned by x, y, z with non-zero bracket $[x, y] = z$. The nilpotency class of $\mathfrak{g}$ is 2. We form the space spanned by the basis elements of $U(\mathfrak{g})$ of weight at most 2. These are

$$1, x, y, z, x^2, xy, y^2.$$

Thus we obtain a 7-dimensional representation for $\mathfrak{g}$. It is straightforward to determine this representation explicitly by computing actions. For example, $y \cdot x = yx = xy - z$ and $x \cdot xy = x^2 y = 0$.

In general, the modules resulting from this simple and very efficient algorithm have very large dimension. In the remainder of this section, we describe two methods to determine a module of smaller dimension from this given module. As a first step, we note that a nilpotent Lie algebra $\mathfrak{g}$ acts faithfully on a module if and only if its center $Z(\mathfrak{g})$ acts faithfully. Thus if I is a left ideal in $U(\mathfrak{g})$ such that $I \cap Z(\mathfrak{g}) = 0$, then $\mathfrak{g}$ acts faithfully on $U(\mathfrak{g})/I$.

For our first method we assume that the considered basis of $\mathfrak{g}$ additionally contains a basis for the center $Z(\mathfrak{g})$. We wish to determine a left ideal I in $U(\mathfrak{g})$ which has possibly small codimension and satisfies that $I \cap Z(\mathfrak{g}) = 0$. Let B be the set of all basis elements of weight at least $c+1$ in $U(\mathfrak{g})$ and initialise $I = \langle B \rangle$. We now iterate the following: let a be one of the finitely many basis elements of $U(\mathfrak{g})$ not contained in I and not contained in $Z(\mathfrak{g})$. If $xa \in I$ for all $x \in \mathfrak{g}$, then we add a to B and thus enlarge I without destroying the property $I \cap Z(\mathfrak{g}) = 0$. We call this algorithm LLIdeal (for Large Left Ideal). We demonstrate it in the following example.

Example 2.3. We continue Example 2.2. We initialise B as the basis elements of weight at least $c+1$. Note that $Z(\mathfrak{g}) = \langle z \rangle$. Thus we consider for a the elements x, y, x^2, xy, y^2. Of those, the elements x^2, xy and y^2 satisfy the condition of the algorithm LLIdeal and thus we move these into B. Now also y satisfies the condition and we also move y into B. Now I is an ideal of codimension 3 in $U(\mathfrak{g})$ and hence we obtain a 3-dimensional faithful representation of $\mathfrak{g}$.

Next we describe a method to construct a faithful quotient for a given faithful $\mathfrak{g}$-module V. It is called FaithfulQuotient and consists of the following steps:

(1) Compute the space $S = \{v \in V \mid x \cdot v = 0 \text{ for all } x \in \mathfrak{g}\}$.
(2) Compute the space $C = \{x \cdot v \mid v \in V,\ x \in Z(\mathfrak{g})\}$.
(3) Set $M = S \cap C$ and let W be a complement to M in S.
(4) If $W = 0$ then the algorithm stops, and the output is V. Otherwise, set $V := V/W$, and return to (1).

We note that any subspace of S is a $\mathfrak{g}$-submodule of V. Therefore the quotient V/W is a $\mathfrak{g}$-module. Let U be a complement to W in V such that $C \subset U$. Then $x \cdot V \subset U$ for all $x \in Z(\mathfrak{g})$. So since $Z(\mathfrak{g})$ acts faithfully on V it acts faithfully on V/W. Hence V/W is a faithful $\mathfrak{g}$-module.

Example 2.4. We consider the module of Example 2.2. Here we get

$$S = \langle z, x^2, xy, y^2 \rangle,$$
$$C = \langle z \rangle,$$
$$W = \langle x^2, xy, y^2 \rangle.$$

After taking the quotient we get a module spanned by (the images of) $1, x, y, z$. For this module we can perform the algorithm again. We get $S = \langle y, z \rangle$, $C = \langle z \rangle$, $W = \langle y \rangle$. So we end up with a faithful module of dimension 3.

Now the complete algorithm, called Quotient, to construct a small-dimensional faithful $\mathfrak{g}$-module runs as follows. First we perform the algorithm LLIdeal to find a (possibly large) left ideal I of $U(\mathfrak{g})$ such that $\mathfrak{g}$ acts faithfully on $U(\mathfrak{g})/I$. Then we perform FaithfulQuotient with input $U(\mathfrak{g})/I$, and obtain the faithful $\mathfrak{g}$-module V. This is the output of our algorithm.

