Some Properties of GM-matrices and their Inverses

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ABSTRACT

The class of GM-matrices is defined by requiring that a positive cycle contain any negative cycle it intersects. Using the cycle structure, a canonical form is developed for irreducible GM-matrices. Relationships between the signs of the principal minors and the cycle are derived. In special cases, results are obtained on the signs of elements of the inverse of a GM-matrix.

1. INTRODUCTION

In [7] James Quirk introduced a class of matrices he called generalized Metzlerian matrices (or GM-matrices for short). He conjectured that the GM-matrices are the largest class of qualitatively defined matrices having the property that the hypothesis of Hicksian stability for the matrix implies that it is stable. We remind the reader that the $n \times n$ matrix $A = [a_{ij}]$ is Hicksian stable if every principal minor of order p has the sign $(-1)^p$ for $0 \leqslant p \leqslant n$.

More recently, in [6] Maybee and Wiener introduced the concept of a matrix having consistently signed principal minors (see Definition 1.2 below). It turns out that a GM-matrix has consistently signed principal minors if and

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only if it is Hicksian stable. Quirk did not address the problem of determining conditions under which a GM-matrix would be Hicksian stable. He used Hicksian stability as an additional hypothesis. Also he failed to prove his conjecture.

We think that it is of some interest to derive sufficient conditions for a GM-matrix to be Hicksian stable (have consistently signed principal submatrices). The methods derived in [6] permit us to do this with the help of an interesting canonical form theorem derived below which shows that very detailed structural results are obtainable for a class of qualitatively defined matrices. We also show how the results of [6] combined with those of [3] can be used to obtain some results about the signs of the elements of A^{-1}. It is our hope that the results we have obtained will provide the additional tools needed to prove or disprove Quirk's conjecture, at least under the sufficient conditions of Sections 5 and 6.

Following Quirk, we consider $n \times n$ matrices $A = [a_{ij}]$. In the usual way, we associate with A the signed digraph $S(A) = (\mathscr{P}, \mathscr{A}, \sigma)$ defined as follows. The point set $\mathscr{P}$ of $S(A)$ consists of n points labeled $1, 2, \ldots, n$. The arc set $\mathscr{A}$ of $S(A)$ contains the arc (i, j) if and only if $a_{ij} \neq 0$ for $i \neq j$. For the arc (i, j) we have $\sigma(i, j) = +$ if $a_{ij} > 0$ and $\sigma(i, j) = -$ if $a_{ij} < 0$. Thus $\sigma : \mathscr{A} \to \{+, -\}$. If $\mathscr{A}_0 \subset \mathscr{A}$, then sign $\mathscr{A}_0$ is the product of the signs of the arcs in $\mathscr{A}_0$ (if $\mathscr{A}_0 = \varnothing$, sign $\mathscr{A}_0 = +$). Here $\mathscr{A}_0$ could be the arc set of a path or cycle of $S(A)$ so that sign $\mathscr{A}_0$ would then be understood as the sign of the corresponding path or cycle. (For our work paths and cycles will always be simple.) It will be convenient to denote by $D(A)$ the underlying digraph of $S(A)$, i.e., $D(A) = (\mathscr{P}, \mathscr{A})$.

When $\mathscr{A}_0 \subset \mathscr{A}$, we shall denote by $\mathscr{P}(\mathscr{A}_0)$ the set of points of $S(A)$ belonging to the arcs of $\mathscr{A}_0$.

We can now define the class of GM-matrices.

DEFINITION 1.1. The real $n \times n$ matrix A is a GM-matrix if $a_{ii} < 0$, $i = 1, 2, \ldots, n$, and whenever c_1 is a negative cycle of $S(A)$ and c_2 is a positive cycle of $S(A)$, then either $\mathscr{P}(c_1) \cap \mathscr{P}(c_2) = \varnothing$ or $\mathscr{P}(c_1) \subset \mathscr{P}(c_2)$.

Observe that the second condition of this definition is purely graph theoretic. We shall show that we can establish a canonical form for all matrices satisfying this condition alone. Indeed, this is the main result of the first portion of our work. Both the result and the methods used in its derivation are new and different from anything in [7].

Next let $I \subset \mathscr{P}$. We use the notation $A[I]$ to denote the principal submatrix of A in the rows and columns I, and $A(I)$ to denote the principal submatrix of A in the rows and column complementary to I.

DEFINITION 1.2. Suppose A is a real $n \times n$ matrix with $a_{ii} < 0$, $i = 1,2,\dots,n$. We say that A has *consistently signed principal minors* if for each $I \subset \mathscr{P}$ the principal minor $\det A[I]$ satisfies

$$\operatorname{sgn} \det A[I] = (-1)^p, \tag{1}$$

where $p = |I|$, the number of points in I (see [6] for a discussion).

Observe that this definition implies both that A^{-1} exists and that $\det A[I] \neq 0$ for all I (take $\det A[I] = 1$ if $I = \varnothing$). Moreover, by virtue of the Jacobi identities between minors of a matrix and those of its inverse, each principal minor of A^{-1} satisfies

$$\operatorname{sgn} \det A^{-1}[I] = \frac{\operatorname{sgn} \det A(I)}{\operatorname{sgn}(\det A)} = \frac{(-1)^{n-p}}{(-1)^n} = (-1)^p,$$

i.e., the condition (1) is also satisfied for A^{-1}.

In the process of studying GM-matrices we shall relate this class to the class of purely qualitative matrices, i.e., *L*-matrices (see [4] and [5]). For this purpose we shall define the $n \times n$ matrix A to be an *L*-matrix if $a_{ii} < 0$, $i = 1,\dots,n$, and $S(A)$ has only negative cycles. Note that every *L*-matrix is a GM-matrix.

2. SOME USEFUL PRELIMINARY RESULTS

It turns out that the GM-condition forces us to examine the way in which various cycles intersect. Thus it will be convenient to have at hand a little special terminology as well as some basic facts about cycles in strongly connected digraphs.