3. Using the dual of the universal enveloping algebra

We have that $\mathfrak{g}$ acts on the dual $U(\mathfrak{g})^*$ by $x \cdot f(a) = f(-xa)$, for $x \in \mathfrak{g}$, $a \in U(\mathfrak{g})$ and $f \in U(\mathfrak{g})^*$. Let $z_1, \dots, z_r$ be a basis of the center of $\mathfrak{g}$, which we assume to be a subset of the basis $x_1, \dots, x_n$. Let $\psi_i \in U(\mathfrak{g})^*$ for $1 \leqslant i \leqslant r$ be defined by $\psi_i(z_i) = 1$ and $\psi_i(a) = 0$ for any PBW-basis element a not equal to z_i (note that this definition depends on the choice of basis of $\mathfrak{g}$).

Let $\bar{\ } : U(\mathfrak{g}) \to U(\mathfrak{g})$ be the antiautomorphism induced by $\bar{x} = -x$ for $x \in \mathfrak{g}$. Then for $a, b \in U(\mathfrak{g})$, $f \in U(\mathfrak{g})^*$ we get $a \cdot f(b) = f(\bar{a}b)$. (In other words, $\bar{\ }$ is the antipode of $U(\mathfrak{g})$.) Note that $\bar{\bar{a}} = a$.

Theorem 3.1. *Let V be the $\mathfrak{g}$-submodule of $U(\mathfrak{g})^*$ generated by $\psi_1, \dots, \psi_r$. Then V is a faithful finite-dimensional $\mathfrak{g}$-module. Moreover, V has no faithful $\mathfrak{g}$-submodules, nor has it faithful quotients.*

Proof. For $k \geqslant 1$ let $U^k(\mathfrak{g})$ be as in Section 2. Let $W = \{f \in U(\mathfrak{g})^* \mid f(U^{c+1}(\mathfrak{g})) = 0\}$. Then W is finite-dimensional (since $U^{c+1}(\mathfrak{g})$ has finite codimension), and a $\mathfrak{g}$-submodule of $U(\mathfrak{g})^*$ (since $U^{c+1}(\mathfrak{g})$ is an ideal). Now $V \subset W$; hence V is finite-dimensional. Let $z = \sum_i \mu_i z_i$ be an element of the center of $\mathfrak{g}$. Then $z \cdot \psi_i(1) = -\psi_i(z) = -\mu_i$. So z acts as zero if and only if all μ_i are zero. So V is a faithful module.

Let $\psi_0 \in U(\mathfrak{g})^*$ be defined by $\psi_0(1) = 1$ and $\psi_0(a) = 0$ for PBW-basis elements a not equal to 1. Then $\psi_0 = -z_1 \cdot \psi_1$, so $\psi_0 \in V$. Let $f \in V$, and suppose there is a PBW-basis element a, not equal to 1, such that $f(a) \neq 0$. Then $\bar{a} \cdot f(1) = f(a) \neq 0$. The conclusion is that ψ_0 spans the space of elements that are killed by $\mathfrak{g}$. Set $M = V/\langle \psi_0 \rangle$. Then M is not a faithful $\mathfrak{g}$-module. Indeed, V has a basis consisting of $a \cdot \psi_i$ for $1 \leqslant i \leqslant r$, and various PBW-basis elements a. Let z lie in the center of $\mathfrak{g}$; then for all PBW-basis elements b we get $z \cdot (a \cdot \psi_i)(b) = \psi_i(\bar{a}b\bar{z})$ which is zero unless $a = b = 1$ (note that $\bar{z} = -z$ also lies in the center of $\mathfrak{g}$). Furthermore, $z \cdot \psi_i = -\mu_i \psi_0$ (where $z = \sum_j \mu_j z_j$). It follows that the center of $\mathfrak{g}$ acts trivially on $V/\langle \psi_0 \rangle$. In particular it is not a faithful $\mathfrak{g}$-module. Now, since every $\mathfrak{g}$-submodule of V must contain ψ_0, it follows that V has no faithful quotients.

Let $M \subset V$ be a faithful $\mathfrak{g}$-submodule. Let $b_{ij} \in U(\mathfrak{g})$ be such that $\{b_{ij}\psi_i\}$ is a basis of V. We assume that $b_{i1} = 1$, and that the $b_{ij} \in \mathfrak{g}U(\mathfrak{g})$ if $j > 1$ (i.e., they have no constant term). Then $z_i \cdot b_{i1}\psi_i = -\psi_0$ and $z_i \cdot b_{kj}\psi_k = 0$ if $k \neq i$ or $j > 1$. So since the center acts faithfully on M it follows that M contains elements of the form

$$\varphi_i = \psi_i + \sum_{\substack{j>1 \\ 1 \leqslant k \leqslant r}} c_{kj} b_{kj} \psi_k,$$

for $1 \leqslant i \leqslant r$. (Here c_{kj} are coefficients in the ground field.) Now we introduce a weight function on $U(\mathfrak{g})^*$. For $k \geqslant 0$ set $F_k = \{f \in U(\mathfrak{g})^* \mid f(U^k(\mathfrak{g})) = 0\}$. Then $0 = F_0 \subset F_1 \subset \cdots$. We set $wgt(f) = k$ if

$f \in F_k$ but $f \notin F_{k-1}$. (For example: $wgt(\psi_0) = 1$.) Let $f \in U(\mathfrak{g})^*$ have weight k, and let $a \in U(\mathfrak{g})$ with $wgt(a) = t$; then a calculation shows that $wgt(a \cdot f) \leqslant k - t$. Hence $b_{ij}\varphi_i$ is equal to $b_{ij}\psi_i$ plus a sum of functions of smaller weight. So if we order the $b_{ij}\psi_i$ according to weight, and express the $b_{ij}\varphi_i$ in the basis $b_{ij}\psi_i$ we get a triangular system. We conclude that the $b_{ij}\varphi_i$ are linearly independent. Hence $\dim M = \dim V$ and $M = V$. $\square$

The algorithm based on this theorem is straightforward: by acting with basis elements of $\mathfrak{g}$ we compute a basis of the $\mathfrak{g}$-module generated by $\psi_1, \ldots, \psi_r$. Then we compute the matrices of the basis elements of $\mathfrak{g}$ with respect to this basis. We illustrate it with an example.

Example 3.2. Let $\mathfrak{g}$ be the Lie algebra of Example 2.2. For a monomial $a \in U(\mathfrak{g})$ we denote by ψ_a the element of $U(\mathfrak{g})^*$ that takes the value 1 on a, and zero on all other monomials. We compute a basis of the submodule of $U(\mathfrak{g})^*$ generated by ψ_z. (Note that the center of $\mathfrak{g}$ is spanned by z.) We have $x \cdot \psi_z(a) = -\psi_z(xa) = 0$ for all monomials a. Secondly, $y \cdot \psi_z(x) = \psi_z(-yx) = \psi_z(-xy + z) = 1$. So we get $y \cdot \psi_z = \psi_x$. Furthermore, $z \cdot \psi_z = -\psi_1$ and $\mathfrak{g} \cdot \psi_1 = 0$, $x \cdot \psi_x = -\psi_1$, $y \cdot \psi_x = z \cdot \psi_x = 0$. So the result is a 3-dimensional $\mathfrak{g}$-module.

Remark 3.3. Since we are working in the dual of an infinite-dimensional space it is not immediately clear how to implement this algorithm. We proceed as follows. Let V be as in Theorem 3.1. From the proof of Theorem 3.1 it follows that $f(U(\mathfrak{g})^{c+1}) = 0$ for all $f \in V$. In other words, for all monomials a with $wgt(a) \geqslant c + 1$ and all $f \in V$ we have $f(a) = 0$. It follows that we can represent an $f \in V$ by the vector containing the values $f(a)$, where a runs through the monomials of weight at most c. This enables us to perform the operations of linear algebra (testing linear dependence, constructing bases of subspaces and so on) with the elements of V. Furthermore, we can compute the action of elements of $\mathfrak{g}$ on V.

We can make this approach considerably more efficient by throwing some of the monomials away. We do this using the following algorithm. Let A be the set of monomials relative to which we represent the elements of $U(\mathfrak{g})^*$. At the start this will be the set of monomials of weight at most c. Let B be the set of all other monomials. So at the outset B spans a left ideal of $U(\mathfrak{g})$ and $f(b) = 0$ for all $f \in V$ and $b \in B$. We move elements from A to B, without changing this last property. Let $a \in A$ be such that $a \notin Z(\mathfrak{g})$ and xa is a linear combination of elements of B for all $x \in \mathfrak{g}$. Then we claim that $f(a) = 0$ for all $f \in V$. In order to see this we use the basis $\{b_{ij}\psi_i\}$ used in the proof of Theorem 3.1. If $j = 1$ then $b_{ij}\psi_i(a) = \psi_i(a) = 0$ as $a \notin Z(\mathfrak{g})$. If $j > 1$ then $b_{ij} \in \mathfrak{g}U(\mathfrak{g})$ and hence $\bar{b}_{ij}a$ is a linear combination of elements in B. Hence $b_{ij}\psi_i(a) = 0$. Also the span of B along with a continues to be a left ideal. We conclude that we can move a from A to B. We continue this process until we do not find such monomials any more.