First let H and K be subdigraphs of the digraph $D = (\mathscr{P}, \mathscr{A})$. We call H and K *disjoint* (otherwise *intersecting*) if $\mathscr{P}(H) \cap \mathscr{P}(K) = \varnothing$. We call them, respectively, *tangent* or *adjacent* if $|\mathscr{P}(H) \cap \mathscr{P}(K)| = 1$ or if $H \cap K$ (graph theoretic intersection) is a path in D. In the last situation, the path $H \cap K$ will be called the *path of adjacency* of H and K. Finally, *K covers H* means $\mathscr{P}(H) \subset \mathscr{P}(K)$.

LEMMA 2.1. *Let H be a strong subdigraph of the strong digraph D. If the arc (h,k) is tangent to H, then there exists a cycle c, containing (h,k), which is either tangent or adjacent to H.*

Proof. Suppose $h \in \mathscr{P}(H)$. Since D is strong, (h, k) is an arc of some cycle b. If b is tangent to H, then let $c = b$. If b is not tangent to H, then (h, k) is contained in a unique subpath $p_1(h \to h')$ of b where $\mathscr{P}(p_1) \cap \mathscr{P}(H) = \{h, h'\}$. [The argument is virtually identical if instead $k \in \mathscr{P}(H)$.] Since H is strong, there exists a path $p_2(h' \to h)$ in H. Thus $\mathscr{P}(p_1) \cap \mathscr{P}(p_2) = \{h, h'\}$, and we can define a cycle $c = p_1(h \to h')p_2(h' \to h)$. ■

COROLLARY 2.2. *If the strong digraph D contains a proper strong subdigraph H, then there exists a cycle c tangent or adjacent to H with $\mathscr{P}(c) - \mathscr{P}(H) \neq \varnothing$.*

Proof. Since H is a proper subdigraph and D is strong, there exists an arc (h, k) tangent to H, i.e., $h \in \mathscr{P}(H)$ and $k \notin \mathscr{P}(H)$. Let c be the cycle from Lemma 2.1. ■

COROLLARY 2.3. *Let $D = (\mathscr{P}, \mathscr{A})$ be strong. If the cycle b is such that $\mathscr{P}(b)$ is a proper subset of $\mathscr{P}$, then there exists a cycle c tangent or adjacent to b with $\mathscr{P}(c) - \mathscr{P}(b) \neq \varnothing$.*

Proof. Let $H = b$ in Corollary 2.2. ■

We will now impose the GM-condition on the strong, signed digraph S. We call such graphs *GM-graphs* for short.

THEOREM 2.4. *Suppose $S = (\mathscr{P}, \mathscr{A}, \sigma)$ is a GM-graph. Then, if S contains a positive cycle, every point in $\mathscr{P}$ belongs to a positive cycle of* S.

Proof. Let $\mathscr{P}_0$ be the subset of points of S belonging to positive cycles. By hypothesis $\mathscr{P}_0 \neq \varnothing$. If $\mathscr{P}_0$ is a proper subset of $\mathscr{P}$, there exists an arc $(h, k) \in \mathscr{A}$ with $h \in \mathscr{P}_0$ and $k \notin \mathscr{P}_0$. Thus (h, k) is tangent to some positive cycle c, where $h \in \mathscr{P}(c)$. Applying Lemma 2.1, we find a cycle b containing (h, k). Since $\mathscr{P}(b) \cap \mathscr{P}(c) \neq \varnothing$ and $\mathscr{P}(c) \not\supset \mathscr{P}(b)$, the GM-condition implies that b must be a positive cycle. But then $\mathscr{P}(b) \subset \mathscr{P}_0$ and so $k \in \mathscr{P}_0$, a contradiction. Thus $\mathscr{P}_0 = \mathscr{P}$. ■

The next lemma points out the special situation occurring when a negative cycle and a positive cycle share an arc. It will be convenient to denote by $\mathscr{C}_-(S)$ the set of negative cycles and by $\mathscr{C}_+(S)$ the set of positive cycles of the GM-graph S.

LEMMA 2.5. *Let S be a GM-graph with $a \in \mathscr{C}_-(S)$ and $b \in \mathscr{C}_+(S)$. If a and b have a common arc (i,k), then any path $p(i \to k)$, other than the arc (i,k) itself, must contain a point from the set $\mathscr{P}(b(k \to i)) - \{i,k\}$. Thus b is the unique cycle of S coinciding with the path $b(k \to i)$.*

Proof. Suppose $\mathscr{P}(p) \cap \{\mathscr{P}(b(i \to k)) - \{i,k\}\} = \varnothing$. Then we can define the cycle $c = p(i \to k)b(k \to i)$. Clearly $c \neq b$, since $p(i \to k) \neq b(i \to k)$. Moreover, $\mathscr{P}(p) - \mathscr{P}(b) \neq \phi$, so $\mathscr{P}(c) \not\subset \mathscr{P}(b)$. Hence c is a positive cycle by the GM-condition. Also by the GM-condition, $\mathscr{P}(a) \subset \mathscr{P}(b)$, so $\mathscr{P}(p) \cap \mathscr{P}(a) = \{i,k\}$ and we can define another cycle $d = p(i \to k)a(k \to i)$. Again $\mathscr{P}(d) \not\subset \mathscr{P}(b)$, so d is a positive cycle. Calculating the sign of the product of the arcs of c and d and of a and b, we see that $0 < \operatorname{sgn} c \cdot \operatorname{sgn} d = \operatorname{sgn} p(i \to k) \operatorname{sgn} b(k \to i) \operatorname{sgn} p(i \to k) \operatorname{sgn} a(k \to i)$, so $\operatorname{sgn} c \operatorname{sgn} d = \operatorname{sgn} b(k \to i) \operatorname{sgn} a(k \to i)$. Now $0 > \operatorname{sgn} a \cdot \operatorname{sgn} d = \operatorname{sgn} b(k \to i) \operatorname{sgn} a(k \to i)$, $c \neq b$, since $p(i \to k)$, since a and b have the common arc (i,k). Therefore we have both $\operatorname{sgn} c \operatorname{sgn} d = -\operatorname{sgn} a \operatorname{sgn} b$ and $\operatorname{sgn} c \operatorname{sgn} d = \operatorname{sgn} c \operatorname{sgn} b$, a contradiction. Hence p must intersect the path $b(k \to i)$ as required. ∎

In a GM-graph S, is it possible that $\mathscr{P}(c) \supset \mathscr{P}(b)$ where $c \in \mathscr{C}_-(S)$ and $b \in \mathscr{C}_+(S)$? Clearly if the negative cycle spans S, the answer is yes. The next theorem shows that the converse also holds.

THEOREM 2.6. *Let S be a GM-graph and $a \in \mathscr{C}_-(S)$. The following are equivalent:*

(i) *$\mathscr{P}(a) \supset \mathscr{P}(b)$ for some $b \in \mathscr{C}_+(S)$ (so actually $\mathscr{P}(a) = \mathscr{P}(b)$);*
(ii) *$\mathscr{P}(a) = \mathscr{P}$;*
(iii) *$\mathscr{P}(a) \supset \mathscr{P}(b')$ for all $b' \in \mathscr{C}_+(S)$.*

Proof. Clearly it suffices to show that (i) implies (ii). Let $b \in \mathscr{C}_+(S)$ with $\mathscr{P}(b) \subset \mathscr{P}(a)$. If $\mathscr{P}(a) \neq \mathscr{P}$, let c be the cycle of Corollary 2.3. Since c intersects a and $\mathscr{P}(c) \not\subset \mathscr{P}(a)$, the GM-condition implies $c \in \mathscr{C}_+(S)$, hence $\mathscr{P}(c) \supset \mathscr{P}(a)$. Thus c is adjacent to a. Let $a(k \to i) = c(k \to i)$ be the path of adjacency of a and c. Since $\mathscr{P}(c(i \to k)) \cap \mathscr{P}(a) = \{i,k\}$ and $\mathscr{P}(a) \subset \mathscr{P}(c)$, we must have $\mathscr{P}(a(k \to i)) = \mathscr{P}(a) \cap \mathscr{P}(c(k \to i))$. Thus $a = (i,k)a(k \to i)$.

By the GM-condition $\mathscr{P}(a) = \mathscr{P}(b)$. Consequently $\mathscr{P}(c(i \to k)) \cap \mathscr{P}(b) = \{i,k\}$, and we can define the cycle $d = c(i \to k)b(k \to i)$. Again $\mathscr{P}(d) \not\subset \mathscr{P}(b)$, since $\mathscr{P}(c(i \to k))$ contains points not in a. Thus $d \in \mathscr{C}_+(S)$ and $\mathscr{P}(a) \subset \mathscr{P}(d)$. It follows that $\mathscr{P}(b) \subset \mathscr{P}(d)$, so $\mathscr{P}(b) = \mathscr{P}(b(k \to i))$ and $b = (i,k)(k \to i)$.

We thus have shown that a and b have the arc (i,k) in common. By Lemma 2.5, b is then the unique cycle containing the path $b(k \to i)$. But, by construction, d is another such cycle. This contradiction forces us to conclude that $\mathscr{P}(a)=\mathscr{P}$. ∎

A GM-graph will be called *degenerate* if every positive cycle c satisfies $\mathscr{P}(c)=\mathscr{P}$. A *nondegenerate* GM-graph contains at least one nonspanning positive cycle. A GM-matrix A will be called degenerate or nondegenerate according as $S(A)$ is or is not degenerate. Theorem 2.6 gave three sufficient conditions for degeneracy. We close with still another condition and an example.

COROLLARY 2.7. *Let S be a GM-graph with $|\mathscr{P}|=n$, and let $a \in \mathscr{C}_-(S)$. If $|\mathscr{P}(a)|=n-1$, then S is degenerate.*

Proof. Suppose $b \in \mathscr{C}_+(S)$. Then $|\mathscr{P}(b)| \geqslant 2$, so $\mathscr{P}(a) \cap \mathscr{P}(b) \neq \varnothing$. Thus $\mathscr{P}(a) \subset \mathscr{P}(b)$ by the GM-condition. By Theorem 2.6, $\mathscr{P}(a)=\mathscr{P}(b)$ is impossible, for otherwise $\mathscr{P}(a)=\mathscr{P}$, contradicting the fact that $|\mathscr{P}(a)|=n-1$ and $|\mathscr{P}|=n$. Thus $\mathscr{P}(a)$ is a proper subset of $\mathscr{P}(b)$, so $|\mathscr{P}(b)|=n$. ∎

The matrix A satisfying

$$\operatorname{sgn} A = \begin{bmatrix} - & - & 0 & 0 & + & + \\ 0 & - & - & 0 & 0 & - \\ 0 & + & - & - & 0 & 0 \\ 0 & 0 & + & - & - & 0 \\ - & 0 & 0 & + & - & + \\ - & + & 0 & 0 & 0 & - \end{bmatrix}$$

is a degenerate GM-matrix. The digraph $S(A)$ has two spanning positive cycles, namely (154326) and (162345), and the spanning negative cycle (123456). All other cycles are negative.

3. MAXIMAL POINT SETS AND CANONICAL FORM OF GM-GRAPHS

The initial ideas in this section apply to an arbitrary signed digraph S. we shall define an equivalence relation on the cycles of S.

DEFINITION 3.1. Let $a, b \in \mathscr{C}_-(S)$. Then $a \sim b$ means there exists a sequence of cycles $c_i \in \mathscr{C}_-(S)$, $1 \leqslant i \leqslant m$, such that $a = c_1$, $b = c_m$, and $\mathscr{P}(c_i) \cap \mathscr{P}(c_{i+1}) \neq \varnothing$, $1 \leqslant i \leqslant m-1$.

It is clear that $\sim$ gives an equivalence relation on $\mathscr{C}_-(S)$. It makes no difference whether the sequence of c_i's is one of distinct cycles or not; transitivity follows easily and the equivalence classes are the same under either definition.

DEFINITION 3.2. If $\bar{I}$ is an equivalence class of $\mathscr{C}_-(S)$ under $\sim$, then we call $I = \mathscr{P}(\bar{I})$ a *maximal point set* of S.

THEOREM 3.3. *For any signed digraph S, the maximal point sets form a partition of $\mathscr{P}(\mathscr{C}_-(S))$, and $\mathscr{P}(S)$ can be expressed as a disjoint union:*

$$\mathscr{P}(S) = I_1 \cup \cdots \cup I_t \cup K = \mathscr{P}(\mathscr{C}_-(S)) \cup K$$

where each I_t is a maximal point set, and K is the set of all points of S not contained in any negative cycle.