We note that the procedure described in the previous remark is exactly the same as the algorithm LLIdeal (see Section 2). So we first perform the algorithm LLIdeal to find a left ideal I, and use the set of PBW-basis elements that are not contained in I to represent elements of the dual of $U(\mathfrak{g})$. The resulting algorithm is called Dual.

4. Affine representations at random

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension d. A homomorphism $\rho : \mathfrak{g} \to \mathfrak{aff}(K^d) \subseteq \mathfrak{gl}_{d+1}(K)$ into the Lie algebra of affine transformations

$$\mathfrak{aff}\big(K^d\big) \simeq \mathfrak{gl}\big(K^d\big) \ltimes K^d$$

is called an *affine representation* of $\mathfrak{g}$. In this section we describe an algorithm that tries to determine a faithful affine representation of $\mathfrak{g}$ of dimension $d + 1$. If the algorithm succeeds, then it returns such a faithful representation of dimension $d + 1$. However, it may also happen that the algorithm fails and does not return a representation. Also, it is worth noting that the algorithm uses random methods and hence different runs of the algorithm may produce different results.

The algorithm uses induction on a central series in $\mathfrak{g}$. Thus we assume by induction that we are given a central ideal I in $\mathfrak{g}$ with $\dim(I) = 1$ and a faithful affine representation

$$\rho : \mathfrak{g}/I \to M_d(K).$$

Let $\{a_1, \ldots, a_d\}$ be a basis of $\mathfrak{g}$ with $I = \langle a_d \rangle$ and let $M_i = \rho(a_i + I)$ for $1 \leqslant i \leqslant d-1$. We assume that every M_i is a lower triangular matrix. Clearly, we can readily extend ρ to an affine representation of $\mathfrak{g}$ with $\rho(a_i) = M_i$ for $1 \leqslant i \leqslant d$ where we set $M_d = 0$ (so that $\rho(a_d) = 0$). This extended representation has kernel I.

Our aim is to extend ρ to a faithful affine representation

$$\psi : \mathfrak{g} \to M_{d+1}(K)$$

such that

$$\psi(a_i) = \begin{pmatrix} M_i & v_i \\ 0 & 0 \end{pmatrix} \quad \text{for } 1 \leqslant i \leqslant d,$$

for certain vectors $v_i \in K^d$. The following lemma shows that the possible values for v_i can be determined using a cohomology computation. From [6], Chapter III, §10, we recall that

$$Z^1(\mathfrak{g}, K^d) = \bigl\{ \nu : \mathfrak{g} \to K^d \text{ linear} \bigm| \nu\bigl([x, y]\bigr) = \rho(x)\nu(y) - \rho(y)\nu(x) \bigr\}$$

is the space of 1-cocycles with values in the $\rho(\mathfrak{g})$-module K^d.

Lemma 4.1. *ψ is a representation of $\mathfrak{g}$ if and only if $v_i = \delta(a_i)$ for $1 \leqslant i \leqslant d$ for some $\delta \in Z^1(\mathfrak{g}, K^d)$.*

Proof. Let $\delta \in Z^1(\mathfrak{g}, K^d)$ with $\delta(a_i) = v_i$. The linearity of δ implies that ψ is linear. The defining condition for maps in $Z^1(\mathfrak{g}, K^d)$ implies that ψ is a Lie algebra representation. The converse follows with similar arguments. $\square$

Note that $Z^1(\mathfrak{g}, K^d)$ is a vector space over K and can be computed readily using linear algebra methods. The computation of $Z^1(\mathfrak{g}, K^d)$ allows to describe all affine representations of $\mathfrak{g}$ extending ρ. It remains to determine the faithful representation among these.

Lemma 4.2. *ψ is faithful if and only if $v_d \neq 0$.*

Proof. If ψ is faithful, then $v_d \neq 0$. Conversely, suppose that $v_d \neq 0$. As ρ is faithful, it follows that $\ker(\psi) \subseteq I$. As $v_d \neq 0$, we find that $\ker(\psi) = 0$. $\square$

These ideas can be combined to the following algorithm, called Affine.