LEMMA 3.4. *If I is a maximal point set of S, then for each pair of points $i, k \in I$, there exists a path $p(i \to k)$, all of whose arcs belong to negative cycles of S.*

Proof. Since I corresponds to an equivalence class, there exist $a, b \in \mathscr{C}_-(S)$ such that $i \in \mathscr{P}(a)$, $k \in \mathscr{P}(b)$, and $a \sim b$. Let $c_1, \ldots, c_m$ be a sequence of cycles satisfying the definition of $\sim$. Assume m is minimal and use induction.

For $m = 1$, we have $a = b$, and $a(i \to k)$ is the required path.

For $m > 1$, let j be defined by

$$\mathscr{P}(a(i \to j)) \cap \bigcup_{l=2}^{m} \mathscr{P}(c_l) = \{j\}.$$

By induction, there is a path $p_1(j \to k)$, all of whose arcs belong to negative cycles of S. The definition of j assures that p_1 is tangent at a. Thus $p_2 = a(i \to j)p_1(j \to k)$ is a path having the required property. ■

REMARK. We shall refer to I as a subdigraph of S with points $\{p : p \in I\}$ and arcs $\{(i, j) : (i, j) \in \mathscr{C}_-(S)\}$.

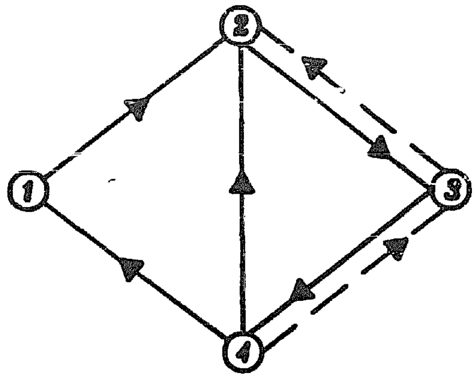

FIG. 1.

Let $\langle I\rangle$ indicate the subdigraph of S generated in the usual way by the point set I and all arcs of S joining these points. Thus $\mathscr{A}(I)=\mathscr{A}(\langle i\rangle)\cap \mathscr{A}(\mathscr{C}_{-}(S))$. Lemma 3.4 implies that I is a strong subdigraph (and so is $\langle I\rangle$ for that matter). The following example should clarify the distinction we want to make.

EXAMPLE 3.5. In the GM-graph shown in Figure 1, $I=\{2,3,4\}$ is a maximal point set. The solid lines in the figure stand for positive arcs, whereas broken lines stand for negative arcs. The subdigraph I has an arc set $\{(23),(32),(34),(43)\}$ because arc (42) does not belong to any negative cycle. Of course, the subdigraph $\langle I\rangle$ would include the arc (42).

We now invoke the GM-condition for the further study of maximal point sets. This will lead us to a canonical form for GM-graphs. An example (Example 3.10) illustrating all of the ideas appears later, in Figure 4. The reader is invited to refer to it beforehand in order to see the ideas more concretely.

LEMMA 3.6. *Let I be a maximal point set of a GM-graph S, and c a cycle of S. If $I\cap\mathscr{P}(c)\neq\varnothing$, then:*

(i) $c\in\mathscr{C}_{-}(S)\ \Rightarrow\ \mathscr{P}(c)\subset I$;
(ii) $c\in\mathscr{C}_{+}(S)\Rightarrow I\subset\mathscr{P}(c)$.

Proof. (i): Since the equivalence classes form a partition of $\mathscr{C}_{-}(S)$, $\mathscr{P}(c)\subset I$.

(ii): Let $\tilde{I}$ be the equivalence class corresponding to I. Since $I\cap\mathscr{P}(c)\neq\varnothing$, there is a negative cycle $a\in\tilde{I}$ such that $\mathscr{P}(a)\subset I$ and $\mathscr{P}(c)\cap\mathscr{P}(a)\neq\varnothing$. By the GM-condition, $\mathscr{P}(a)\subset\mathscr{P}(c)$. Now choose any $b\in\tilde{I}$. Because $a\sim b$, there exists (negative) cycles $c_1, \ldots, c_m$ satisfying the intersection properties in Definition 3.1. By the GM-condition, if $\mathscr{P}(c_i) \subset \mathscr{P}(c)$, then $\mathscr{P}(c_{i+1}) \subset \mathscr{P}(c)$, $i \leqslant i \leqslant m$. Thus $\mathscr{P}(c_1) = \mathscr{P}(a) \subset \mathscr{P}(c)$ implies $\mathscr{P}(b) = \mathscr{P}(c_m) \subset \mathscr{P}(c)$, and $I \subset \mathscr{P}(c)$. ∎

We come now to a result of fundamental importance.

THEOREM 3.7. *If the GM-graph S is not a maximal point set, then for each maximal point set I of S, there exist $i, j \in I$ such that:*

(1) *there is a unique path $p(j \to i)$ in S from j to i;*
(2) *for all $b \in \mathscr{C}_+(S)$ which cover I, $\mathscr{P}(b(j \to i)) = I$.*

Proof. (1): We first construct a cycle $b \in \mathscr{C}_+(S)$ and find points $i, j \in I$ for which $\mathscr{P}(b(j \to i)) = I$. Then we will show that $b(j \to i)$ is the unique path in (1), from which (2) readily follows.

Since I, by the remark above, is a strong, proper subdigraph of S, we can use Corollary 2.2 to find a cycle b which is adjacent to I and not covered by I. Thus Lemma 3.6 implies that $b \in \mathscr{C}_+(S)$ and b covers I. Let $b(j \to i)$ be the path of adjacency, which obviously gives $\mathscr{P}(b(j \to i)) \subset I$. By definition of adjacency, $\mathscr{P}(b(i \to j)) \cap I = \{i, j\}$, so $I \subset \mathscr{P}(b)$ forces $I \subset \mathscr{P}(b(j \to i))$. Therefore, $\mathscr{P}(b(j \to i)) = I$. Note also that $\mathscr{P}(b(i \to j)) - I \neq \varnothing$.

CLAIM. *$b(j \to i)$ is a path in the subdigraph I, i.e., each arc of $b(j \to i)$ belongs to some negative cycle a, which necessarily has its point set in I by the GM-condition.*

Let $u(j \to i)$ be a path of the type specified in Lemma 3.4. If $b(j \to i) \neq u(j \to i)$, then let $k \neq i$ be the first point in $b(j \to i)$ for which $b(j \to k) = u(j \to k)$, but $(k, k_1) \subset b$ and $(k, k_2) \subset u$ with $k_1 \neq k_2$. Thus $v = u(j \to k_2) b(k_2 \to i)$ is a path from j to i (Figure 2).

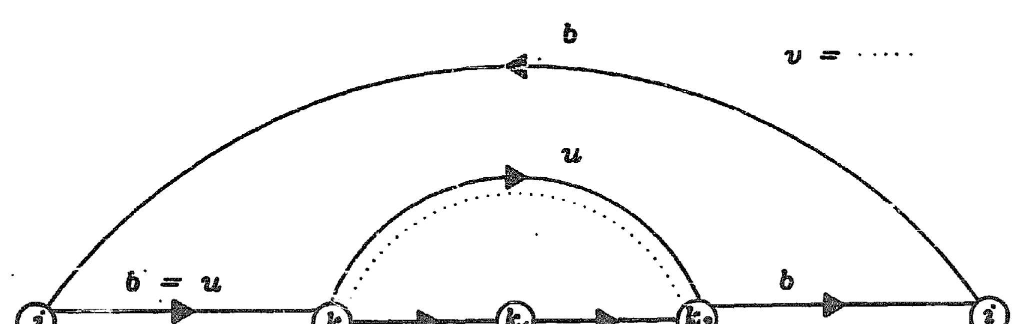

FIG. 2.

Moreover, $c = b(i \to j)v(j \to i)$ is a cycle not covered by I, since $\mathscr{P}(b(i \to j)) - I \neq \varnothing$. Again, by Lemma 3.6, $I \subset \mathscr{P}(c)$ and thus $I \subset \mathscr{P}(v(j \to i))$. But $k_1 \notin \mathscr{P}(v(j \to i))$. However, $k \neq i$ and $(k, k_1) \subset b$ imply $k_1 \in I$. This contradiction proves the claim.

The claim has established only that $b(j \to i)$ is the unique path in the subdigraph I from j to i, but not yet in S. The existence of other paths in S from j to i must next be eliminated.

Let $u(j \to i)$ be a path in S. If $b(j \to i) \neq u(j \to i)$, then as was done above, we define the points k, k_1, and k_2. Again $k_1 \in I$, since $k \neq i$. We consider several cases.

Case 1: $k_2 \in I$. As in the proof of the claim, we derive a contradiction from the cycle $c = u(j \to k_2)b(k_2 \to j)$.

Case 2: $k_2 \notin I$. Define $l \in I$ by $\mathscr{P}(u(k \to l)) \cap I = \{k, l\}$. Let $T = \mathscr{P}(b(i \to j)) \cap \mathscr{P}(u(k \to l))$. There are two possibilities:

(i) $T - \{k, l\} = \varnothing$. In this situation, $c = u(k \to l)b(l \to k)$ is a cycle not covered by I, so $I \subset \mathscr{P}(c)$, whence $k_1 \in \mathscr{P}(c)$. The definition of l forces k_1 to be a point in the path $b(l \to k)$. Since (k, k_1) is an arc of b, it follows that $k_1 = l$, i.e., (k, l) is an arc of b. By the claim, (k, l) is thus an arc of a negative cycle. Since it is also an arc of the positive cycle c, we can apply Lemma 2.5, which leads to the contradiction that $T - \{k, l\} \neq \varnothing$.

(ii) $T - \{k, l\} \neq \phi$. Define k_3 as the first point along $b(i \to j)$ such that $\mathscr{P}(b(i \to k_3)) \cap \mathscr{P}(u(k \to l)) = \{k_3\}$. (See Figure 3.) Then $v = b(i \to k_3)u(k_3 \to l)$ is a path such that $\mathscr{P}(v) \cap I = \{i, l\}$ and $\mathscr{P}(v) - I \neq \varnothing$. Thus the cycle $c = v(i \to l)b(l \to i)$ is not covered by I. As before, $I \subset \mathscr{P}(c)$, but since $k \notin \mathscr{P}(c)$, we have a contradiction.

In conclusion, $u(j \to i) = b(j \to i)$, and $b(j \to i)$ is the unique path in S from j to i.

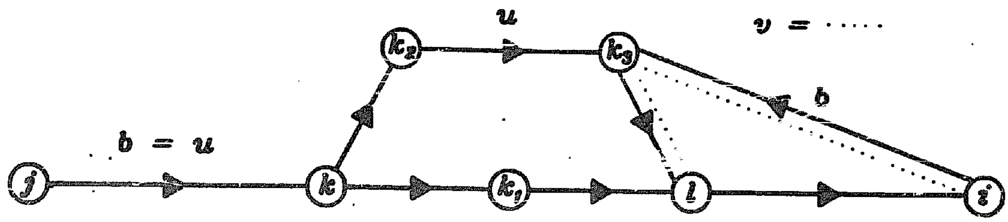

FIG. 3.

(2): Let $c \in \mathscr{C}_+(S)$ cover I. Then $i, j \in \mathscr{P}(c)$, and $c(j \to i)$ is a path in S from j to i. By (1), $c(j \to i) = b(j \to i)$, wherefore $\mathscr{P}(c(j \to i)) = \mathscr{P}(b(j \to i)) = I$. ∎

We point out here the relationship between the class of nondegenerate GM-graphs and the restricted class of the last result.

COROLLARY 3.8. *Let S be a GM-graph, and suppose* $|\mathscr{C}_+(S)| > 1$. *Then S is degenerage (every positive cycle c has* $\mathscr{P}(c) = \mathscr{P}$*) if and only if* $\mathscr{P}(S)$ *is a maximal point set.*

Proof. If S is itself a maximal point set, then S is clearly a degenerate GM-graph. So let I be a maximal point set of S, and suppose that $\mathscr{P}(S) - I \neq \varnothing$. Let $b, c \in \mathscr{C}_+(S)$ with $b \neq c$, and assume both b and c span S.