(1) Choose a central series $\mathfrak{g} = \mathfrak{g}_0 > \mathfrak{g}_1 > \cdots > \mathfrak{g}_d > \mathfrak{g}_{d+1} = 0$ of ideals in $\mathfrak{g}$ such that $\dim(\mathfrak{g}_i/\mathfrak{g}_{i+1}) = 1$.
(2) For $i = 1, \ldots, d$ extend a faithful affine representation from $\mathfrak{g}/\mathfrak{g}_i$ to $\mathfrak{g}/\mathfrak{g}_{i+1}$, using the following steps:
   - Compute $Z^1(\mathfrak{g}/\mathfrak{g}_{i+1}, K^i)$.
   - Choose a $\delta \in Z^1(\mathfrak{g}/\mathfrak{g}_{i+1}, K^i)$ with $\delta(a_i) \neq 0$.
   - If no such δ exists, then return fail.
   - If δ exists, then extend ρ to $\mathfrak{g}/\mathfrak{g}_{i+1}$.

If $\mathfrak{g}$ has a faithful affine representation of dimension $d+1$, then this algorithm can in principle find it. However, it may be that a "wrong" choice of a δ at a certain step may cause the algorithm to fail at a later step.

The algorithm is based on linear algebra only and hence is very effective. It often succeeds in finding a faithful representation in dimension $d+1$ if it exists.

5. A series of filiform nilpotent Lie algebras

Let K be a field of characteristic zero. In this section we define a filiform Lie algebra $\mathfrak{f}_n$ in each dimension $n \geqslant 13$ having interesting properties (see Proposition 5.5) concerning Lie algebra cohomology, affine structures and faithful representations. In fact, we believe that the algebras $\mathfrak{f}_n$ are counter examples to the conjecture of Milnor mentioned in the introduction, i.e., that $\mu(\mathfrak{f}_n) \geqslant n+2$ holds. Hence it is interesting to compute the invariants $\mu(\mathfrak{f}_n)$.

The ideas behind the construction of $\mathfrak{f}_n$ are explained in [3], where a family of Lie algebras $\mathfrak{A}_n^2(K)$ is defined, of which $\mathfrak{f}_n$ is a specialization.

Define an index set $\mathcal{I}_n$ by

$$\mathcal{I}_n^0 = \bigl\{(k,s) \in \mathbb{N} \times \mathbb{N} \bigm| 2 \leqslant k \leqslant [n/2],\ 2k+1 \leqslant s \leqslant n\bigr\},$$

$$\mathcal{I}_n = \begin{cases} \mathcal{I}_n^0 & \text{if } n \text{ is odd}, \\ \mathcal{I}_n^0 \cup \{(\frac{n}{2}, n)\} & \text{if } n \text{ is even}. \end{cases}$$

Now fix $n \geqslant 13$. We define a filiform Lie algebra $\mathfrak{f}_n$ of dimension n over K as follows. For $(k,s) \in \mathcal{I}_n$ let $\alpha_{k,s}$ be a set of parameters, subject to the following conditions: all $\alpha_{k,s}$ are zero, except for the following ones:

$$\alpha_{\ell,2\ell+1} = \frac{3}{\binom{\ell}{2}\binom{2\ell-1}{\ell-1}}, \quad \ell = 2,3,\ldots,\left\lfloor \frac{n-1}{2} \right\rfloor,$$

$$\alpha_{3,n-4} = 1,$$

$$\alpha_{4,n-2} = \frac{1}{7} + \frac{10}{21}\,\frac{(n-7)(n-8)}{(n-4)(n-5)},$$

$$\alpha_{4,n} = \begin{cases} \frac{22105}{15246} & \text{if } n = 13, \\ 0 & \text{if } n \geqslant 14, \end{cases}$$

and

$$\alpha_{5,n} = \frac{1}{42} - \frac{70(n-8)}{11(n-2)(n-3)(n-4)(n-5)} + \frac{25}{99}\,\frac{(n-6)(n-7)(n-8)}{(n-2)(n-3)(n-4)} + \frac{5}{66}\,\frac{(n-5)(n-6)}{(n-2)(n-3)} - \frac{65}{1386}\,\frac{(n-7)(n-8)}{(n-4)(n-5)}.$$

Let $(e_1,\ldots,e_n)$ be a basis of $\mathfrak{f}_n$ and define the Lie brackets as follows:

$$[e_1,e_i] = e_{i+1}, \quad i = 2,\ldots,n-1,$$

$$[e_i,e_j] = \sum_{r=1}^{n} \left(\sum_{\ell=0}^{\lfloor \frac{j-i-1}{2} \rfloor} (-1)^{\ell} \binom{j-i-\ell-1}{\ell} \alpha_{i+\ell,r-j+i+2\ell+1} \right) e_r, \quad 2 \leqslant i < j \leqslant n.$$

In order to show that this defines a Lie bracket we need the following lemma which follows from the Pfaff–Saalschütz sum formula:

Lemma 5.1. *We have the following identities for all $n \geqslant 13$:*

$$\sum_{\ell=3}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{\ell-1} \binom{n-\ell-5}{\ell-2} \alpha_{\ell,2\ell+1} = \frac{(n-7)(n-8)}{(n-4)(n-5)},$$

$$\sum_{\ell=5}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{\ell} \binom{n-\ell-5}{\ell-4} \alpha_{\ell,2\ell+1} = -\frac{1}{70} + \frac{12(n-8)}{(n-2)(n-3)(n-4)(n-5)},$$

$$\sum_{\ell=3}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{\ell} \binom{n-\ell-3}{\ell-2} \alpha_{\ell,2\ell+1} = -\frac{(n-5)(n-6)}{(n-2)(n-3)}.$$

Proposition 5.2. *The Jacobi identity is satisfied, so that $\mathfrak{f}_n$ is a Lie algebra for any $n \geqslant 13$.*

Proof. Let $n \geqslant 14$ and choose the parameters $\alpha_{k,s}$ as follows. Consider $\alpha_{k,2k+1}$, $k = 3, \dots, \lfloor \frac{n-1}{2} \rfloor$ and $\alpha_{4,n-2}, \alpha_{5,n}$ as free variables. Let the remaining parameters be zero, except for $\alpha_{2,5} = 1$, $\alpha_{3,7} \neq 0$ and $\alpha_{3,n-4} = 1$. The Jacobi identity is equivalent to a system of polynomial equations in the free parameters. First we obtain the equation $\alpha_{3,7}(10\alpha_{3,7} - \alpha_{2,5}) = 0$, so that $\alpha_{3,7} = \frac{1}{10}$. More generally we see that

$$(\ell-1) \cdot \alpha_{\ell,2\ell+1} = (4\ell+2) \cdot \alpha_{\ell+1,2\ell+3}, \quad \ell = 2, 3, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor.$$

This implies the given explicit formula for all $\alpha_{\ell,2\ell+1}$. Secondly we obtain

$$\alpha_{4,n-2} = \frac{\alpha_{4,9}}{\alpha_{3,7}} + \frac{\alpha_{4,9}}{3\alpha_{3,7}^2} \sum_{\ell=3}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{\ell-1} \binom{n-\ell-5}{\ell-2} \alpha_{\ell,2\ell+1},$$

$$\alpha_{5,n} = \frac{1}{\alpha_{4,9} + \alpha_{3,7} - 2\alpha_{2,5}} \left(-4\alpha_{4,9} + \sum_{\ell=5}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{\ell} \binom{n-\ell-5}{\ell-4} \alpha_{\ell,2\ell+1} \right)$$
$$+ \frac{1}{\alpha_{4,9} + \alpha_{3,7} - 2\alpha_{2,5}} \left(\alpha_{4,n-2} \left(13\alpha_{4,9} + \sum_{\ell=3}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{\ell} \binom{n-\ell-3}{\ell-2} \alpha_{\ell,2\ell+1} \right) \right).$$

This amounts to the given formulas in the definition of $\mathfrak{f}_n$, if we substitute the identities from Lemma 5.1. Conversely this also shows that the Jacobi identity is satisfied if the free parameters are given in this way.

For $n = 13$ there is one difference. The parameter $\alpha_{4,n}$ coincides with the parameter $\alpha_{4,13}$, which is given by

$$\alpha_{4,13} = \frac{\alpha_{3,9}(-\alpha_{5,13} + 6\alpha_{4,11} - 5\alpha_{3,9})}{\alpha_{3,7} + 2\alpha_{2,5}},$$

and cannot be chosen to be zero. For $n \geqslant 14$ the choice $\alpha_{4,n} = 0$ is consistent with the Jacobi identity. $\square$

Example 5.3. The parameters for $\mathfrak{f}_{13}$ are given as follows:

$$\alpha_{2,5}=1, \quad \alpha_{3,7}=\frac{1}{10}, \quad \alpha_{4,9}=\frac{1}{70}, \quad \alpha_{5,11}=\frac{1}{420}, \quad \alpha_{6,13}=\frac{1}{2310},$$
$$\alpha_{3,9}=1, \quad \alpha_{4,11}=\frac{43}{126}, \quad \alpha_{4,13}=\frac{22105}{15246}, \quad \alpha_{5,13}=\frac{313}{3388}.$$

The algebras $\mathfrak{f}_n$ belong to the family of filiform Lie algebras $\mathfrak{A}_n^2(K)$ defined in [3]. Let us recall the following definition (cf. [3]).