By Theorem 3.7, $b(j \to i) = c(j \to i)$ for some $i, j \in I$ such that $\mathscr{P}(b(j \to i)) = I$. So if $b \neq c$, there exists $k \in \mathscr{P}(b(i \to j)) - \{j\}$ such that $b(i \to k) = c(i \to k)$ with (k, k_1) an arc of b, (k, k_2) an arc of c, and $k_1 \neq k_2$.

Define the cycle $d = c(i \to k_2)b(k_2 \to i)$. Then $k_1 \notin \mathscr{P}(d)$, so d does not span S. Also $k_2 \notin I$, so $I \not\supset \mathscr{P}(d)$. Thus sign $d = +$ and S is not degenerate. ∎

We note that by removing arc (4,2) from the digraph in Figure 1 we obtain a degenerate GM-graph with maximal point set $\{2,3,4\} \not\subseteq \mathscr{P}(S)$.

With reference to the decomposition of $\mathscr{P}(S)$ in terms of its maximal point sets as given by Theorem 3.3, we see that in a nondegenerate GM-graph, either (a) $t > 1$ or (b) $t = 1$ with $K \neq \varnothing$ (and at least two positive cycles). The remaining results will then be phrased in terms of nondegeneracy of S, and any previous result requiring, equivalently, that S not be a maximal point set will be invoked without further mention of Corollary 3.8.

Let I be a maximal point set of a nondegenerate GM-graph S. We denote by $I(j, k)$ the unique path of S from j to i, where i, j are as in Theorem 3.7. This path makes I a linearly ordered set with minimum element j and maximum element i. It follows that for $k, k' \in I$ with $k < k'$ (in the ordering of I), there exists a unique path from k to k' in S, namely the subpath $I(k, k')$ of the path $I(j, i)$.

Every positive cycle properly covering I must leave I at i and enter I at j. Thus i is the unique exit vertex of I, and j is the unique entrance vertex of I.

We next describe the structure of a negative cycle of S relative to the maximal point set to which it belongs.

COROLLARY 3.9. *Let* $a \in \mathscr{C}_-(S)$ *and* $a \in I$. *If S is a nondegenerate GM-graph, then there exist unique* $l < k$ *in I such that* $a = (k, l)I(l, k)$.

Proof. By Theorem 3.7, I is linearly ordered. Let k be the maximum point of $\mathscr{P}(a)$ in the I-ordering. Then $l < k$, and $a = (k, l)I(l, k)$. ■

We can now describe the canonical form for a nondegenerate GM-graph S.

Denote the maximal point sets of S by I_p, $1 \leq p \leq t$, and label their entrance and exit points, respectively, as j_p, i_p. All arcs emanating from $k \in I_p - \{i_p\}$ must belong to a negative cycle of I_p. For arcs of the form (i_p, k) there are several possibilities:

(a) $k \notin I_p$. Then either $k \in K$ or $k = j_q$ for some $q \neq p$.

(b) $k \in I_p$. If $k \neq j_p$, then (i_p, k) belongs to a negative cycle of I_p. If $k = j_p$, then there are two subcases:

(1) (i_p, j_p) *is an arc of the digraph* I_p *only*. Then $(i_p, j_p)I_p(j_p, i_p)$ is a negative cycle covering I_p.

(2) (i_p, j_p) *is not an arc of* I_p *only*. Then $(i_p, j_p)I_p(j_p, i_p)$ is a positive cycle covered by I_p. A similar analysis can be made for arcs of the form (k, j_p).

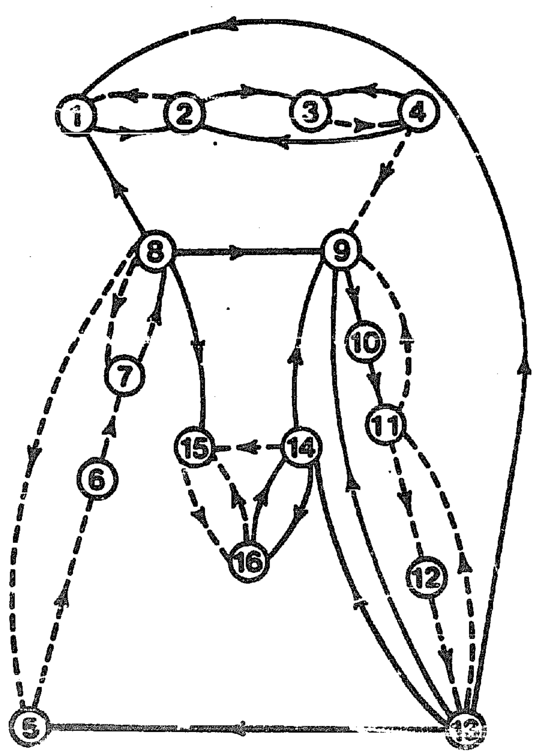

FIG. 4. A GM-graph in canonical form.

From the above, we see that the maximal point sets of a nondegenerate GM-graph S fall into two categories, depending upon whether or not they contain a cycle $b \in \mathscr{C}_+(S)$.

The following example illustrates all of the results we have obtained.

EXAMPLE 3.10. Observe that, in the diagraph shown in Figure 4 we have maximal point sets $I_1 = \{1,2,3,4\}$, $I_2 = \{5,6,7,8\}$, $I_3 = \{9,10,11,12,13\}$, and we have $K = \{14,15,16\}$. The entrance points are 1,5,9, and the exit points 4,8,13. I_1 and I_2 do not cover a positive cycle, but I_3 does.

4. CANONICAL FORM FOR A GM-MATRIX

We can now summarize all of our results and classify GM-matrices. Let A be an irreducible GM-matrix. We identify four subclasses. Note that in this section we make use of the hypothesis $a_{ii} < 0$ for the first time.

I. All cycles of $S(A)$ are negative. Then A is an irreducible L-matrix. This is a widely studied class, and we shall not discuss it further here.

II. All cycles of $S(A)$ are positive. In this case we can find a signature matrix D, i.e., $D = \operatorname{diag}[d_1, \ldots, d_n]$ with each $d_i = \pm 1$, such that $D^{-1}AD$ has nonnegative off-diagonal elements. Therefore, $-D^{-1}AD$ is a Z-matrix and has consistently signed principal minors if and only if it is an M-matrix. Since these matrices have also been widely studied, we shall also not discuss them further here (see [1] and [8]).