Definition 5.4. Let $\mathfrak{g}$ be a filiform nilpotent Lie algebra of dimension n. A 2-cocycle $\omega \in Z^2(\mathfrak{g}, K)$ is called *affine*, if $\omega: \mathfrak{g} \wedge \mathfrak{g} \to K$ does not vanish on $Z(\mathfrak{g}) \wedge \mathfrak{g}$. A class $[\omega] \in H^2(\mathfrak{g}, K)$ is called *affine* if every representative is affine.

The cohomology class $[\omega] \in H^2(\mathfrak{g}, K)$ of an affine 2-cocycle ω is affine and non-zero. If a filiform Lie algebra $\mathfrak{g}$ of dimension $n \geqslant 6$ has second Betti number $b_2(\mathfrak{g}) = 2$, then there exists no affine cohomology class, see [3].

We have shown in [3] that a filiform Lie algebra $\mathfrak{g}$ which has an affine cohomology class, admits a central extension

$$0 \to \mathfrak{a} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

with some Lie algebra $\mathfrak{h}$ and $\iota(\mathfrak{a}) = Z(\mathfrak{h})$, and has an affine structure. In particular, such a Lie algebra has a faithful representation of dimension $n+1$.

We can conclude from the results in [3] that the Lie algebras $\mathfrak{f}_n$ do *not* have an affine structure arising this way.

Proposition 5.5. *The algebras $\mathfrak{f}_n$, $n \geqslant 13$ have second Betti number $b_2(\mathfrak{f}_n) = 2$. Hence there exists no affine cohomology class $[\omega] \in H^2(\mathfrak{g}, K)$. In particular there is no central Lie algebra extension as above.*

For Lie algebras in $\mathfrak{A}_n^2(K)$ the second Betti number is 3 or 2, depending on whether a certain polynomial identity $\alpha_{3,n-4} = P$ in the free parameters does hold or does not hold (see [3]). For $\mathfrak{f}_n$ we have chosen the parameters in such a way that $P = 0$ and $\alpha_{3,n-4} = 1$. This implies that $b_2(\mathfrak{f}_n) = 2$.

It follows that a very natural way to obtain a faithful representation of dimension $n+1$ does not work. In fact, we believe that there is no such representation at all for these algebras:

Conjecture 5.6. *The Lie algebras $\mathfrak{f}_n$, $n \geqslant 13$ do not have any faithful representation of dimension $n+1$, i.e., $\mu(\mathfrak{f}_n) \geqslant n+2$.*

For $n = 13$ a very complicated analysis of possible faithful representations seems to confirm this conjecture. In general our methods are not sufficient to prove this for all $n \geqslant 14$. Even more difficult of course is the determination of $\mu(\mathfrak{f}_n)$. For computations of faithful representations for the algebras $\mathfrak{f}_n$, for small n, see Table 3.

6. Practical experiences

We implemented all the algorithms described above in the computer algebra system GAP. In this section we report on the application of these implementations to various examples. We have the algorithms Quotient (Section 2), Dual (Section 3), and Affine (Section 4). We note that the first two algorithms share the first basic step, which is the algorithm LLIdeal (see Section 2).

In all our experiments Quotient and Dual returned faithful representations of the same dimension (with Dual being slightly faster). This is illustrated in Table 3. We believe that there must be an

Table 1
Running times (in seconds) for $U_n(\mathbb{F})$.

n	$\mathbb{F}$	$\dim(U_n(\mathbb{F}))$	LLIdeal		Dual		Affine	
			time	dim	time	dim	time	dim
4	$\mathbb{F}_2$	6	0.0	7	0.1	5	0.0	7
5	$\mathbb{F}_2$	10	0.25	15	0.3	11	0.3	11
6	$\mathbb{F}_2$	15	3.4	35	3.6	17	3.5	16
7	$\mathbb{F}_2$	21	65	79	66	35	45	22
4	$\mathbb{F}_3$	6	0.0	7	0.0	5	0.0	7
5	$\mathbb{F}_3$	10	0.2	15	0.3	11	0.3	11
6	$\mathbb{F}_3$	15	3.4	35	3.6	17	3.7	16
7	$\mathbb{F}_3$	21	65	79	67	35	46	22
4	$\mathbb{Q}$	6	0.0	7	0.0	5	0.0	7
5	$\mathbb{Q}$	10	0.2	15	0.3	11	0.3	11
6	$\mathbb{Q}$	15	3.0	35	3.2	17	3.6	16
7	$\mathbb{Q}$	21	66	79	67	35	45	22

Table 2
Running times (in seconds) for $N_{n,c}(\mathbb{Q})$.

n	c	$\mathbb{F}$	$\dim(N_{n,c}(\mathbb{F}))$	LLIdeal		Dual		Affine	
				time	dim	time	dim	time	dim
2	5	$\mathbb{Q}$	14	0.2	20	0.3	20	0.5	15
2	6	$\mathbb{Q}$	23	0.9	34	1.3	34	8.4	24
2	7	$\mathbb{Q}$	41	3.2	65	4.8	65	☹	☹
2	8	$\mathbb{Q}$	71	14	117	21	117	☹	☹
3	4	$\mathbb{Q}$	32	0.8	41	1.7	41	54	33
3	5	$\mathbb{Q}$	80	11.5	113	17.5	113	☹	☹
4	3	$\mathbb{Q}$	30	0.9	36	1.3	36	37	31
4	4	$\mathbb{Q}$	90	13	113	19.7	113	☹	☹

intrinsic reason for this to happen, such as one module being the dual of the other. But we have no proof of that. We only exhibit the results of Dual in Tables 1 and 2, noting that the results for Quotient are similar in all cases.

All computations were done on a 2 GHz processor with 1 GB of memory for GAP.

6.1. Upper triangular matrix Lie algebras

The upper triangular matrices in $M_n(\mathbb{F})$ form a nilpotent Lie algebra $U_n(\mathbb{F})$ with $n-1$ generators and class $n-1$. We applied our algorithms to some Lie algebras of this type. The results are recorded in Table 1.

Table 1 exhibits that the underlying field does not have much impact on the runtime or the result. The larger the dimension of the considered Lie algebra is, the more superior is Affine. It yields small-dimensional representations and is the fastest of all methods.

6.2. Free nilpotent Lie algebras

Next we consider the free nilpotent Lie algebras with n generators of class c over the field $\mathbb{F}$, denoted $N_{n,c}(\mathbb{F})$.

Table 2 displays the time in seconds for the three algorithms, with input $N_{n,c}$. The ☹ in the last two columns indicates that the algorithm Affine did not succeed, either because it made the “wrong” choice at some stage, or due to Memory problems: for its cohomology computation it has to solve a system of linear equations which is of the size $O(\dim(\mathfrak{g})^2)$ and this can be time and space consuming.

Table 3
Running time (in seconds) for the Lie algebras $\mathfrak{f}_n$.

n	LLIdeal		Quotient		Dual		Affine
	time	dim	time	dim	time	dim	
13	8.6	85	14	43	12.3	43	☹
14	17	105	28	53	24.7	53	☹
15	33	145	63	64	50	64	☹
16	64	185	125	77	102	77	☹
17	123	256	323	94	218	94	☹
18	234	316	731	111	461	111	☹
19	487	433	1844	134	1162	134	☹
20	920	538	4009	158	3039	158	☹

6.3. The Lie algebras $\mathfrak{f}_n$

Finally, we consider the Lie algebras $\mathfrak{f}_n$ of the previous section. The results of that are contained in Table 3.

Table 3 displays the time in seconds for the algorithms Quotient and Dual, with input $\mathfrak{f}_n$. The ☹ in the last column indicates that the algorithm Affine did not succeed. In this case, this was due to the fact that Affine did not find any possible faithful representation of dimension $n+1$. Of course, if our conjecture on $\mathfrak{f}_n$ holds, then it cannot succeed.

Note that the dimensionals of the determined modules for $\mathfrak{f}_n$ are significantly larger than $n+1$. However, they do not seem to grow very fast. Some naive tests with least squares fits seem to suggest that the dimensions grow quadratically or cubically.

6.4. Some comments

From the above tables we conclude that if Affine succeeds, then it usually finds a module of significantly smaller dimension than Quotient or Dual. This supports the suggested strategy to try this algorithm first.

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