III. $S(A)$ is a degenerate GM-graph with $|\mathscr{C}_+(S)| \geqslant 1$ and $|\mathscr{C}_-(S)| \geqslant 1$.

IV. $S(A)$ is a nondegenerate GM-graph with $|\mathscr{C}_+(S)| \geqslant 1$ and $|\mathscr{C}_-(S)| \geqslant 1$.

The remainder of our work will be concerned with matrices in classes III and IV. In particular, we now can write a canonical form for nondegenerate GM-matrices having both positive and negative cycles.

Let $I_1, I_2, \ldots, I_r$ be maximal point sets of $S(A)$ not containing any positive cycles, and let $I_{r+1}, I_{r+2}, \ldots, I_{r+s}$ be maximal point sets of $S(A)$ each containing a unique positive cycle c_k, $k = 1, \ldots, s$, with the length of c_k equal to $|I_{r+k}|$. Finally let $K = \{1,2,\ldots,n\} - (\cup_{i=1}^{r+s} I_i$. Now suppose $\mathscr{P}$ is a permutation matrix permuting the set I_1 into $\{1,2,\ldots,|I_1|\}$, I_2 into $\{|I_1| + 1, \ldots, |I_1| + |I_2|\}$, etc. Without loss of generality we may suppose $j_1 = 1$, $i_1 = |I_1|$, $j_2 = |I_1| + 1$, $i_2 = |I_1| + |I_2|, \ldots$. We may then assert that the irreducible, nondegenerate GM-matrix $\hat{A} = P^T AP$ has the form

$$\hat{A} = \left[A_{pq} \right],$$

$1 \leqslant p, q \leqslant r+s+1$, where the following conditions hold on the blocks:

(1) Each block A_{pp}, $p=1,2,\ldots,r$, is a square irreducible, lower Hessenberg L-matrix.

(2) Each block A_{pp}, $p=r+1,\ldots,r+s$, is a square, lower Hessenberg matrix with negative elements on the principal diagonal. If we write this block in the form $A_{pp}=[a_{ij}^{k}]$, $1 \leqslant i, j \leqslant |I_p|$ $(=|I_{r+k}|)$, we have $c_k = a_{12}^{k}a_{23}^{k} \cdots a_{|I_p|-1,|I_p|}^{k}a_{|I_p|,1}^{k}$, and all other cycles are negative. Moreover, the block obtained from A_{pp} by setting $a_{|I_p|,1}=0$ is irreducible.

(3) Each block A_{pq} with $p \neq q$ and $1 \leqslant p, q \leqslant r+s$ is a zero block except possibly for a nonzero entry in the first column and last row.

(4) Each block $A_{p,r+s+1}$ is a zero block except possibly for nonzero entries in the last row.

(5) Each block $A_{r+s+1,q}$ is a zero block except possibly for nonzero entries in the first column.

(6) The block $A_{r+s+1,r+s+1}$ is square with negative principal diagonal and all positive cycles. As noted above, we may suppose it has nonnegative elements a_{ij}^{r+s+1} for $i \neq j$. Thus we may suppose this block is a Minkowski matrix (see [1] or [8]).

Now $\hat{A}$ itself must be irreducible. Let $\hat{D}(\hat{A})$ be the digraph with point set $V=\{1,2,\ldots,r+s+1\}$ and an arc (p,q) if $A_{pq} \neq 0$, $p \neq q$. Then, under the hypothesis that $A_{ar+s+1,r+s+1}$ is itself irreducible, $\hat{A}$ is irreducible if and only if $\hat{D}(\hat{A})$ is strongly connected. Observe also that the set K may be empty, in which case $r+s \geqslant 2$. On the other hand, in case IV we must always have $r+s \geqslant 1$.

5. DEGENERATE GM-MATRICES

For the degenerate GM-matrices we consider conditions under which they will be consistently signed.

If I is a proper subset of $\{1,2,\ldots,n\}$, then the principal submatrix $A[I]$ has the property that $S(A[I])$ has only negative cycles, i.e., $A[I]$ is an L-matrix. It follows that every principal minor or order less than n of a degenerate GM-matrix is consistently signed qualitatively. Thus it remains to consider det A itself.

Since $a_{ii}<0$ for $i=1,2,\ldots,n$, we may conveniently think of $S(A)$ as being enhanced by a loop at each point. Thus any factor f of $S(A)$ will, in general, consist of a spanning set of disjoint cycles including loops. Every factor of $S(A)$ corresponds to a term in the expansion of det A having sign $(-1)^n$ except for those factors consisting of a single positive cycle (of length n) of $S(A)$. They each contribute a term to the expansion of $\det A$ having sign $(-1)^{n+1}$. Now, in general, if $D(A)$ has the 1-factors $f_k = \{c_{k1}, c_{k2}, \dots, c_{km_k}\}$, $k = 1, 2, \dots, q$, then we have

$$\det A = (-1)^n \sum_{k=1}^{q} (-1)^{m_k} A[c_{k1}] A[c_{k2}] \cdots A[c_{km_k}],$$

(see, for example, [2]). Here $A[c_{kj}]$ is the product of elements of A corresponding to the cycle c_{kj} in $D(A)$ [or in $S(A)$]. Observe that when $m_k > 1$, each of the factors $A[c_{kj}] = -|A[c_{kj}]|$ and the sign of the term is $(-1)^n$ as cited above. But when $m_k = 1$ and $A[c_k] > 0$ we have sign $(-1)^{n+1}$ as asserted.

Let us write (1) in the form

$$\det A = (-1)^{n+1} A_n^+ + (-1)^n \sum_k (-1)^{m_k} A[c_{k1}] A[c_{k2}] \cdots A[c_{km_k}],$$

where the second term ranges over all k such that $m_k > 1$. [Note that such terms exist by virtue of the fact that $a_{ii} < 0$, $i = 1, 2, \dots, n$, and $|\mathscr{C}_-(S)| \geq 1$ by hypothesis.] Thus we have

$$\det A = (-1)^{n+1} A_n^+ + (-1)^n \sum_{m_k \geq 2} |A[c_{k1}] \cdots [c_{km_k}]|.$$

It follows that sign det $A = \operatorname{sign}(-1)^n$ if and only if

$$\sum_{\substack{k \\ m_k \geq 2}} |A[c_{k1}] A[c_{k2}] \cdots A[c_{km_k}]| > A_n^+,$$

where A_n^+ is the sum of the values of the positive cycles (all of length n) of A. Thus the degenerate GM-matrix has consistently signed principal minors if and only if the above inequality holds.

6. CONSISTENTLY SIGNED NONDEGENERATE GM-MATRICES

For GM-matrices in the class IV we shall use the concepts introduced in [6]. This involves the classification of the elements of the matrix A into friendly, unfriendly, and neutral elements and the introduction of the matrix $\hat{A}_{\min}$.

In each of the first r diagonal blocks of A the only elements belonging to positive cycles are those above the principal diagonal. But all of these elements also belong to (at least one) negative cycles; hence they are neutral elements. Similarly the elements above the principal diagonal in the block A_{pp} for $p = r + k$, $k = 1, \ldots, s$, are all neutral. For $p = 1, \ldots, r$, all nonzero elements below the principal diagonal of A_{pp} are friendly, so we can take $A_{pp,\min}$ for $p = 1, 2, \ldots, r$, to be the matrix with first superdiagonal and principal diagonal elements the same as those of A and all other elements equal to zero. For $p = r + 1, \ldots, r + s$ we can define $A_{pp,\min}$ in exactly the same way except for the element in the last row and column, which is also the same as the corresponding element of A_{pp}. Thus we may take $\hat{A}_{\min}$ as having the form

$$\hat{A}_{\min} = \left[A_{pq,\min}\right],$$

where

(α) $A_{pp,\min}$ has been defined above for $1 \leqslant p \leqslant r + s$, and
(β) $A_{pq,\min} = A_{pq}$ for all other values of p, q,

since all of these blocks contain only unfriendly elements.

In summary, then, all of the elements on the first superdiagonal of $\hat{A}_{\min}$ within the blocks $A_{pp,\min}$, $p = 1, 2, \ldots, r + s$, are neutral elements. The remaining nonzero elements of $\hat{A}_{\min}$ not on the principal diagonal are all unfriendly elements. Of course, all elements of $\hat{A} - \hat{A}_{\min}$ are friendly elements of $\hat{A}$. Thus we have completely classified the elements of $\hat{A}$.

Now by Theorem 3 of [6], $\hat{A}$ is consistently signed if $\hat{A}_{\min}$ is consistently signed. Also, by Theorem 4 of [6], if $\hat{A}_{\min}$ is diagonally dominant, it is consistently signed. Thus we have simple conditions for $\hat{A}$ to be consistently signed.

To investigate the matter more closely, let us denote the exit points of the diagonal blocks by $i_1, i_2, \ldots, i_{r+s}$, respectively. Let $K = \{n - i_{r+s}, i_{r+s} + 1, \ldots, n\}$, and consider the set $I_0 = \{i_1, i_2, \ldots, i_{r+s} + 1, i_{r+s} + 2, \ldots, n\}$. It satisfies

$$1 < i_1 < i_2 < \cdots < i_{r+s} < i_{r+s} + 1 < \cdots < n.$$

Now for $j \notin I_0$ the diagonal dominance condition is

$$-a_{jj} > |a_{j,j+1}|.$$

On the other hand, for $j \in I_0$ the condition takes the form

$$-a_{jj} > \sum_{k \in I_j} |a_{jk}|.$$

Here $|I_j| \geq 2$ and $I_j \subseteq J_i = \{j_1, j_2, \ldots, j_{r+s}, i_{r+s}+1, \ldots, n\}$, where j_p is the entrance point of the pth diagonal block for $p = 1, 2, \ldots, r+s$. (Note that $j_1 = 1$ and

$$j_1 < j_2 < \cdots < j_{r+s} < j_{r+s} + 1 < \cdots < n.)$$

Now it is clear that, if the set I is disjoint from I_0, then the principal minor $\det \hat{A}[I]$ is qualitatively consistently signed. But we can say more. If

(i) $I \cap I_{r+s+1} = \varnothing$,
(ii) I does not contain I_q for $q = r+1, \ldots, r+s$, and
(iii) I contains at most one of the sets I_q for $q = 1, \ldots, r$,

then $\det \hat{A}[I]$ is also qualitatively consistently signed.

Finally, the following interesting fact is worth noting. For the diagonal submatrices $A_{pp,\min}$, $p = r+1, \ldots, r+s$, we have that $\det A_{pp,\min}$ will be consistently signed if and only if the product of the absolute values of the diagonal elements is larger than the value of the (unique) positive cycle in $A_{pp,\min}$. In general, this condition may differ from the diagonal dominance condition applied to this submatrix.

7. INVERSES

We shall suppose now that the GM-matrix in class IV has consistently signed principal minors. The paper [6] contains a theorem about the inverse of such a matrix in general. From that result we obtain the following information about inverses of GM-matrices of class IV.

If we partition A^{-1} in the same way as A itself, then:

(i) Each diagonal block A_{pp}^{-1}, $p = 1, \ldots, r+s$, has all elements on and above the principal diagonal nonzero and with qualitatively determined signs; all elements on the first subdiagonal have indeterminate signs; and all other elements α_{ij} below the principal diagonal have qualitatively determined signs if and only if i and j do not both belong to the same negative cycle of I_p.

(ii) The nonzero elements of $A_{r+s+1,r+s+1}^{-1}$ are all negative.

(iii) If symmetrically placed elements in any diagonal block of A^{-1} are both sign determined, they will have the same sign, and if the block A_{pp} satisfies the condition that $a_{ij}^p \neq 0$ for all $i > j$, then A_{pp}^{-1} has all elements below the principal diagonal with undeterminate signs.

(iv) Each of the blocks A_{pq}^{-1} for $p \neq q$, $1 \leqslant p, q \leqslant r+s$, is a quasi-Morishima matrix in normal form [3] and with all elements different from zero.

(v) The elements in $A_{r+s+1,q}^{-1}$ vary in sign only with q, and those of $A_{p,r+s+1}^{-1}$ vary only with p, and all elements in each block are nonzero and have the same sign.

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